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REMARKS ON A COPULA-BASED CONDITIONAL VALUE AT RISK

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ABSTRACT. We deal with a multivariate Conditional Value at Risk. Compared to the usual notion for the single random variable, a multivariate Value at Risk is concerned with several variables, and thus the relation between each risk factors should be taken into account. We here introduce a new definition of copula-based conditional Value at Risk, (CCVaR), which is ready to be computed. Copulas are known to provide a flexible method for investigating a possible nonlinear structure; copulas may be naturally involved in the theory of Value at Risk. We derive a formula of our CCVaR in the case of Archimedean copulas. Examples show that our proposed definition works effectively.

1. INTRODUCTION

It is well aclaimed that Value at Risk (VaR) provides one of central risk measures in the area of risk management. Because of its usefulness, VaR plays a principal role in measuring various risk factors. We refer to Duffie and Pan [2] for instance. See also [4]. One of drawbacks is that VaR does not satisfy the axioms of coherent risk measure, and to remedy this point, the so-called conditional Value at Risk (CVaR) is introduced. See Section 2 for the details.

VaR is defined for a single random variable, and there has been much effort such that the definition is extended to involve multivariate random vectors. Indeed, in the pioneering work of [12], Prékopa considers a vector valued multivariate Value at risk (MVaR). We may wonder, however, whether MVaR really serves as a risk measure; in other words, whether MVaR characterizes effectively the risk structure of multiple random variables, especially, the nonlinear dependence relation between each risk factors. The answer is partially yes and partially still under developing.

Copulas, we have to recall at this point, are well recognized functions, which provide a useful tool for understanding the dependence relation among random variables (see for example [5]). Because of their flexibility, copulas are now widely employed in the research of dependence structure of random variables. It is then natural and desirable to define MVaR through the formulation of copulas. Along this prospect, Krzemienowski and Szymczyk [8] has introduced the concept of copula-based conditional Value at Risk. Their definition is, however, somewhat complicated and it seems that the computation is hard and requires much task.

Here we introduce a new definition of copula-based conditional Value at Risk (CCVaR), which is rather simple, easy to calculate, and also enjoys nice properties. It is noted that our CCVaR extends the multivariate conditional Value at Risk introduced by Lee and Prékopa

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[9]. Moreover, in the case of Archimedean copulas, which is a one parameter family of copulas, a handy formula of our CCVaR is obtained. Examples show that the formula works well to estimate the nonlinear relation between risk factors.

The paper is organized as follows: Section 2 gives basic definition and properties of Value at Risk and copulas. New notion of copula-based Value at Risk is presented in Section 3. Our main results with examples are addressed in Section 4. Section 5 concludes with discussions.

2. Preliminary

2.1. **Value at Risk.** We begin with recalling the notion of Value at Risk of the single variable for completeness.

Let X be a random variable, which is assumed to be continuous for simplisity, and let $F_X(x) = P(X \leq x)$ denote the distribution function. The Value at Risk (VaR_β) at the confidence level β ($0 \leq \beta < 1$) is defined by

$$\mathrm{VaR}_\beta(X) := F_X^{(-1)}(\beta) = \inf\{u \,|\, F_X(u) \geq \beta\}.$$

The conditional Value at Risk (CVaR_β) is then formulated as

$$\begin{aligned}\mathrm{CVaR}_\beta(X) &:= \frac{1}{1-\beta}\int_\beta^1 \mathrm{VaR}_t(X)dt = \frac{1}{1-\beta}\int_\beta^1 F_X^{(-1)}(t)dt \\ &= \frac{1}{1-\beta}\int_{F_X^{(-1)}(\beta)}^{\infty} u dF_X(u).\end{aligned}$$

It is known that VaR is not coherent but CVaR is so. Here we recall that a risk measure $\rho(X)$ for a random variable X in some specified set is said to be coherent if it verifies the next conditions. See Artzner et al. [1]. We note that the conditions is modified in our setting.

(i) $\rho(0) = 0$;
(ii) $\rho(X + k) = \rho(X) + k \quad (k \in \mathbb{R})$;
(iii) $X_1 \leq X_2$ implies $\rho(X_1) \leq \rho(X_2)$;
(iv) $\rho(sX) = s\rho(X) \quad (s > 0)$;
(v) $\rho(X_1 + X_2) \leq \rho(X_1) + \rho(X_2)$.

We remark that VaR fails in (v) in general. It is also recognaized that a risk measure is believed to had better satisfy these properties.

2.2. **Copula.** Next we recall the definition of copulas in the case of bivariate joint distribution. For a general reference, we refer to Durante and Sempi [3], Joe [7], and Nelsen [11] for examples. See also [6][14].

Definition (copula) A function C defined on $\mathbb{I}^2 := [0,1] \times [0,1]$ and valued in $\mathbb{I} := [0,1]$ is said to be a copula if the following conditions are satisfied.

(i) For every $(u, v) \in \mathbb{I}^2$,

$$\begin{aligned} &C(u,0) = C(0,v) = 0, \\ &C(u,1) = u \quad \text{and} \quad C(1,v) = v. \end{aligned} \tag{2.1}$$

(ii) For every $(u_i, v_i) \in \mathbb{I}^2$ $(i = 1, 2)$ with $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$C(u_1, v_1) - C(u_1, v_2) - C(u_2, v_1) + C(u_2, v_2) \geq 0. \tag{2.2}$$

The requirement (2.2) is referred to as *the 2-increasing condition*. It is noted that a copula is a continuous function by its definition.

The well-known result due to Sklar [13], who employed the term "copula" almost for the first time, gives the basic property of copulas. We here recall Sklar's theorem in bivariate case, for completeness of our presentation.

Theorem 1. *(Sklar's theorem) Let H be a bivariate joint distribution function with marginal distribution functions F_1 and F_2; that is,*

$$\lim_{x\to\infty} H(x,y) = F_2(y), \qquad \lim_{y\to\infty} H(x,y) = F_1(x).$$

Then there exists a copula, which is uniquely determined on $\mathrm{Ran}\, F_1 \times \mathrm{Ran}\, F_2$, such that

$$H(x,y) = C(F_1(x), F_2(y)). \tag{2.3}$$

Conversely, if C is a copula and F and G are distribution functions, then the function H defined by (2.3) is a bivariate joint distribution function with margins F and G.

An important class of copulas is given by the so-called Archimedean copulas. We recall for completeness what are the Archimedean copulas.

Let $\varphi : \mathbb{I} \to [0,\infty]$ be a convex function such that φ is strictly decreasing and verifies $\varphi(1) = 0$. Let $\varphi^{(-1)}$ denote the pseudo-inverse of φ; that is, $\mathrm{Dom}\,\varphi^{(-1)} = [0,\infty]$, $\mathrm{Ran}\,\varphi^{(-1)} = \mathbb{I}$, and

$$\varphi^{(-1)}(t) = \begin{cases} \varphi^{-1}(t) & (0 \le t \le \varphi(0)) \\ 0 & (\varphi(0) \le t \le \infty). \end{cases}$$

It is then possible to prove that the function C defined on $\mathbb{I}^2$ by

$$C(u,v) = \varphi^{(-1)}(\varphi(u) + \varphi(v)) \tag{2.4}$$

provides a copula. Copulas of the form (2.4) are called Archimedean copulas and the function φ is called a generator of the copula.

The class of Archimedean copula finds a wide range of applications, because it is determined through single generator. Some examples are exhibited in §4. For a general reference concerning Archimedean copulas, we refer for instance to a book by Nelsen [11].

3. Copula-based conditional Value at Risk

Value at Risk is typically defined for a single random variable. It is our intension that multivariate random variables should be incorporated into the definition of Value at Risk, which will be more useful to application. Several attempts have been already undertaken. For example, Prékopa [12] introduce a multivariate Value at Risk for random vector, which is vector valued. However, because of the fact that the measure is vector valued, the order relation becomes slightly indirect.

Recently, Krzemienowski and Szymczyk [8] introduced a nice idea of copula-based conditional Value at Risk. Here we recall their definition in the bivariate case for the readers' sake.

Let $\mathbf{X} = (X_1, X_2)$ be a bivariate random vector with the distribution functions $F_{X_j}(t) = P(X_j \le t)$ $(j = 1,2)$. Given a copula C, $H(x,y) = C(F_{X_1}(x), F_{X_2}(y))$ becomes a joint distribution function. Let

$$\mathcal{U}_\beta^{KS} = \{(u,v) \in \mathbb{R}^2 \,|\, C(u,v) = \beta\}.$$

A copula-based conditional Value at Risk ($\mathrm{CCVaR}_\beta^{KS}(\mathbf{X})$) due to Krzemienowski and Szymczyk is then defined through

$$\mathrm{CCVaR}_\beta^{KS}(\mathbf{X}) = \frac{1}{\beta} \min_{(u,v)\in\mathcal{U}_\beta^{KS}} \int_0^u \int_0^v (F_{X_1}^{(-1)}(p) + F_{X_2}^{(-1)}(q)) dC(p,q). \tag{3.1}$$

However, since the risk measure involves the minimum procedure, the computation may become messy. For example, if $C(u,v) = \Pi(u,v) = uv$, namely, X_1 and X_2 are independent, then we see that

$$\begin{aligned}\mathrm{CCVaR}_\beta^{KS}(\Pi) &= \frac{1}{\beta} \min_{\beta\le u\le 1} \int_0^u \int_0^{\frac{\beta}{u}} (F_{X_1}^{(-1)}(p) + F_{X_2}^{(-1)}(q)) dpdq \\ &= \frac{1}{\beta} \min_{\beta\le u\le 1} \Big(\frac{\beta}{u} \int_0^u F_{X_1}^{(-1)}(p)dp + u \int_0^{\frac{\beta}{u}} F_{X_2}^{(-1)}(q)dq \Big),\end{aligned}$$

taking into account of the fact

$$\mathcal{U}_\beta^{KS}(\Pi) = \left\{ \left(u, \frac{\beta}{u}\right) \in \mathbb{R}^2 \,|\, \beta \le u \le 1 \right\}.$$

Here we propose another slightly different definition of copula-based multivariate conditional Value at Risk. We confine ourselves to the bivariate case as before for simplicity and let $\mathbf{X} = (X_1, X_2)$ be a random vector with the joint distribution function $H(x,y) = P(X_1 \le x, X_2 \le y)$ as well as marginal distribution functions $F_{X_j}(x) = P(X_j \le x)$ $(j = 1,2)$. Observing the definition of multivariate conditional Value at Risk introduced by Lee and Prékopa [9], we now formulate our definition as follows:

Definition For a random vector $\mathbf{X} = (X_1, X_2)$, a copula-based conditional Value at Risk ($\mathrm{CCVaR}_\beta(\mathbf{X})$) at the confidence level β $(0 \le \beta < 1)$ is defined by

$$\mathrm{CCVaR}_\beta(\mathbf{X}) = \frac{\iint_{\mathcal{U}_\beta} (\lambda F_{X_1}^{(-1)}(u) + (1-\lambda) F_{X_2}^{(-1)}(v)) dC(u,v)}{\iint_{\mathcal{U}_\beta} dC(u,v)}, \tag{3.2}$$

where $0 < \lambda < 1$ and we have put

$$\mathcal{U}_\beta := \{(u,v) \,|\, C(u,v) \ge \beta\}.$$

The constant λ represents the portfolio aspect of X_1 and X_2. If both X_1 and X_2 follows the same distribution, then the impact due to λ will be irrelvant. We also remark that for some C the denominator is zero and/or for some (X_1, X_2) the numerator is infinite.

It is to be noted that if we write temporally for abuse of notation

$$E^C[f] = \iint_{\mathbb{I}^2} f(u,v) dC(u,v),$$

then our CCVaR can be written as

$$\mathrm{CCVaR}_\beta(\mathbf{X}) = E^C[\lambda^t \mathbf{F}_{\mathbf{X}}^{(-1)} \,|\, \mathcal{U}_\beta]$$

where $\lambda^t = (\lambda, 1-\lambda)$, which indicates that our Definition above extends Definition 3 of Lee and Prékopa [9].

Our CCVaR of (3.2) is simpler than CCVaR^{KS} of (3.1). Nevertheless, our definition seems work well as a risk measure, which will be assured by the computation of examples in the next section.

4. Main results

First we begin with making sure about basic properties which our CCVaR satisfies.

Proposition 2. *A copula-based conditional Value at Risk* CCVaR *defined by (3.2) verifies* (i)(ii)(iv) *above; that is,*
(i) $\mathrm{CCVaR}_\beta(\mathbf{0}) = 0$,
(ii) $\mathrm{CCVaR}_\beta(\mathbf{X} + k\mathbf{e}) = \mathrm{CCVaR}_\beta(\mathbf{X}) + k \qquad (k \in \mathbb{R}, \mathbf{e} = (1,1))$,
(iv) $\mathrm{CCVaR}_\beta(s\mathbf{X}) = s\mathrm{CCVaR}_\beta(\mathbf{X}) \quad (s > 0)$.

The proof is performed along the similar line of that for VaR and we may safely omit the details.

Several remarks are in order. Concerning the monotonicity (iii), we need to clarify the meaning of the order between $\mathbf{X}_1$ and $\mathbf{X}_2$; we had better avoid unfavorable assumption and we do not treat it here. For the subadditivity (v), it does not seem to be true in the general setting. Observe Theorem 8 and the example in [9].

Now we state our main theorem of this article, which shows what our CCVaR is if the copula C is Archimedean.

Theorem 3. *Let $\mathbf{X} = (X_1, X_2)$ be a nonnegative random vector, whose joint distribution function is provided by an Archimedean copula C of the form (2.4), where the generator φ is C^1-class. Then our proposed copula-based conditional Value at Risk (*CCVaR*) of (3.2) is expressed as*

$$\mathrm{CCVaR}_\beta(\mathbf{X}) = \frac{\int_\beta^1 (\lambda F_{X_1}^{(-1)}(t) + (1-\lambda) F_{X_2}^{(-1)}(t)) \left(1 - \frac{\varphi'(t)}{\varphi'(\beta)}\right) dt}{1 - \beta + \frac{\varphi(\beta)}{\varphi'(\beta)}}. \tag{4.1}$$

Proof. The proof is implemented in an elementary fashion. By the standard approximation argument, we may assume that φ is C^2-class. If the copula C is Archimedean of the above form (2.4), we learn that

$$dC(u,v) = \frac{-\varphi''(t)}{(\varphi'(t))^3} \varphi'(u)\varphi'(v) du dv,$$

where we have put $t = \varphi^{(-1)}(\varphi(u) + \varphi(v))$. Taking into account of the symmetry of u, v, we have

$$\begin{aligned} &\mathrm{CCVaR}_\beta(\mathbf{X}) \\ &= \frac{1}{\iint_{\{\varphi(u)+\varphi(v) \le \varphi(\beta)\}} \frac{-\varphi''(t)}{(\varphi'(t))^3} \varphi'(u)\varphi'(v) du dv} \\ &\quad \cdot \left\{ \iint_{\{\varphi(u)+\varphi(v) \le \varphi(\beta)\}} (\lambda F_{X_1}^{(-1)}(u) + (1-\lambda) F_{X_2}^{(-1)}(u)) \frac{-\varphi''(t)}{(\varphi'(t))^3} \varphi'(u)\varphi'(v) du dv \right\}. \end{aligned}$$

Now applying the change of variables

$$(u,v) \to (u,t) \quad \text{where} \quad t = \varphi^{(-1)}(\varphi(u) + \varphi(v)),$$

we infer that

$$\begin{aligned}
&\mathrm{CCVaR}_\beta(\mathbf{X}) \\
&= \frac{1}{\iint_{\{\beta\le t\le u\}} \frac{-\varphi''(t)}{(\varphi'(t))^2}\varphi'(u)dudt} \\
&\quad \cdot \Big\{ \iint_{\{\beta\le t\le u\}} (\lambda F_{X_1}^{(-1)}(u) + (1-\lambda)F_{X_2}^{(-1)}(u)) \frac{-\varphi''(t)}{(\varphi'(t))^2}\varphi'(u)dudt \Big\} \\
&= \frac{\int_\beta^1 \Big((\lambda F_{X_1}^{(-1)}(u) + (1-\lambda)F_{X_2}^{(-1)}(u))\varphi'(u) \int_\beta^u \frac{-\varphi''(t)}{(\varphi'(t))^2}dt \Big) du}{\int_\beta^1 \varphi'(u)du \int_\beta^u \frac{-\varphi''(t)}{(\varphi'(t))^2}dt} \\
&= \frac{\int_\beta^1 (\lambda F_{X_1}^{(-1)}(u) + (1-\lambda)F_{X_2}^{(-1)}(u))\varphi'(u) \Big[\frac{1}{\varphi'(t)}\Big]_\beta^u du}{\int_\beta^1 \varphi'(u) \Big[\frac{1}{\varphi'(t)}\Big]_\beta^u du} \\
&= \frac{\int_\beta^1 (\lambda F_{X_1}^{(-1)}(u) + (1-\lambda)F_{X_2}^{(-1)}(u)) \Big(1 - \frac{\varphi'(u)}{\varphi'(\beta)}\Big) du}{1-\beta+\frac{\varphi(\beta)}{\varphi'(\beta)}},
\end{aligned}$$

which implies the theorem. □

We remark that a similar calculation for the denominator is already employed in the literature (see for instance Theprem 4.3.4 in Nelsen [11]).

If the generator is $\varphi(t) = -\log t$, then the corresponding Archimedean copula is $\Pi(u, v) = uv$, that is, the product copula which represents the independence relation. In this case, the relevant CCVaR reduces to the multivariate conditional Value at Risk (MCVaR) due to Lee and Prékopa [9], namely,

$$\mathrm{MCVaR}_\beta(\mathbf{X}) = \frac{\int_\beta^1 (\lambda F_{X_1}^{(-1)}(t) + (1-\lambda)F_{X_2}^{(-1)}(t)) \Big(1 - \frac{\beta}{t}\Big) dt}{1-\beta+\beta\log\beta}.$$

We then obtain the next property immediately from Theorem.

Corollary 4. *If the generator φ verifies for $\beta \le t \le 1$*

$$\begin{gathered}
\frac{1-\frac{\varphi'(t)}{\varphi'(\beta)}}{1-\beta+\frac{\varphi(\beta)}{\varphi'(\beta)}} \ge \frac{1-\frac{\beta}{t}}{1-\beta+\beta\log\beta} \\
\Bigg(\textit{resp.}, \frac{1-\frac{\varphi'(t)}{\varphi'(\beta)}}{1-\beta+\frac{\varphi(\beta)}{\varphi'(\beta)}} \le \frac{1-\frac{\beta}{t}}{1-\beta+\beta\log\beta}\Bigg),
\end{gathered} \tag{4.2}$$

then the corresponding CCVaR *is not less than (resp., not greater than) the one for the independent relation. Precisely stated, we have*

$$\begin{gathered}
\mathrm{CCVaR}_\beta(\mathbf{X}) \ge \mathrm{MCVaR}_\beta(\mathbf{X}) \\
(\textit{resp.}, \mathrm{CCVaR}_\beta(\mathbf{X}) \le \mathrm{MCVaR}_\beta(\mathbf{X})).
\end{gathered}$$

This Corollary implies that the employment of copulas makes it possible to estimate the total risk of several risk factors effectively, in comparison with the standard assumption of the independence.

We here present examples to illustrate our observations.

Example 5. Let the generator be $\varphi(t) = \log(t^{-1}(1-\theta(1-t)))$ for $\theta \in [-1,1)$; that is, we consider the Ali-Mikhail-Haq family, which yields

$$C(u,v) = \frac{uv}{1-\theta(1-u)(1-v)}.$$

We then see that the corresponding CCVaR^{AMH} becomes

$$\mathrm{CCVaR}_\beta^{AMH}(\mathbf{X}) = \frac{\int_\beta^1 (\lambda F_{X_1}^{(-1)}(t) + (1-\lambda)F_{X_2}^{(-1)}(t))\Big(1 - \frac{\beta(1-\theta(1-\beta))}{t(1-\theta(1-t))}\Big)dt}{1 - \beta + \frac{\beta(1-\theta(1-\beta))}{1-\theta}\log\frac{\beta}{1-\theta(1-\beta)}}$$

We further compute that, for the denominator

$$\frac{\varphi(\beta)}{\varphi'(\beta)} = \frac{\beta(1-\theta(1-\beta))}{1-\theta}\log\frac{\beta}{1-\theta(1-\beta)} \le \beta\log\beta,$$

and, for the numerator with $\beta \le t \le 1$

$$\frac{\varphi'(t)}{\varphi'(\beta)} = \frac{\beta(1-\theta(1-\beta))}{t(1-\theta(1-t))} \le \frac{\beta}{t}.$$

Thus it follows that

$$\mathrm{CCVaR}_\beta^{AMH}(\mathbf{X}) \ge \mathrm{MCVaR}_\beta(\mathbf{X}).$$

Example 6. Let the generator be $\varphi(t) = \log(1-\theta\log t)$ for $0 < \theta \le 1$; that is, we consider the Gumbel-Hougaard family, which yields

$$C(u,v) = uv\exp(-\theta\log u\log v).$$

We then see that the corresponding CCVaR^{GH} becomes

$$\mathrm{CCVaR}_\beta^{GH}(\mathbf{X}) = \frac{\int_\beta^1 (\lambda F_{X_1}^{(-1)}(t) + (1-\lambda)F_{X_2}^{(-1)}(t))\Big(1 - \frac{\beta(-\log t)^{\theta-1}}{t(-\log\beta)^{\theta-1}}\Big)dt}{1-\beta-\frac{\beta}{\theta}(1-\theta\log\beta)\log(1-\theta\log\beta)}.$$

We further compute that, for the denominator, with a little complicated calculation

$$\frac{\varphi(\beta)}{\varphi'(\beta)} = -\frac{\beta}{\theta}(1-\theta\log\beta)\log(1-\theta\log\beta) \ge \beta\log\beta,$$

and, for the numerator with $\beta \le t \le 1$ (we note that $0 < \theta \le 1$)

$$\frac{\varphi'(t)}{\varphi'(\beta)} = \frac{\beta}{t}\frac{1-\theta\log\beta}{1-\theta\log t} \ge \frac{\beta}{t}.$$

Thus it follows that

$$\mathrm{CCVaR}_\beta^{GH}(\mathbf{X}) \le \mathrm{MCVaR}_\beta(\mathbf{X}).$$

Example 7. Let the generator be $\varphi(t) = (-\log t)^\theta$ for $1 \le \theta < \infty$; that is, we consider the Gumbel family, which yields

$$C(u,v) = \exp(-((-\log u)^\theta + (-\log v)^\theta)^{1/\theta}).$$

We then see that the corresponding CCVaR^G becomes

$$\mathrm{CCVaR}_\beta^{G}(\mathbf{X}) = \frac{\int_\beta^1 (\lambda F_{X_1}^{(-1)}(t) + (1-\lambda)F_{X_2}^{(-1)}(t))\Big(1 - \frac{\beta(-\log t)^{\theta-1}}{t(-\log\beta)^{\theta-1}}\Big)dt}{1-\beta+\frac{\beta}{\theta}\log\beta}.$$

We further compute that, however, for the denominator

$$\frac{\varphi(\beta)}{\varphi'(\beta)} = \frac{\beta}{\theta} \log \beta \geq \beta \log \beta,$$

and, for the numerator with $\beta \leq t \leq 1$

$$\frac{\varphi'(t)}{\varphi'(\beta)} = \frac{(-\log t)^{\theta-1}}{(-\log \beta)^{\theta-1}} \frac{\beta}{t} \leq \frac{\beta}{t}.$$

Thus the order relation between CCVaR^G and MCVaR is not clear. Indeed the inequalities (4.2) is subject to the value of β.

5. Conclusion

We have developed a new risk measure of copula-based conditional Value at Risk (CCVaR). The measure is defined for a multivariate random vector and thus the effect of the relation between each risk component is taken into account. The copula function is involved to this purpose, which is known to provide a flexible and handy tool to investigate possible nonlinear relations among risk factors.

Compared to the previous definition due to Krzemienowski and Szymczyk [8], our introduced measure is rather simple and ready to be computed. We have established the formula of our CCVaR in the case of Archimedean copulas through the use of its generator. Examples show that our introduced quantity works well for a multivariate random vector under the presence of dependence structure. In particular, the difference from the standard assumption of the independence is able to be characterized.

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