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Complexity yields benefits: An example of a heterogeneous duopoly market

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Abstract

This paper investigates an economic implication of chaotic fluctuations. To this end, it constructs a nonlinear, discrete-time Cournot model in which duopoly firms are heterogeneous and have U-shaped reaction functions with different microeconomic foundations. Due to the U-shaped reaction functions, output adjustment process generates chaotic fluctuations. This paper demonstrates that duopoly firms can eventually benefit from chaotic output adjustment more than from all possible equilibria when external effects on production are asymmetric. The result indicates a possibility that chaotic fluctuations are beneficial for heterogeneous markets.

1 Introduction

Research in the past two decades has demonstrated that a Cournot adjustment process of output may be chaotic if reaction functions are non-monotonic; see Rand [14], Dana and Montrucchio [4] and Witteloostuijn and Lier [15] (W-L hereafter), to name a few. At least two questions arise naturally: (i) Under what economic circumstances do oligopoly firms react non-monotonically to a change in the rival's behavior? (ii) Is chaotic adjustment of output a favorable economic phenomenon?

We have already had two possible answers to the first question. One is given by Puu [12, 13] who shows that profit maximization with linear production technologies and a hyperbolic market demand may result in unimodal reaction functions. The other is by Kopel [8] who confirms that, under a linear demand function and nonlinear cost functions, profit maximization also generates unimodal reaction functions. The second question is partly answered for a one-dimensional dynamic process. Huang [7] uses a cautious cobweb model and gives an affirmative answer that perpetual fluctuations may be preferable to a competitive equilibrium from the long-run perspective. Kopel [8], on the other hand, gives a negative answer. He constructs a simple model of evolutionary dynamics to show that various performance measures such as aggregate profits, aggregate sales revenues, mean sales and the rate of return indicate the inferiority of chaotic dynamics to equilibrium on the average. Matsumoto [9] provides an equivocal result in a simple exchange model with two agents and two goods that the long-run average performance measure (i.e., utility or profit) taken along a chaotic trajectory can be greater than the corresponding measure calculated at an equilibrium for one agent but less for the other.

In this paper, we aim to provide an answer to the second question in a multi-dimensional dynamic process. In particular, we consider the long-run statistical behavior of Cournot duopoly firms. This paper is a continuation of Matsumoto and Nonaka [10] (M-N hereafter) which also investigates an economic implication of chaotic adjustment of Cournot duopoly. M-N demonstrate through an example that chaotic dynamics can be more beneficial than stationary states in terms of the long-run average profits. They provide a weak result in which chaos is beneficial from the viewpoint of industry profits. In other words, the sum of the long-run average profits of duopoly firms taken along chaotic trajectories is larger than the sum of profits obtained at stationary states. In this paper, however, we will provide the stronger result that for each of duopoly firm, the long-run average profit is strictly larger than any possible equilibrium profits if firms are heterogeneous and asymmetric.

The paper is organized as follows. Section 2 constructs a dynamic model of Cournot duopoly. Section 3 examines existence and stability of equilibria numerically as well as analytically. Through some numerical experiments, Section 4 investigates the long-run outcome of the output adjustment of duopoly firms. Section 5 gives concluding remarks.

2 Model

Consider a market in which two firms, firm 1 and firm 2, compete by producing homogeneous outputs x and y. Assuming that inverse demand function is linear and decreasing:

$$p = a - b(x + y), \quad a > 0 \text{ and } b > 0. \tag{1}$$

Without loss of generality, we normalize the set of outputs of firms to a unit interval $I \equiv [0, 1]$. Equation (1) implies that the demand of one firm is externally affected by the other's output level and this external effect is always negative. In general, a downsloping demand curve implies a negative external effect (negative externality hereafter).

Our primary interest is in the cases where an externality exists not only on demand but also on production. To describe production externality we follow Kopel's approximation [8] and make the following assumption.

Assumption 1 *Each firm has a marginal cost of production that is constant with respect to its own output but varies with respect to the rival's output. Let C_i be the cost function of firm i. Then*

$$C_1(x, y) = c_1(y)x \quad \text{and} \quad C_2(x, y) = c_2(x)y \tag{2}$$

where c_i denotes the marginal cost of firm i.

We specify each c_i as follows:

$$\begin{aligned} c_1(y) &= a - by - 2b(\alpha y - \alpha + 1)^2 \\ c_2(x) &= a - bx - 2b(\beta x - 1)^2. \end{aligned} \tag{3}$$

The last terms in (3) imply that duopoly firms have marginal costs of production which are unequal nonlinear functions of rival's output in different manners. Therefore production externalities are nonlinear for both firms and heterogeneous between the firms. In particular, if the parameters have relatively high values, both of the marginal costs are unimodal to the rival's output. That is, for each firm, the production externality will change from negative to positive for sufficient increment of the rival's output. This negative-positive production externality may come from the hypothesis of *labor hire competition of duopoly firms*[1]. Since it can be positive or negative, production externality determines the total external effects that result in the reaction patterns of each firm.

From (1), (2) and (3) The profit functions of the firms become accordingly,

$$\Pi_1 = px - c_x(y)x \quad \text{and} \quad \Pi_2 = py - c_y(x)y. \tag{4}$$

Since profit of each firm is a quadratic polynomial of its own output, solving the first-order condition of the profit maximization problem always gives unique solution for a given value of the rival's output. This optimal output is known as the *reaction function.* Let r_i be the reaction function of firm i, so that

$$r_1(y^e) = \arg\max_x \Pi_1(x, y^e) \quad \text{and} \quad r_2(x^e) = \arg\max_y \Pi_1(x^e, y) \tag{5}$$

where x^e and y^e are output expectations. Substituting (4) into (5), we have

$$r_1(y) = (\alpha y - \alpha + 1)^2, \quad \text{and} \quad r_2(x) = (\beta x - 1)^2. \tag{6}$$

If $1 \leq \alpha, \beta \leq 2$ both of r_1 and r_2 map from I to I. We restrict the combinations of the parameters, α and β , to the following set A:

$$A \equiv \{(\alpha, \beta) | 1 \leq \alpha, \beta \leq 1\} \tag{7}$$

and assume that $a \geq 3b$ to keep the marginal costs nonnegative for arbitrary x and y in I. Due to the presence of total external effects, the reaction function can be upward- or downward-sloping according to whether the external effect is positive or negative.

[1]Suppose that firms demand labor for output production, and there are many people who want to work. If firm 2 produces only a small amount of output, it employs only a few people as employment and output production are positively correlated. Thereby, there are many people left unemployed and firm 1 can hire at a lower wage rate. While firm 2 expands output, unemployed people can increase their negotiating power so that firm 1 inevitably offers a higher wage rate to employ more people. However, if firm 2 employs most of the people, the remaining unemployed people have less negotiating power than before and thus firm 1 can hire them at a lower wage rate again.

If $\alpha = 1$, r_1 is monotonically increasing with y. As α becomes larger, the shape of r_1 becomes U-shaped. On the contrary, r_2 is monotonically decreasing with x with $\beta = 1$ and takes U-shaped profile for further increment of β.

About the expectation of each firm toward the rival's future output, we make a simple assumption — the so called *naive expectation* — that is "tomorrow's expected output level is today's observed output level". Suppose therefore that firms have naive expectations, $x_t^e = x_{t-1}$ and $y_t^e = y_{t-1}$. The process of output adjustment is

$$\begin{aligned} x_{t+1} &= (\alpha y_t - \alpha + 1)^2, \\ y_{t+1} &= (\beta x_t - 1)^2. \end{aligned} \tag{8}$$

Intersections of these two reaction functions are the Cournot-Nash equilibria at which each firm's expectation toward the other firm's behavior is confirmed by its actual behavior. If the expectation is not correct, then firms adjust their outputs. We assume the *Markov-Perfect-Equilibrium* (MPE hereafter) process of adjustment, which implies that firms choose their outputs iteratively as the best response to the rival's output in the previous period. Therefore, if the adjustment process of a Cournot duopoly market follows MPE process, the state of economy (i.e. the output bundle of duopoly firms) is always defined in the union of the reaction curves[2]

3 Equilibria

In this section, we investigate the conditions of existence and stability of equilibrium points of dynamical system (8). Since (8) consists of asymmetric quadratic equations, we can not easily identify the explicit conditions for equilibrium analytically. Instead, we approximate the implicit conditions numerically. Section 3.1 considers the existence of equilibrium points and Section 3.2 examines the stability and bifurcations of these points.

3.1 Existence of equilibrium points

Since each r_i is a quadratic polynomial of the rival's output, (8) may have four equilibria S_1, S_2, S_3 and S_4. To identify these equilibria, we first combine r_1 with r_2 to define a one-dimensional map F:

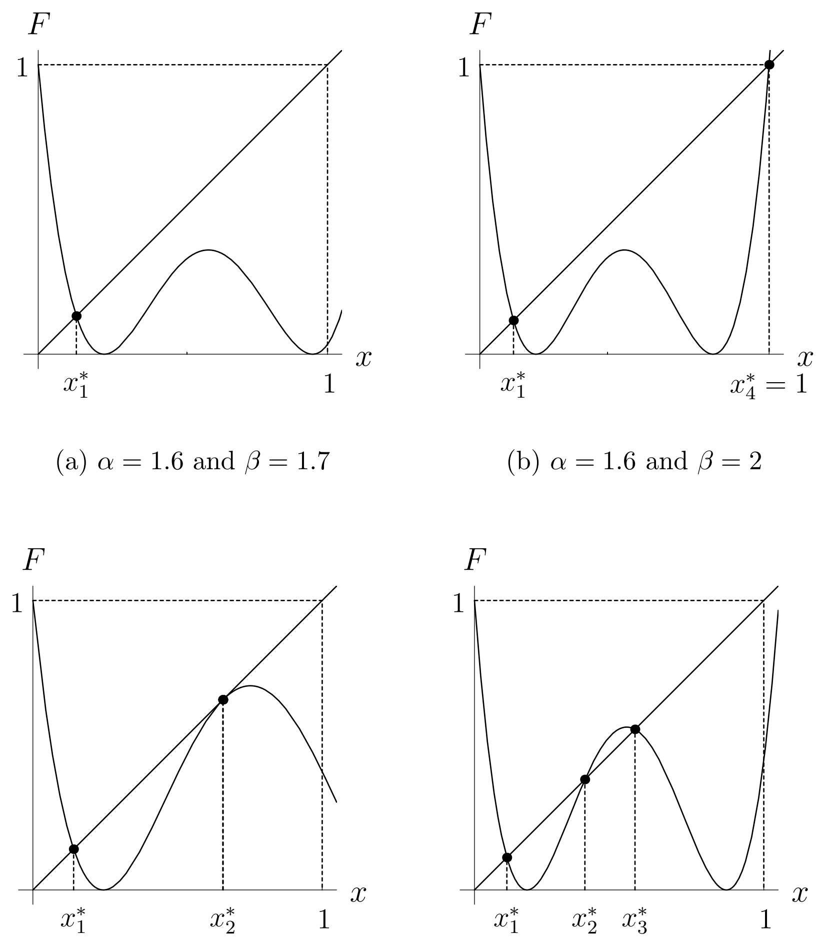
$$F(x) \equiv r_1 \circ r_2(x) = (1 + \alpha\beta x(\beta x - 2))^2 \tag{9}$$

Here $F : I \to I$, since each reaction function $r_i : I \to I$. A fixed point of F in I corresponds to an equilibrium point of (8). Since it is a forth-order polynomial, F possibly has four fixed points. Denoting a fixed point of F in I by x_j^* $(j = 1, \dots, 4)$, we assume that $x_1^* < x_2^* < x_3^* < x_4^*$ for convenience. Therefore for S_j (j=1, ... ,4),

$$S_j = (x_j^*, y_j^*) \text{ such that } x_j^* = F(x_j^*) \text{ and } y^* = r_2(x_2^*) \tag{10}$$

[2]Although we assume that firms simultaneously update their outputs, the adjustment process follows MPE process whenever we restrict the initial condition on the union of the reaction curves.

To identify the existence of each x_i^*, we first examine the fixed points of F from more general viewpoint through following lemma.



(a) $\alpha = 1.6$ and $\beta = 1.7$

(b) $\alpha = 1.6$ and $\beta = 2$

(c) $\alpha \simeq 1.84$ and $\beta = 1.33$

(d) $\alpha = 1.75$ and $\beta = 1.9$

Figure 1: Determinations of fixed points of F.

Lemma 1 *F possesses four fixed points. Moreover,*

(i) the minimal fixed point of F always exists in the interval $(0, 1)$ for arbitrary parameter combinations in A and

(ii) the maximal fixed point exists in I and is equal to unity if and only if $\beta = 2$.

Proof First we set

$$F'(x) \equiv \frac{\partial F}{\partial x} = 4\alpha\beta(\beta x - 1)(\alpha\beta x^2 - 2\alpha\beta x + 1) \tag{11}$$

and

$$F''(x) \equiv \frac{\partial^2 F}{\partial x^2} = 4\alpha\beta^2(1 + \alpha(3\beta^2 x^2 - 6\beta x + 2). \tag{12}$$

(i) $F(0) = 1$ for arbitrary parameter combination in A. In addition, $F'(x) < 0$ and $F''(x) > 0$ for all x in $(-\infty, 0]$. Therefore F has no fixed point in $(-\infty, 0]$. Solving $F(x) = 0$, we have $x = (\alpha \pm \sqrt{\alpha(\alpha-1)})/\alpha\beta$. Let $\hat{x} = (\alpha - \sqrt{\alpha(\alpha-1)})/\alpha\beta$. Then $0 < \hat{x} \leq 1$ since $1 \leq \alpha, \beta \leq 2$. This immediately leads to the conclusion that $F(x)$ always cuts once 45-degree line from above in the interval $(0, \hat{x})$ and then this intersection is the minimal fixed point of F.

(ii) $F(1) = 1$ if $\alpha = 2/\beta(2-\beta)$ or $\beta = 2$. In the former case, $\alpha = 2$ with $\beta = 1$ and $\alpha > 2$ with $1 < \beta \leq 2$. Then assume that $\alpha = 2$ and $\beta = 1$. Here $x = 1$ is a fixed point of F and $F'(1) = 0$ and $F''(1) = -8$. However for sufficiently large x, both $F'(x)$ $F''(x)$ become positive and this implies that F has at least a fixed point in $(1, \infty)$. Considering the latter case, now suppose that $\beta = 2$. Then $F'(x) = 8\alpha(2x-1)(4\alpha x(x-1)+1) > 0$ and $F''(x) = 16\alpha(1+12\alpha x(x-1)+2\alpha) > 0$ for all x in $[1, \infty)$. Thus F has no fixed point in $(1, \infty)$, that is, $x = 1$ is the maximal fixed point of F if and only if $\beta = 2$. ■

From Theorem 1, we get $0 < x^* < 1$ and $x^*4 = 1$. (See Figure 1(a) and 1(b).) Considering the existence of x_2^* and x_3^*, let us start from $(\alpha, \beta) = (1, 1)$. Here only x_1^* exists since $\hat{x} = 1$. For increment of α and β x_2^* appears in $(\hat{x}, 1]$ if F tangents to 45-degree line from below, see Fig. 1(c). If α or β increases further, then x_3^* appears in $(x_2^*, 1]$, see Fig. 1(d).

To identify the implicit conditions about the existence of x_2^* and x_3^* define a mapping ϕ:

$$(\phi(\beta), \beta) \in A \;\text{ such that }\; F'(x_2^*) = 1 \;\text{ and }\; F''(x_2^*) < 0. \tag{13}$$

Then A can be divided into the following three subsets A_1, A_2 and A_3:

$$\begin{aligned} A_1 &\equiv \{(\alpha, \beta) \in A \mid \alpha < \phi(\beta)\}, \\ A_2 &\equiv \{(\alpha, \beta) \in A \mid \alpha = \phi(\beta)\}, \\ A_3 &\equiv A \setminus (A_1 \cup A_2). \end{aligned} \tag{14}$$

Regarding the existence of x_2^* and x_3^* (i.e. S_2 and S_3) we have following states:

(i) If $(\alpha, \beta) \in A_1$, then neither x_2^* nor x_3^* exists,

(ii) if $(\alpha, \beta) \in A_2$, then only x_2^* exists and

(iii) if $(\alpha, \beta) \in A_3$, then both of x_2^* and x_3^* exist.

From Lemma 1, x_1^* always exists in all of the three cases above and x_4^* exists only if $\beta = 2$. These three subsets are illustrated in Figure 2.

To conclude this subsection we state following theorem.

Theorem 1 *There exists a subset of A in which all possible fixed points of F exist in I. Therefore the adjustment process of (8) has four distinct equilibria.*

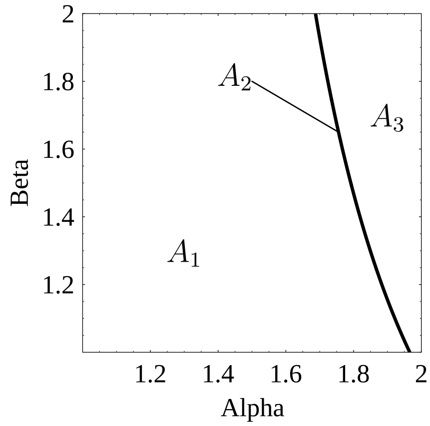


Figure 2: Existence of the equilibrium points.

Proof $F'(x) = 0$ if $x = \frac{1}{\beta}$ and then $F(\frac{1}{\beta}) = (1-\alpha)^2$ and $F''(\frac{1}{\beta}) = 4\alpha(1-\alpha)\beta^2 \le 0$. Solving $(1-\alpha)^2 \ge \frac{1}{\beta}$, we have

$$\alpha < 1 - \frac{1}{\sqrt{\beta}} \quad \text{or} \quad \alpha > 1 + \frac{1}{\sqrt{\beta}}. \tag{15}$$

Therefore for arbitrary β in $[1, 2]$, x_2^* and x_3^* appear whenever α increases toward 2. From Lemma 1, x_1^* always exists and x_4^* exists if $\beta = 2$. Thus if $\alpha > 1 + \frac{1}{\sqrt{2}}$ and $\beta = 2$, then F has four fixed points in I, that is, all of S_1, S_2, S_3 and S_4 exist. ■

3.2 Stabilities and bifurcations of equilibrium points

To identify the stability condition of each equilibrium point we linearize the system (8) around the equilibria. Suppose that J is the Jacobian matrix of (8) such that

$$J \equiv \begin{pmatrix} 0 & \dfrac{\mathrm{d}r_1}{\mathrm{d}y} \\ \dfrac{\mathrm{d}r_2}{\mathrm{d}x} & 0 \end{pmatrix} \tag{16}$$

with eigenvalues $\lambda_1 = \lambda_2 = \lambda$. Then an equilibrium point S_i is stable if and only if

$$|\lambda| \equiv \left| \frac{\mathrm{d}r_1}{\mathrm{d}y} \frac{\mathrm{d}r_2}{\mathrm{d}x} (x_j^*, y_j^*) \right| = |F'(x_j^*)| < 1 \tag{17}$$

where the derivatives are evaluated at S_j.

Here we use again the one-dimensional map F for checking the stabilities of the equilibrium points. $S_i = (x_j^*, r_2(x_j^*))$ is a stable equilibrium point if and only if x_j^* is a stable fixed point of F. From Figure 1 we can see that x_2^* and x_4^* (i.e. S_2 and S_4) are unstable whenever they exist. In addition, it can be numerically checked that x_1^* (i.e. S_1) is unstable for arbitrary parameter combinations in A. Therefore we only examine the stability of x_3^* (i.e. S_3).

It is known that a dynamical system with logistics-like maps has flip bifurcations when it loses stability. That is, for the system in this paper, (8) loses stability if $\lambda \leq -1$. Here we define a mapping ψ:

$$(\psi(\beta), \beta) \in A \quad \text{such that} \quad F'(x_3^*) = -1. \tag{18}$$

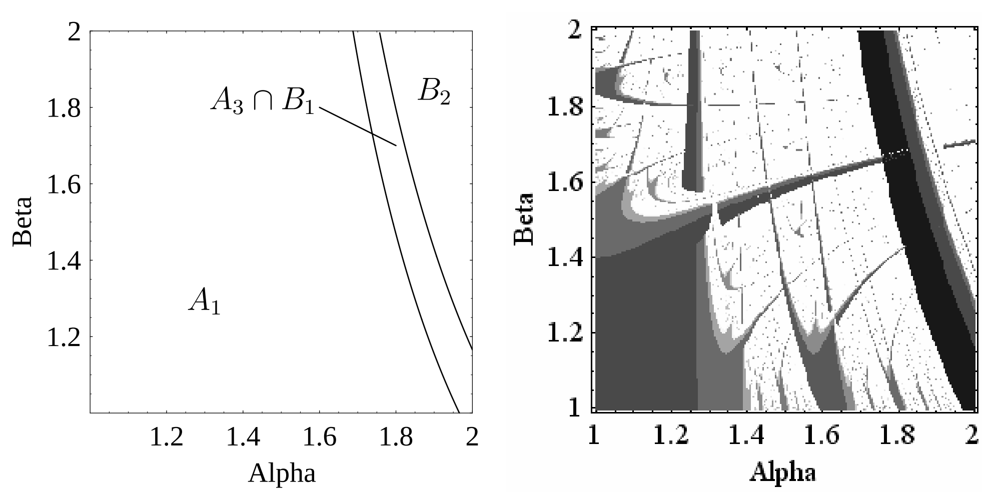
Now A can be divided into following two subsets, B_1 and B_2:

$$B_1 \equiv \{(\alpha, \beta) \in A \mid \alpha < \psi(\beta)\} \quad \text{and} \quad B_2 \equiv A \setminus B_1. \tag{19}$$

Regarding the stability condition of x_3^* (i.e. S_3) we have the following states:

(i) If $(\alpha, \beta) \in B_1 \cap A_3$, then S_3 is stable and

(ii) if $(\alpha, \beta) \in B_2$, then S_3 is unstable.



(a) stability condition of x_3^*

(b) bifurcation diagram

Figure 3: Stability and bifurcation of the equilibrium points.

In Figure 3 we calculate the implicit stability condition of x_3^* (Fig. 3(a)) and the bifurcation diagram of F with respect to α and β (Fig. 3(b)). A dark-colored belt in Fig. 3(b) corresponds to $B_1 \cap A_3$ in Fig. 3(a). In white-colored regions in Fig. 3(b), the dynamics generates non-periodic fluctuations, that is, MPE process of output may be chaotic.

4 Statistical Dynamics

In this section we investigate the long-run statistical behavior of the system (8). Since analytical investigations are difficult or not possible, numerical experiments are used to calculate the profits taken at the equilibriums (equilibrium profits hereafter) and the long-run average profits taken along MPE process (average profits hereafter). We compare the equilibrium profits and the average profits and then check that whether firms can eventually benefit from chaotic fluctuations instead of settling at possible equilibria. In Section 4.1 we focus on the case in which $\alpha = \beta = 2$. In Section 4.2 we extend the calculation to entire region of A.

4.1 Ergodic Chaos

Suppose that $\alpha = \beta = 2$. Now the system (8) consists of symmetric reaction functions $r_1(y) = (2y-1)^2$ and $r_2(x) = (2x-1)^2$ and has four equilibrium points, so that equivalently, F has four fixed points in I such that

$$x_1^* = \frac{3-\sqrt{5}}{8},\ x_2^* = \frac{1}{4},\ x_3^* = \frac{3+\sqrt{5}}{8} \text{ and } x_4^* = 1. \tag{20}$$

We evaluate profits of duopoly firms at each equilibrium point $S_j = (x_j^*, r_2(x_j^*))$. These are

$$\begin{aligned} \Pi_1^1 &= \Pi_2^3 = \frac{(\sqrt{5}-3)^2 b}{64}, \\ \Pi_1^2 &= \Pi_2^2 = \frac{b}{16}, \\ \Pi_1^3 &= \Pi_2^1 = \frac{(\sqrt{5}+3)^2 b}{64} \text{ and} \\ \Pi_1^4 &= \Pi_2^4 = b. \end{aligned} \tag{21}$$

where Π_i^j denotes equilibrium profit of firm i at S_j.

It is known that if $\alpha = \beta = 2$, the chaotic fluctuation generated by the adjustment process is ergodic. From M-N [10], if chaos is ergodic, the average profits with MPE process can be analytically derived. Let $\bar{\Pi}_i$ be the average profit of firm 1 with MPE process, then

$$\begin{aligned} \bar{\Pi}_i &= \lim_{T\to\infty} \frac{1}{T} \sum_{i=0}^{T-1} \Pi_i(x_t, y_t) \\ &= \frac{1}{2}\left(\int_0^1 \Pi_i(r_1(u), u)\Phi(u)\mathrm{d}u + \int_0^1 \Pi_i(u, r_2(u))\Phi(u)\mathrm{d}u \right) \\ &= \frac{b}{4} \end{aligned} \tag{22}$$

where

$$\Phi(u) = \frac{1}{\pi\sqrt{u(1-u)}}. \tag{23}$$

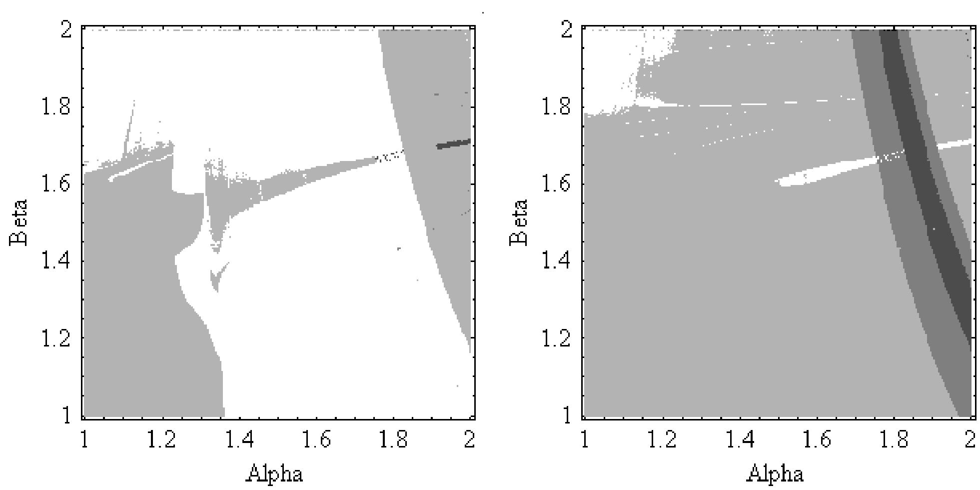
Constructing a profit ordering of each firm, we have

$$\begin{aligned} \Pi_1^1 < \Pi_1^2 < \bar{\Pi}_1 < \Pi_1^3 < \Pi_1^4 \\ \Pi_2^3 < \Pi_2^2 < \bar{\Pi}_2 < \Pi_2^1 < \Pi_2^4. \end{aligned} \tag{24}$$

Therefore duopoly firms cannot always benefit from chaotic fluctuations when $\alpha = \beta = 2$.

4.2 Beneficial Chaos

Now we extend the analysis to entire region of A. From Fig. 3(b), we can see that dynamical system (8) generates chaotic fluctuations although neither α nor β equals 2. However, since ergodicity does not hold in these cases, the average profits generated by chaotic fluctuations with these parameter combinations can not be bounded by theoretically derived values. Instead, we check the actual value of the average profits with sufficiently large number of periods. For each combination of the parameters we calculate possible equilibrium profits and the average profit under chaos and then check the profit ordering of each firm.



(a) firm 1

(b) firm 2

Figure 4: Variation of the profit order.

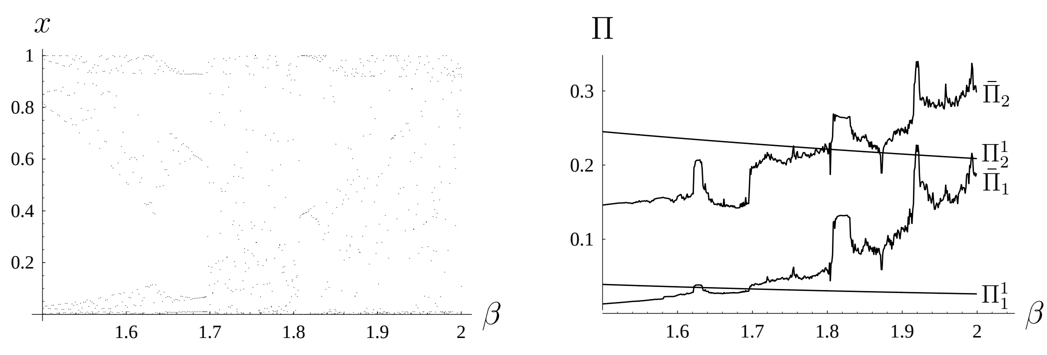
Figure 4 illustrates the variation of the profit ordering of each firm[3]. The colors of Fig.4 correspond to the preference ordering of firms about the average profit under chaos. That is, each color depicts how many equilibrium points exist, at which the profit is larger than the average profit. In white-colored regions no equilibrium profit

[3]For each parameter combination, we calculate the average profit for 1000 periods after a 2000-period iterations.

is larger than the average chaotic profit (i.e. $\bar{\Pi}_i \geq \Pi_i^j > \cdots$). As the color gets darker, the preference ordering about the average profit becomes worse. In the black-colored region, therefore, the average chaotic profit is less than all possible equilibrium profits (i.e. $\bar{\Pi}_i < \Pi_i^j < \cdots$).

Comparing Fig. 4(a) and Fig. 4(b) we can say that there exists a subset of A in which, for both firms, the average chaotic profits are larger than all possible equilibrium profits. Since this subset is in A_1, only S_1 exists and we have the profit ordering:

$$\Pi_1^1 < \bar{\Pi}_1 \quad \text{and} \quad \Pi_2^1 < \bar{\Pi}_2. \tag{25}$$



(a) bifurcation cascade

(b) variation of the average profits

Figure 5: An example of beneficial chaos ($\alpha = 1.1$).

An example of beneficial chaos is illustrated in Figure 5, where α is fixed at $\bar{\alpha} = 1.1$ and β varies ($1.5 \leq \beta \leq 2$). As β increases MPE process becomes chaotic, (see Fig. 5(a)) and then the average profits of both firms becomes larger than the equilibrium profits at S_1, see Fig. 5(b).

5 Concluding Remarks

This paper presents two main results. Firstly, a Cournot duopoly market may be chaotic if firms have U-shaped reaction functions. To construct an economic foundation of U-shaped reaction functions, the investigation takes account of two patterns of nonlinear, negative-positive production externalities. Secondly, heterogeneous duopoly firms can eventually benefit from chaotic adjustment of output when production externalities are asymmetric. To demonstrate beneficial chaos, some numerical experiments calculate the long-run average profits taken along chaotic trajectories and then compare them with profits taken at possible equilibrium points. The comparison shows that there exists a subset of parameter combinations for which, and for both the firms, the long-run average profits are strictly larger than equilibrium profits. In that sense, we can conclude that chaotic fluctuation can be beneficial in a heterogeneous market.

Since chaotic fluctuations do not converge to an equilibrium state, they are thought of as an unfavorable phenomenon in traditional economics. However, the results obtained in this paper, in particular, the last numerical result, throws an interesting light on the nature of complex chaotic dynamics and suggests a further study on duopoly dynamics involving chaos worthy of consideration.

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