Dynamic Goodwin's Business Cycles with Fixed and Continuously Distributed Time Delays*

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Abstract

Dynamic extensions of the delayed Goodwin's business cycle model are examined. Conditions for the local stability of the stationary state will be determined in the case of fixed and continuously distributed time lags. These conditions and the stability regimes will be compared with different types of weighting functions. We will show that the stability regions for continuously distributed time delay are always larger than in the case of fixed time lags and they converge to the stability region with fixed time lag as the variance of the delay converges to zero..

Key words: fixed time delay, continuously distributed time delay, business cycle

1 Introduction

In this paper the business cycle model of Goodwin (1951) will be examined in which a time delay is introduced between decision to invest and the corresponding outlays. There is always a time lag due to the decision making process and the decision implementation. Time delays can be modeled in two different ways. In the first case fixed time delays are considered, and the characteristic equations of the corresponding dynamic equations are the mixtures of polynomial and exponential terms resulting in an infinite spectrum. The stability theory of such dynamic equations is much more complicated than that of ordinary differential equations without delays. A comprehensive summary of the relevant results is presented, for example, in Kuang (1993). The alternative approach is to consider continuously distributed time lags, when at each time period t, a weighted average of all data from zero up to t is used in the dynamic equations.

*This paper was prepared when the first author visited the Department of Systems and Industrial Engineering of the University of Arizona. He appreciated its hospitality over his stay. The usual disclaimer applies.

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This method is very realistic, since the length of a time delay is usually uncertain, so if it is considered a random variable, continuously distributed time lags represent average delayed data. Very often firms do not want to react to sudden changes, they prefer to react to an average of past information. In the case of continuously distributed time delays with special weighting functions, we have a finite spectrum similarly to models without time delays.

In this paper we will compare the delayed Goodwin's model with fixed and continuously distributed time lags. We will examine local stability of the stationary state and in the case of stability loss the possibility of the birth of limit cycles will be examined via Hopf bifurcation. In Matsumoto and Szidarovszky (2009), a linearized version of Goodwin's model with fixed lag is compared with the one with continuously distributed lag having the exponentially declining weighting function. It is demonstrated that both models generate the same dynamics when the lags are small and different dynamics when they are large. In this study we focus on more general weighting functions

The paper develops as follows. After the basic model without time delay will be introduced and solved, fixed time lags will be considered. We will determine the stability region in the parameter space of the model. Then the case of continuously distributed time delay will be examined first with exponential and gamma type weighting functions. In addition to deriving stability conditions and determining the stability regions, we will compare these results for different types of weighting functions. In the last section conclusions will be drawn.

2 Goodwin's Business Cycle Model

This section is divided into three parts. In Section 2.1, we recapitulate the basic elements of Goodwin's model. We, then, adopt an explicit treatment of the investment lag into the model. We will examine fixed time delays in Section 2.2 and continuously distributed time delays in Section 2.3, and we will show how such a delay affects the characteristics of the cyclic dynamics.

2.1 Basic Model

Goodwin (1951) presents five different versions of the nonlinear accelerator model. The first version assumes a piecewise linear function with three levels of investment, which can be thought as the crudest or simplest version of the non-linear accelerator. This is a text-book model that can give a simple exhibition on how nonlinearities give rise to endogenous cycles without relying on structurally unstable parameters, exogenous shocks, etc. The second version replaces the piecewise linear investment function with a smooth nonlinear investment function. Although persistent cyclical output oscillations are shown to exist, the second version includes a unfavorable phenomenon, namely, discontinuous investment jump, which is not realistic in the real economic world. "In order to come close to reality" (p.11, Goodwin (1951)), an investment lag is introduced in the third version. However, no analytical considerations are given to this version. The existence of a business cycle is confirmed in the fourth version, which is a linear approximation of the third version with respect to the investment lag. And finally, alternation of autonomous expenditure over time is taken into account in the fifth version.

To find out how nonlinearity can generate endogenous cycles, we review the second version of Goodwin's model, which we call the *basic model*,

$$\begin{cases} \varepsilon \dot{y}(t) = \dot{k}(t) - (1-\alpha) y(t), \\ \\ \dot{k}(t) = \varphi(\dot{y}(t)). \end{cases} \tag{1}$$

Here k is the capital stock, y the national income, α the marginal propensity to consume, which is positive and less than unity, and the reciprocal of ε a positive adjustment coefficient. The dot over variables stands for time differentiation. The first equation of (1) defines an adjustment process of the national income. Accordingly, national income rises or falls if investment is larger or smaller than savings. The second equation, in which $\varphi(\dot{y}(t))$ denotes the induced investment, describes an accumulation process of capital stock based on the acceleration principle. According to this principle, investment depends on the rate of changes in the national income. A distinctive feature of Goodwin' model is to introduce a nonlinearity into the investment function in such a way that the investment is proportional to the change in the national income in the neighborhood of the equilibrium income but becomes inflexible (i.e., less elastic) for extremely larger or smaller values of the income. This *nonlinear* acceleration principle is crucial in obtaining endogenous cycles in Goodwin's model. We will next retain this nonlinearity assumption and specify its explicit form. On the other hand, we depart from Goodwin's non-essential assumption of positive autonomous expenditure and will work with zero autonomous expenditure for the sake of simplicity. A direct consequence of this assumption is that an equilibrium solution or a stationary point of the basic model is $y(t) = \dot{y}(t) = 0$ for all t.

Inserting the second equation of (1) into the first one and moving the terms on the left hand side to the right gives the single dynamic equation for the national income y,

$$\varepsilon \dot{y}(t) - \varphi(\dot{y}(t)) + (1-\alpha) y(t) = 0. \tag{2}$$

This is a nonlinear differential equation. Although it is one-dimensional, its nonlinearity prevents deriving an explicit form of the solution. In spite of this simple form, it is possible to detect local dynamics by examining its linearized version in a neighborhood of the stationary point and global dynamics by performing numerical simulations.

The linear version of the income dynamic equation (2) is

$$\varepsilon \dot{y}(t) - \nu \dot{y}(t) + (1-\alpha) y(t) = 0, \tag{3}$$

where $\nu = \varphi'(0)$ is the slope of the investment function at the stationary point.

This is a first-order ordinary differential equation. Applying separation of variables gives a complete solution,

$$y(t) = y_0 e^{\lambda t} \text{ with } \lambda = \frac{1-\alpha}{\nu - \varepsilon}, \tag{4}$$

where y_0 is an initial condition. The stationary point is locally asymptotically stable or unstable according to whether the eigenvalue λ is negative or positive. Since $1-\alpha$ is the positive marginal propensity to save, the sign of the eigenvalue depends on whether the numerator is positive or negative. Thus the stationary point is locally asypmptotically stable if $\nu < \varepsilon$ and unstable if $\nu > \varepsilon$.

2.2 Delay Model with Fixed Time Lags

Due to the fact that in real economy, plans and their realizations need time to take effects, Goodwin (1951) introduces the investment lag, θ, between decisions to invest and the corresponding outlays. Inserting θ into the investment function of the basic model yields the third version of his model,

$$\varepsilon \dot{y}(t) - \varphi(\dot{y}(t-\theta)) + (1-\alpha)y(t) = 0. \tag{5}$$

This is a *neutral delayed nonlinear differential equation*, which we call the *fixed delay model*. Goodwin does not analyze dynamics generated by this fixed delay model. Furthermore, to the best of our knowledge, no analytical solutions of the delayed model are available yet. However, it is possible, again, to investigate dynamics of the delayed model by using linearization for local dynamics and numerical simulations for global dynamics. Since a cyclic oscillation has been shown to exist in the basic model, our main concern is to see how the presence of the investment lag affects characteristics of such a slow-rapid cycle. To this end, we analytically investigate the stability of the cycle generated in the linearized model and numerically detect what effects are caused by the lag on cyclical dynamics.

The fixed delay model is autonomous and its special solution is constant (i.e., $y(t) = 0$) so that its linearized version takes the form of a *linear neutral autonomous delay* differential equation,

$$\varepsilon \dot{y}(t) - v\dot{y}(t-\theta) + (1-\alpha)y(t) = 0. \tag{6}$$

It is well known that if the characteristic polynomial of a linear neutral equation has roots only with negative real parts, then the stationary point is locally asymptotically stable. The normal procedure for solving this equation is to try an exponential form of the solution. Substituting $y(t) = y_0 e^{\lambda t}$ into (6) and rearranging terms, we obtain the corresponding characteristic equation:

$$\varepsilon\lambda - \nu\lambda e^{-\lambda\theta} + (1-\alpha) = 0.$$

To check stability, we determine conditions under which all roots of this characteristic equation lie in the left or right half of the complex plane. Dividing both sides of the characteristic equation by ε and introducing the new variables $A = \frac{1-\alpha}{\varepsilon}$ and $B = -\frac{\nu}{\varepsilon}$, we rewrite the characteristic equation as

$$\lambda + A + B\lambda e^{-\lambda\theta} = 0. \tag{7}$$

Kuang (1993) derives explicit conditions for stability/instability of the n-th order linear real scalar neutral differential difference equation with a single delay. Since (7) is a special case of the n-th order equation, applying the result of Kuang (1993, Theorem 1.2) implies that the real parts of the solutions of equation (7) are positive for all θ if $|B| > 1$. Hence we have the following result.

Theorem 1 *If $\nu > \varepsilon$, then the stationary point of (6) is unstable for all $\theta > 0$.*

If $v < \varepsilon$ (i.e., $|B| < 1$), (7) has at most finitely many eigenvalues with positive real part. The roots of the characteristic equation are functions of the delay. As the lengths of the delay change, the roots may change their signs from positive to negative or vise versa so that the stability of the solution may also change. Such phenomena are often referred to as *stability switches*. We will next show that such stability switchings cannot take place in the fixed delayed model.

For the following discussion we assume that $v < \varepsilon$. The case $v = \varepsilon$ will be treated later as a critical case. It can be checked that $\lambda = 0$ is not a solution of (7) because substituting $\lambda = 0$ yields $A = 0$ that contradicts $A > 0$. In the case of $v < \varepsilon$, Kuang (1993, Theorem 1.4) shows that if the stability switches at $\theta = \bar{\theta}$, then (7) must have a pair of pure conjugate imaginary roots with $\theta = \bar{\theta}$. Thus to find the critical value of $\bar{\theta}$, we assume that $\lambda = i\omega$, with $\omega > 0$ being a root of (7) for $\theta = \bar{\theta}$, $\bar{\theta} \geq 0$. Substituting $\lambda = i\omega$ into (7), we have

$$A + B\omega \sin \omega\theta = 0,$$

and

$$\omega + B\omega \cos \omega\theta = 0.$$

Moving A and ω to the right hand side and adding the squares of the resultant equations, we obtain

$$A^2 + (1 - B^2)\omega^2 = 0.$$

Since $A > 0$ and $1 - B^2 > 0$ as $|B| < 1$ is assumed, there is no ω that satisfies the above equation. In other words, there are no roots of (7) crossing the imaginary axis when θ increases. Therefore, there are no stability switches for any θ.

In case $\varepsilon = \nu$ in which $|B| = 1$, the characteristic equation becomes

$$\lambda(1 - e^{-\lambda\theta}) + A = 0. \tag{8}$$

It is clear that $\lambda = 0$ is not a solution of (8) since $A > 0$. Thus we can assume that a root of (8) has non-negative real part, $\lambda = u + iv$ with $u \geq 0$ for some $\theta > 0$. From (8), we have

$$(u + A)^2 + v^2 = e^{-2u\theta}(u^2 + v^2) \leq (u^2 + v^2),$$

where the last inequality is due to $e^{-2u\theta} \leq 1$ for $u \geq 0$ and $\theta > 0$. Hence

$$2uA + A^2 \leq 0,$$

where the direction of inequality contradicts the assumption that $u \geq 0$ and $A > 0$. Hence it is impossible for the characteristic equation to have roots with nonnegative real parts. Therefore, all roots of (8) must have negative real parts for all $\theta > 0$. Summarizing the above discussions gives the following theorem.

Theorem 2 *In case of $\nu \leq \varepsilon$, the the stationary point of (6) is asymptotically stable for all $\theta > 0$.*

Combining Theorems 1 and 2, we can state that the stationary state is asymptotically stable if $\nu \leq \varepsilon$ and unstable if $\nu > \varepsilon$. The stationary state is asymptotically stable on the boundary of the stable region.

2.3 Delay Model with Continuously Distributed Lags

Continuously distributed time delay is an alternative approach to deal with a time lag in investment. If the expected change of national income is denoted by $\dot{y}^e(t)$ at time t and is based on the entire history of the actual changes of the national income from zero to t, then the dynamic system can be written as the system of integro-difference equations,

$$\begin{aligned} &\varepsilon\dot{y}(t) - \varphi(\dot{y}^e(t)) + (1-\alpha)y(t) = 0, \\ &\dot{y}^e(t) = \int_0^t w(t-s,\theta,m)\dot{y}(s)ds, \end{aligned} \tag{9}$$

where the weighting function is

$$w(t-s,\theta,m) = \begin{cases} \dfrac{1}{\theta}e^{-\frac{t-s}{\theta}} & \text{if } m = 0, \\ \dfrac{1}{m!}\left(\dfrac{m}{\theta}\right)^{m+1}(t-s)^m e^{-\frac{m(t-s)}{\theta}} & \text{if } m \geq 1. \end{cases}$$

Here m is a nonnegative integer and θ is a positive real parameter, which is associated with the length of the delay. We call this dynamic system the *distributed delay model*.

To examine local dynamics of the above system in the neighborhood of the stationary point, we consider the linearized version,

$$\varepsilon\dot{y}(t) - \nu\int_0^t w(t-s,\theta,m)\dot{y}(s)ds + (1-\alpha)y(t) = 0.$$

Looking for the solution in the usual exponential form

$$y(t) = y_0 e^{\lambda t} \text{ and } \dot{y}(t) = \lambda y_0 e^{\lambda t},$$

substituting them into the linearized version, we obtain

$$\varepsilon\lambda-\nu\lambda\int_0^t w(t-s,\theta,m)e^{-\lambda(t-s)}ds+(1-\alpha)=0.$$

Introducing the new variable $z=t-s$ simplifies the integral as

$$\int_0^t w(t-s,\theta,m)e^{-\lambda(t-s)}ds=\int_0^t w(z,\theta,m)e^{-\lambda z}dz.$$

By letting $t\to\infty$ and assuming that $\mathrm{Re}(\lambda)+\frac{m}{\theta}>0$, we have

$$\int_0^\infty \frac{1}{\theta}e^{-\frac{z}{\theta}}e^{-\lambda z}dz=(1+\lambda\theta)^{-1} \text{ if } m=0,$$

and

$$\int_0^\infty \frac{1}{m!}\left(\frac{m}{\theta}\right)^{m+1}z^m e^{-\frac{mz}{\theta}}e^{-\lambda z}dz=(1+\frac{\lambda\theta}{m})^{-(m+1)} \text{ if } m>1.$$

That is,

$$\int_0^\infty w(z,\theta,m)e^{-\lambda z}ds=\left(1+\frac{\lambda\theta}{q}\right)^{-(m+1)}$$

with

$$q=\begin{cases}1 & \text{if } m=0,\\ m & \text{if } m\geq 1.\end{cases}$$

Then the characteristic equation becomes

$$(\varepsilon\lambda+(1-\alpha))\left(1+\frac{\lambda\theta}{q}\right)^{m+1}-\nu\lambda=0. \tag{10}$$

If there are no time delays, $\theta=0$, then the above equation is reduced to the same characteristic equation as the one we have already derived above. We will next examine some simple cases in which analytical results can be obtained.

As mentioned in the Introduction, the case of $m=0$ is rigorously discussed in Matsumoto and Szidarovszky (2009), we examine stability in cases with $m\geq 1$. We expand the characteristic equation (10) by using the binomial theorem to obtain,

$$a_0\lambda^{m+2}+a_1\lambda^{m+1}+...+a_{m+1}\lambda+a_{m+2}=0, \tag{11}$$

where the coefficients a_i are defined as

$$\begin{aligned}
a_0 &= \varepsilon\theta^{m+1}>0,\\
a_k &= \left\{\binom{m+1}{k}m\varepsilon+\binom{m+1}{k-1}(1-\alpha)\theta\right\}m^{k-1}\theta^{m+1-k}>0 \text{ for } k=1,2,...m,\\
a_{m+1} &= m^m\{m\varepsilon+(m+1)(1-\alpha)\theta-m\nu\}\gtreqless 0,\\
a_{m+2} &= m^{m+1}(1-\alpha)>0.
\end{aligned}$$

According to the Routh-Hurwitz stability criterion, the necessary and sufficient conditions that all roots of the characteristic equation (11) have negative real parts are the following:

(1) the coefficients are positive, $a_k > 0$ for $k = 1, 2, ...2m + 1$,

(2) the principle minors of the Routh-Hurwitz determinant are positive,

$$D_2^m = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} > 0,\ D_3^m = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_0 \\ a_5 & a_4 & a_3 \end{vmatrix} > 0,\ D_4^m = \begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ a_5 & a_4 & a_3 & a_2 \\ a_7 & a_6 & a_5 & a_4 \end{vmatrix} > 0, ...$$

Case 1. $m = 1$

Substituting $m = 1$ into (11) yields

$$a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{12}$$

where

$$\begin{aligned} a_0 &= \varepsilon\theta^2 > 0, \\ a_1 &= (2\varepsilon + (1-\alpha)\theta)\theta > 0, \\ a_2 &= \varepsilon + (1-\alpha)2\theta - \nu \gtreqless 0, \\ a_3 &= 1 - \alpha > 0. \end{aligned}$$

It can be seen that the sign of a_2 is not determined. In addition to $a_2 > 0$, the Routh-Hurwitz criterion requires that the following second- and third-order Routh-Hurwitz determinants are positive,

$$D_2^1 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} > 0 \text{ and } D_3^1 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & 0 & a_3 \end{vmatrix} > 0.$$

Since $D_3^1 = a_3 D_2^1$ and $a_3 = 1 - \alpha > 0$, we have

$$\text{sign}\left(D_3^1\right) = \text{sign}\left(D_2^1\right),$$

where

$$D_2^1 = \theta\left\{2(\varepsilon + (1-\alpha)\theta)^2 - (2\varepsilon + (1-\alpha)\theta)\nu\right\}. \tag{13}$$

Notice that $D_2^1 = a_1a_2 - a_0a_3 > 0$ requires that $a_2 > 0$ since all other coefficients are positive. Therefore the only stability condition is $D_2^1 > 0$, that is,

$$\nu < \frac{2(\varepsilon + (1-\alpha)\theta)^2}{2\varepsilon + (1-\alpha)\theta}.$$

Therefore we have an explicit equation for the partition line,

$$\nu = \frac{2(\varepsilon + (1-\alpha)\theta)^2}{2\varepsilon + (1-\alpha)\theta}. \tag{14}$$

Let $g_1(\theta)$ denote the right hand side, then $g_1(0) = \varepsilon$ and for $\theta > 0$, both derivatives $g_1'(\theta)$ and $g_1''(\theta)$ are positive. Figure 4 with $m = 1$ shows the curve of this function, and the stationary state is locally asymptotically stable if the point (θ, ν) is under this partition line.

We now return to equation (12) to show the possibility of the birth of a limit cycle with continuously distributed delay by applying the Hopf bifurcation theorem. According to the theorem, we can establish the existence if the cubic characteristic equation has a pair of pure imaginary roots and the real part of these roots vary with a bifurcation parameter. We select ν as the bifurcation parameter and then calculate its value at the point for which loss of stability just occurs. Substituting (14) into (12), we can obtain a factorized expression of the characteristic equation along the partition line,

$$(2\varepsilon + (1-\alpha)\theta + \varepsilon\theta\lambda)(1-\alpha + (2\varepsilon\theta + (1-\alpha)\theta^2)\lambda^2) = 0,$$

which can be explicitly solved for λ. One of the characteristic roots is real and negative and the other two are pure imaginary:

$$\begin{aligned}
\lambda_1 &= -\frac{2\varepsilon + (1-\alpha)\theta}{\varepsilon\theta} < 0, \\
\lambda_{2,3} &= \pm i\sqrt{\frac{1-\alpha}{2\varepsilon\theta + (1-\alpha)\theta^2}} = \pm i\omega.
\end{aligned}$$

In order to apply the Hopf bifurcation theorem, we need to check whether the real part of the conjugate complex roots change its sign as the bifurcation parameter passes through its critical value. Suppose that λ depends on ν, $\lambda(\nu)$, and then implicit-differentiation of (12) shows that

$$\left(3\varepsilon\theta^2\lambda^2 + 2\left(2\varepsilon\theta + (1-\alpha)\theta^2\right)\lambda + \varepsilon + (1-\alpha)2\theta - \nu\right)\frac{d\lambda}{d\nu} = \lambda.$$

Thus

$$\begin{aligned}
sign\left[\frac{d(\mathrm{Re}\,\lambda)}{d\nu}\right]_{\lambda=i\omega} &= sign\left[\mathrm{Re}\left(\frac{d\lambda}{d\nu}\right)^{-1}\right]_{\lambda=i\omega} \\
&= sign\left[2(2\varepsilon\theta + (1-\alpha)\theta^2)\right]
\end{aligned}$$

where we used the facts that the terms with λ are imaginary and the constant terms are real. Therefore we have

$$\left.\frac{d(\mathrm{Re}\,\lambda)}{d\nu}\right|_{\lambda=i\omega} > 0.$$

This implies that the roots cross the imaginary axis at $i\omega$ from left to right as ν increases. Therefore the Hopf bifurcation theorem applies, and thus there is the possibility of the birth of limit cycles around the stationary point. The left part of Figure 3 illustrates a limit cycle in a 3D space when the stationary state is unstable, and the right part shows an orbit approaching the stationary state when it is stable.[1]

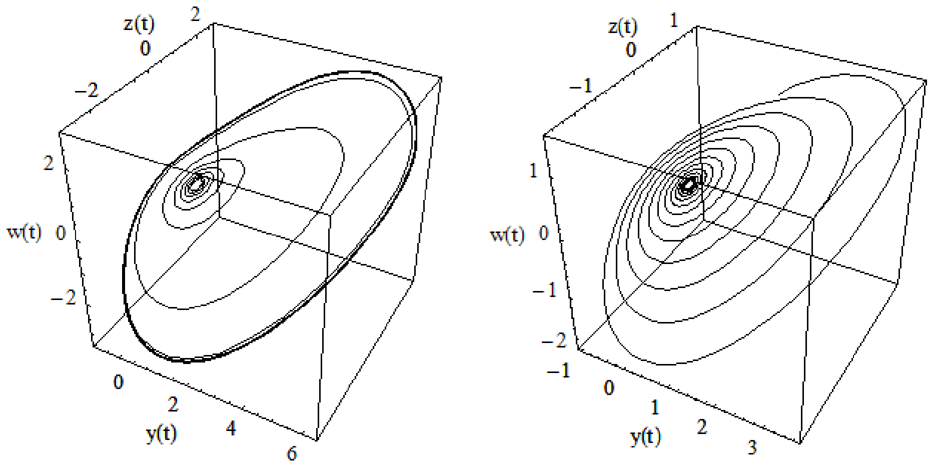

Figure1. Limit cycle and stable trajectory in 3D system

Case 2. $m = 2$

The characteristic equation is quartic in λ,

$$a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0$$

with coefficients

$$\begin{aligned} a_0 &= \varepsilon\theta^3 > 0, \\ a_1 &= (6\varepsilon + \theta(1-\alpha))\theta^2 > 0, \\ a_2 &= (2\varepsilon + \theta(1-\alpha))6\theta > 0, \\ a_3 &= 4(2\varepsilon + 3(1-\alpha)\theta) - 2\nu) \gtreqless 0, \\ a_4 &= 8(1-\alpha) > 0 \end{aligned}$$

[1] $z(t)$ and $w(t)$ are defined as follows:

$$z(t) = \int_0^t \frac{1}{\theta^2}(t-s)\, e^{-\frac{t-s}{\theta}}\dot{y}(s)ds$$

and

$$w(t) = \int_0^t \frac{1}{\theta} e^{-\frac{t-s}{\theta}}\dot{y}(s)ds,$$

furthermore ν is selected as

$$\frac{2(\varepsilon + (1-\alpha)\theta)^2}{2\varepsilon + (1-\alpha)\theta} + 0.05 \text{ in the unstable case,}$$

and

$$\frac{2(\varepsilon + (1-\alpha)\theta)^2}{2\varepsilon + (1-\alpha)\theta} - 0.05 \text{ in the stable case.}$$

All coefficients are positive except a_3 whose sign is not determined. The Routh-Hurwitz determinants can be defined in the same way as before,

$$D_2^2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}, \ D_3^2 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & a_4 & a_3 \end{vmatrix} \text{ and } D_4^2 = \begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_4 \end{vmatrix},$$

where

$$\begin{aligned} D_2^2 &= 2\theta^3 \left\{ 32\varepsilon^2 + 3(1-\alpha)^2\theta^2 + \varepsilon(18\theta(1-\alpha) + 4\nu) \right\} > 0, \\ D_3^2 &= 8\theta^3 \left\{ 8(2\varepsilon + (1-\alpha)\theta)^3 - 2(28\varepsilon^2 + 12(1-\alpha)\varepsilon\theta + 3(1-\alpha)^2\theta^2)\nu - 8\varepsilon\nu^2 \right\}, \\ D_4^2 &= 8(1-\alpha)D_3^2. \end{aligned}$$

Notice that condition $D_3^2 = a_3 D_2^2 - a_4 a_1^2 > 0$ requires for a_3 to be positive, since all other quantities are positive. Therefore, the only stability condition is $D_3^2 > 0$, that is,

$$g_2(\theta) = 8\theta^3 \left\{ 8(2\varepsilon + (1-\alpha)\theta)^3 - 2(28\varepsilon^2 + 12(1-\alpha)\varepsilon\theta + 3(1-\alpha)^2\theta^2)\nu - 8\varepsilon\nu^2 \right\} > 0,$$

so we obtained an explicit expression of the partition line

$$8(2\varepsilon + (1-\alpha)\theta)^3 - 2(28\varepsilon^2 + 12(1-\alpha)\varepsilon\theta) + 3(1-\alpha)^2\theta^2)\nu - 8\varepsilon\nu^2 = 0. \quad (15)$$

Since $g_2(\theta)$ is a concave parabola with $g_2(0)$, there is a unique positive root $\nu(\theta)$, and the stationary state is locally asymptotically stable if $\nu < \nu(\theta)$, that is, the point (θ, ν) is under the partition line. It is shown in Figure 4 with $m = 2$. Simple, but lengthy calculation shows that the delay model with $m = 1$ has a larger stable region that with $m = 2$ by verifying that

$$g_2\left(\frac{2(\varepsilon + (1-\alpha)\theta)^2}{2\varepsilon + (1-\alpha)\theta} \right) < 0.$$

By the same procedure as in the case of $m = 1$ above, we can show the birth of a limit cycle in the case of $m = 2$ as well.

After we repeat the above procedure for the values of $m = 1, 2, 3, 4, 5$, the five partition lines with m from 1 to 5 are depicted in Figure 4. It can be seen that all lines cross the vertical axis for $\nu = \varepsilon$ and their slopes become smaller as m increases. Notice that the dotted horizontal line is the partition line in the case of fixed time delay. This implies that the stable region becomes smaller as the value of m increases and converges to the region defined with the fixed time delay when m tends to infinity.

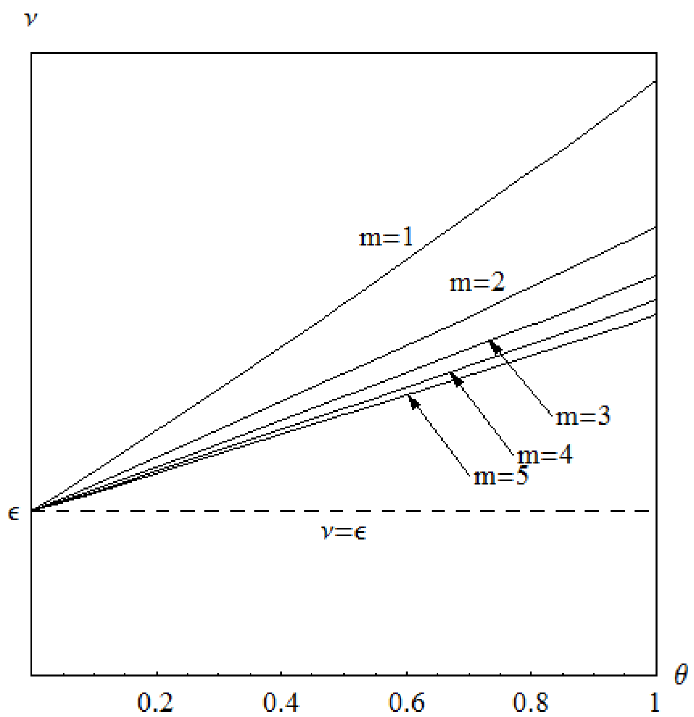

Figure 2. Partition lines

3 Conclusions

This paper examines the dynamic delayed Goodwin's business cycle model. Sufficient and necessary stability condition was derived first in the case of fixed lags. The stability conditions and the stability regions depend on the type of the weighting function in the case of continuously distributed time lags. These stability region shrinks as the variance of the delay decreases, and it converges to the stability region of fixed delays as the variance converges to zero. In the case of continuously distributed time lags, we also showed that in the case of stability loss, Hopf bifurcation occurs giving the possibility of the birth of limit cycles around the stationary state.

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