Discussion Paper No.248

On the Comparison of Discrete and Continuous Dynamic Systems

Akio Matsumoto
Chuo University

Ferenc Szidarovszky
University of Pécs

February 2015

INSTITUTE OF ECONOMIC RESEARCH
Chuo University
Tokyo, Japan

On the Comparison of Discrete and Continuous Dynamic Systems*

Akio Matsumoto† Ferenc Szidarovszky‡

Abstract

The stability conditions of continuous and discrete dynamic systems are compared showing simple reason why continuous systems are more stable than their discrete counterparts.

Keywords. Continuous and discrete systems, linear and nonlinear systems, Delay dynamic system

*This paper has been prepared while the second author was a visiting professor at the Department of Economics of Chuo University. The authors highly appreciate the financial supports from the MEXT-Supported Program for the Strategic Research Foundation at Private Universities 2013-2017, the Japan Society for the Promotion of Science (Grant-in-Aid for Scientific Research (C) 24530202, 25380238 and 26380316). The usual disclaimers apply.

†Professor, Department of Economics, Senior Researcher, International Center for further Development of Dynamic Economic Research, Chuo University, 742-1, Higashi-Nakano, Hachioji, Tokyo, 192-0393, Japan; akiom@tamacc.chuo-u.ac.jp

‡Visting Professor of Department of Economics at Chuo University, Professor, Department of Applied Mathematics, University of Pécs, Ifjúság, u. 6, H-7624, Pécs, Hungary; szidarka@gmail.com

1 Introduction

It is a well-known that in examining dynamic economic systems that continuous models are more stable than discrete systems. This is demostrated in many works including Bischi et al. (2010), where the discrete and continuous versions of the different extensions of the Cournot oligopoly model are examined. In this note we give general mathematical reasons for this fact by comparing the eigenvalues of the Jacobians as well as comparing the condtitions on the characteristic polynomial coefficients in the two-dimensional case.

2 Stability Conditions

2.1 n dimensional system

Considere an n-dimensional continuous system

$$\dot{x}_i(t) = f_i\left(x_1(t), x_2(t), ..., x_n(t)\right) \ \ (i = 1, 2, ..., n) \tag{1}$$

where funcions f_i are continuously differentiable in the neighborhood of a steady state $\bar{\boldsymbol{x}} = (\bar{x}_1, \bar{x}_2, ...\bar{x}_n)$. The linearized system around this steady state has the form

$$\dot{x}_{i\delta}(t) = \sum_{j=1}^{n} \frac{\partial f_i(\bar{\boldsymbol{x}})}{\partial x_j} x_{j\delta}(t) \tag{2}$$

where $x_{i\delta}(t) = x_i(t) - \bar{x}_i$ for all i. This is a linear system and it is well-known (see for example, Szidarovszky and Bahill, 1998) that the steady state $\bar{\boldsymbol{x}}$ is asymptotically stable if and only if all eigenvalues of the coefficient matrix have negative real parts. Notice that the coefficient matrix is the Jacobian of system (1) at the steady state are given by

$$\boldsymbol{J_C} = \begin{pmatrix} \frac{\partial f_1(\bar{\boldsymbol{x}})}{\partial x_1} & \cdots & \frac{\partial f_1(\bar{\boldsymbol{x}})}{\partial x_n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial f_n(\bar{\boldsymbol{x}})}{\partial x_1} & \cdots & \frac{\partial f_n(\bar{\boldsymbol{x}})}{\partial x_n} \end{pmatrix}. \tag{3}$$

where the subscritp "C" stands for continuous.

The corresponding discrete system is obtained by replacing the derivatives $\dot{x}_i(t)$ by the increments $x_i(t+1) - x_i(t)$ which results in the discrete system,

$$x_i(t+1) = x_i(t) + f_i\left(x_1(t), x_2(t), ..., x_n(t)\right) \ \ (i = 1, 2, ..., n). \tag{4}$$

The linearized version of this equation is clearly

$$x_{i\delta}(t+1) = x_{i\delta}(t) + \sum_{j=1}^{n} \frac{\partial f_i(\bar{\boldsymbol{x}})}{\partial x_j} x_{j\delta}(t) \tag{5}$$

where we use the simple fact that systems (1) and (4) have the same steady state. It is also well-known that $\bar{\boldsymbol{x}}$ is asymptotically stable if and only if all eigenvalues of the coefficient matrix of system (5) are inside the unit circle. The coefficient matrix of this system is

$$\boldsymbol{J_D} = \begin{pmatrix} 1 + \frac{\partial f_1(\bar{\boldsymbol{x}})}{\partial x_1} & \cdots & \frac{\partial f_1(\bar{\boldsymbol{x}})}{\partial x_n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \frac{\partial f_n(\bar{\boldsymbol{x}})}{\partial x_1} & \cdots & 1 + \frac{\partial f_n(\bar{\boldsymbol{x}})}{\partial x_n} \end{pmatrix} \tag{6}$$

where the subscrit "D" stands for discrete. The coefficient matrices in (3) and (6) apparently satisfy the relation,

$$\boldsymbol{J_D} = \boldsymbol{I} + \boldsymbol{J_C} \tag{7}$$

where $\boldsymbol{I}$ is the $n \times n$ identity matrix. Therefore λ_D is an eigenvalue of $\boldsymbol{J_D}$ if and only if $\lambda^D = 1 + \lambda^C$ with some eigenvalue λ^C of $\boldsymbol{J_C}$. Concerning stability of these two dynamic systems, we have the following two results:

(I) If $\left|\lambda^D\right| < 1$, then $\mathrm{Re}(\lambda^C) < 0$.

(II) $\mathrm{Re}(\lambda^C) < 0$ does not necessarily implies $\left|\lambda^D\right| < 1$.

We first prove result (I) by assuming that the discrete system is asymptotically stable. Inequality $\left|\lambda^D\right| < 1$ means that λ^D is inside the unity circle. In Figure 1, the stable region of the discrete system is the union of the red region and the part of the yellow region surrounded by the dotted and real curves. So $1 + \lambda^C$ is in the unit circle that is the dotted circle shifted to left by a unity, therefore λ^C is inside the circle with unit radius and center -1 as shown as the yellow region in Figure 1(A).[1] It is clear that the real parts of all points of this circle are negative, implying that the stability of the discrete system implies the stability of the continuous system. This proves the statement (I).

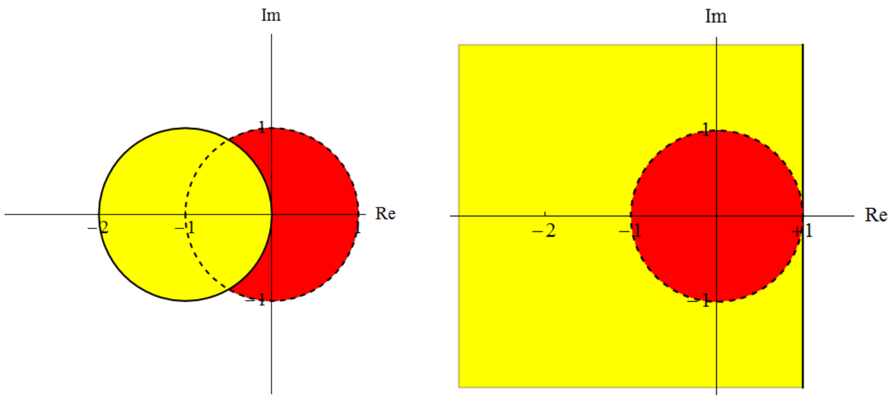
We proceed to prove result (II). Assume that the continuous system is stable. Then $\mathrm{Re}(\lambda^C) < 0$, so $1 + \lambda^C$ belongs to the region located to the left of the vertical line $\mathrm{Re}(\lambda) = +1$ as shown in Figure 1(B), where we also indicate the dotted unit cycle in which the discrete system is stable. It is clear that the unit circle is only a small part of the stability region of the continous system, which is the union of the yellow and red regions. So stability of the continuous system

[1] If $1 + \lambda^C = 1 + \alpha + i\beta$, then $\left|1 + \lambda^C\right| < 1$ can be rewritten as

$$(1+\alpha)^2 + \beta^2 < 1^2,$$

and it is necssary that $|1+\alpha| < 1$ holds, that is, $-2 < \alpha < 0$.

does not necessarily imply the same for the corresponding discrete system.



(A). Discret system

(B). Continuous system

Figure 1. Stability regions

In the following two subsections, we turn attention to the special cases, that is, two and three dimensional systems and convert the stability results obtained just above to the ones in terms of the coefficients of the characteristic equations.

2.2 Two dimensional systems

In the two dimensional case we will show the same conclusions on stability of the systems based on the coefficients of the characteristic polynomials. Assume that the coefficient martix of the continuous system is given by

$$\boldsymbol{J_C} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{8}$$

and then the coefficient matix of the discrete system turns to be

$$\boldsymbol{J_D} = \boldsymbol{I} + \boldsymbol{J_C} = \begin{pmatrix} a+1 & b \\ c & d+1 \end{pmatrix}. \tag{9}$$

The characteristic polynomial of $\boldsymbol{J_C}$ can be written as

$$\det(\boldsymbol{J_C} - \lambda \boldsymbol{I}) = \lambda^2 - (a+d)\lambda + (ad - bc). \tag{10}$$

The roots have negative real parts if and only if

$$\begin{aligned} A &= a + d < 0 \\ B &= ad - bc > 0 \end{aligned} \tag{11}$$

as shown in Szidaovszky and Bahill (1998). The characteristic polynomial of $\boldsymbol{J_D}$ has the following form:

$$\begin{aligned}\det(\boldsymbol{J_D} - \lambda \boldsymbol{I}) &= \lambda^2 - (a + d + 2)\lambda + (ad + a + d + 1 - bc) \\ &= \lambda^2 - (A + 2)\lambda + (A + B + 1).\end{aligned} \tag{12}$$

The roots are inside the unit circle if and only if

$$\begin{gathered} A + B + 1 < 1, \\ (A + 2) + (A + B + 1) + 1 > 0, \\ -(A + 2) + (A + B + 1) + 1 > 0, \end{gathered} \tag{13}$$

as shown in Bischi *et al*. (2010). Relations (11) show that point (A, B) is in the second quadrant, however (13) can be written as

$$\begin{gathered} A + B < 0, \\ 2A + B + 4 > 0, \\ B > 0. \end{gathered} \tag{14}$$

In Figure 3 we illustrate the region of the points (A, B) satisfying (11) by the union of the yellow and red regions and the region of the point satisfying (14) with the red region. It is clear that the red region is only a small subset of the union of the yellow and red regions showing that the stability of the discrete system implies stability for the continuous system but not the other way round.

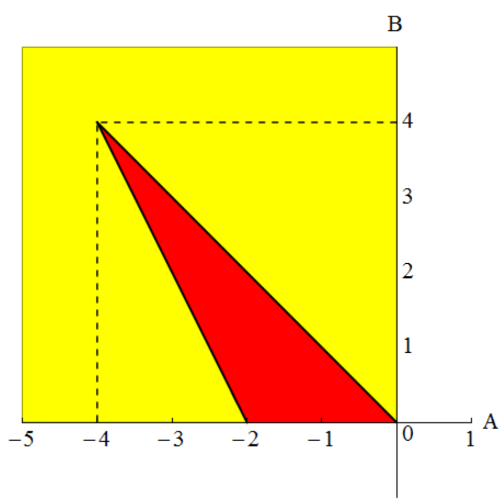


Figure 2. Comparison of stability regions

2.3 Three Dimensional Systems

Consider next a three-dimensional discrete system with characteristic polynomial,

$$\varphi(\lambda) = (\lambda)^3 + a_1(\lambda)^2 + a_2\lambda + a_3. \tag{15}$$

As shown by Farebrother (1973), the eigenvalues are inside the unit circle if and only if

$$\begin{gathered} 1 + a_1 + a_2 + a_3 > 0, \\ 1 - a_1 + a_2 - a_3 > 0, \\ 1 - a_2 + a_1 a_3 - a_3^2 > 0, \\ 3 - a_2 > 0. \end{gathered} \tag{16}$$

The eigenvalues of the corresponding continuous system are $\lambda^C = \lambda^D - 1$ due to equation (7). So replacing λ in the right hand side of equation (15) with $\lambda + 1$ yields the characteristic equation of the continuous system,

$$\begin{aligned} &(\lambda + 1)^3 + a_1(\lambda + 1)^2 + a_2(\lambda + 1) + a_3 \\ &= \lambda^3 + (3 + a_1)\lambda^2 + (3 + 2a_1 + a_2)\lambda + (1 + a_1 + a_2 + a_3). \end{aligned}$$

So the system is asymptotically stable if and only if

$$\begin{gathered} 3 + a_1 > 0, \\ 3 + 2a_1 + a_2 > 0, \\ 1 + a_1 + a_2 + a_3 > 0, \\ (3 + a_1)(3 + 2a_1 + a_2) - (1 + a_1 + a_2 + a_3) > 0. \end{gathered} \tag{17}$$

as a consequnce of the Routh-Hurwitz criterion. Although the first condition in (16) and the third condition in (17) are identical, it might be challenging to analytically check the inclusion relation between these two conditions. We graphically confirm it in Figure 3 in which the saddle shaped red body is constructed by the four inequality conditions in (16) while the three dimensional space surrounded by yellow-wise surfaces are constructed by the four conditions in (17). It is clearly seen that the stability region of the discrete system is included in the stable region of the continuous system.

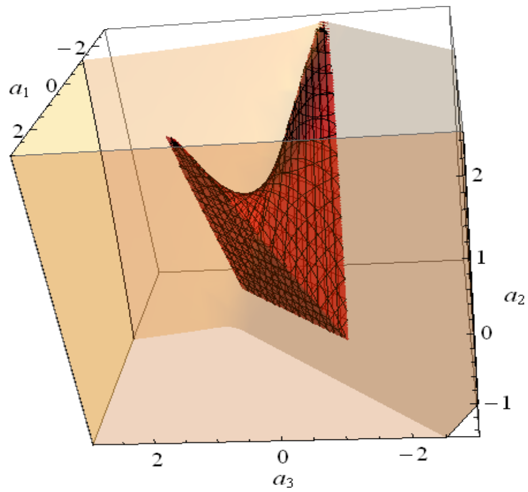
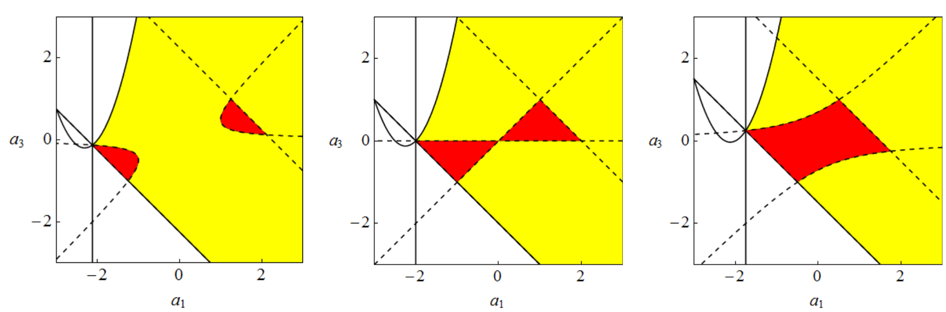


Figure 3. Stability regions

To see the same results from a different view point, we will numerically show that the conditions in (16) imply the conditions in (17). To this end, taking $a_2 = 1.25$, $a_2 = 1$, and $a_2 = 0.5$, respectively, we horizontally cut the 3D box in Figure 3 at each particular value of a_2 parallel to the (a_1, a_2) plane and then project the cross-section views onto it. The results are shown in Figure 4 in which, as before, the red region is the stable region of the discrete system and the yellow region is the stable region of the continuous system. Results (I) and (II) holds in the three dimensional models.



(A) $a_2 = 1.25$ (B) $a_2 = 1$ (C) $a_2 = 0.5$

Figure 4. Stability regions in the (a_1, a_3) plane with the fixed value of a_2

3 Examples

In this section, we confirm the stability conditions obtained above in actual economic dynamic models.

Example 1: Theocharis Problem

We start with an n dimensional model. Theocharis (1960) shows a provocative result (often called "Theocharis problem") of the Cournot quantity adjustment procss: in the case of linear single-product oligopolies without product differentiation, stability of the discrete-time model depends only on the number of firms in the market.[2] In particular, it is stable if the number is two, marginally stable if three and unstable if the number is more than three. Rebuilding the essential part of his discrete model, we convert it into a continuous-time framework to compare the stability regions.

The price function is linear

$$p = a - bQ$$

where $a > 0,\ b > 0$ and Q denotes the total output in the market. If x_j is firm j's output, then Q is defined by

$$Q = \sum_{j=1}^{n} x_j.$$

Firm j has a linear cost function $C_j(x_j) = c_j x_j$ and its profit is

$$\pi_j = [a - b(x_j + Q_{-j})]\, x_j - c_j x_j$$

where $Q_{-j} = Q - x_j$. Solving the first-order condition of the profit maximization yields a best reply of firm j,

$$R_j(Q_{-j}) = -\frac{1}{2}Q_{-j} + \frac{a - c_j}{2b}. \tag{18}$$

Summing up equation (18) for all $j = 1, 2, ..., n$ and solving it for Q present the equilibrium value of the total output

$$Q^* = \frac{na - C}{(n+1)b} \text{ with } C = \sum_{j=1}^{n} c_j.$$

Substituting Q^* into the best reply (18) and then solving the resultan equation for x_j give the equilibrium output level of firm j

$$x_j^* = \frac{1 + C - (n+1)c_i}{(n+1)b}.$$

[2] The Theocharis problem in a differentiated oligopoly is recently reconsidered by Matsumoto and Szidarovszky (2014a).

Assuming naive expectation, we consider stability of the equilibrium. The output adjustment is given by a system of difference equations,

$$x_j(t+1) = R_j(Q_{-j}(t))$$

or

$$x_j(t+1) = -\frac{1}{2}\sum_{i \neq j} x_i(t) + \frac{a - c_j}{2b} \text{ for } j = 1, 2, ..., n. \tag{19}$$

The coefficient matrix of the dynamic system is given by

$$\boldsymbol{J_D} = \begin{pmatrix} 0 & -1/2 & \cdots & -1/2 \\ -1/2 & 0 & \cdots & -1/2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ -1/2 & -1/2 & \cdots & 0 \end{pmatrix}. \tag{20}$$

Applying Lemma E.1 of Bischi *et al.* (2010), we can obtain the following characteristic polynmoial,[3]

$$\det(\boldsymbol{J_D} - \lambda \boldsymbol{I}) = (-1)^n \left(\lambda - \frac{1}{2}\right)^{n-1} \left(\lambda + \frac{n-1}{2}\right)$$

where the characteristic roots are

$$\lambda_1^D = ... = \lambda_{n-1}^D = \frac{1}{2} \text{ and } \lambda_n^D = -\frac{n-1}{2}.$$

Stability depends on the value of λ_n^D that is equal to

$$-1/2 \text{ if } n = 2,$$

$$-1 \text{ if } n = 3,$$

$$-3/2 \text{ if } n = 4$$

$$\text{and smaller than } -4/3 \text{ for } n \geq 5.$$

[3] Let A be a matrix having the following form

$$\boldsymbol{A} = \begin{pmatrix} a_1 & b_1 & \cdots & b_1 \\ b_2 & a_2 & \cdots & b_2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ b_n & b_n & \cdots & a_n \end{pmatrix}.$$

Then the characteristic polynomial of matrix $\boldsymbol{A}$ is given by

$$\det(\boldsymbol{A} - \boldsymbol{\lambda I}) = \prod_{k=1}^{n} (a_k - b_k - \lambda) \left(1 + \sum_{k=1}^{n} \frac{b_k}{a_k - b_k - \lambda}\right)$$

. This result is repeatedly used in the later part of this paper.

Hence the oligopoly model is stable in the duopoly market, marginally stable (i.e., cyclic fluctuations) in the triopoly market while it is unstable in the quartropoly market and the market with $n \geq 5$ is also unstable.

We can construct the corresponding continuous-time Cournot model by replacing $x(t+1) - x(t)$ with $\dot{x}(t)$. Let $\boldsymbol{J}_C$ be the coefficient matrix of the continuous time system

$$\boldsymbol{J_C} = \begin{pmatrix} -1 & -1/2 & \cdots & -1/2 \\ -1/2 & -1 & \cdots & -1/2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ -1/2 & -1/2 & \cdots & -1 \end{pmatrix}. \tag{21}$$

The characteristic equation is

$$\det(\boldsymbol{J_C} - \lambda \boldsymbol{I}) = (-1)^n \left(\lambda + \frac{1}{2}\right)^{n-1} \left(\lambda + \frac{n+1}{2}\right) = 0$$

and the characteristic roots are

$$\lambda_1^C = ... = \lambda_{n-1}^C = -\frac{1}{2} < 0 \text{ and } \lambda_n^C = -\frac{n+1}{2} < 0.$$

This result is also obtained via the relation (7) in which $\boldsymbol{J}_D = \boldsymbol{I} + \boldsymbol{J}_C$ implies that λ^C is equal to $\lambda^D - 1$. Hence the continuous system is always stable irrespective of the number of n. Thus in the large size market in which many firms participate (more precisely, $n > 3$), two dynamic system show a sharp difference, the discrete-time model is always unstable and the continuous-time model is always stable.

Example 2: Adjustment toward best responses

We take up a two dimensional model. Consider a duopoly with linear price and linear cost function as in Example 1. Equation (18) with $n = 2$ yields the best reply of firm j

$$R_j(x_{3-j}) = \frac{a - c_j}{2b} - \frac{x_{3-j}}{2} \text{ for } j = 1, 2.$$

The continuous dynamic system with adjustment toward best responses has the form[4]

$$\dot{x}_j(t) = k_j \left(R_j(x_{3-j}) - x_j\right) \text{ for } j = 1, 2 \tag{22}$$

where k_j is a adjustment coefficient. A cofficient matrix is

$$\boldsymbol{J_C} = \begin{pmatrix} -k_1 & -\dfrac{k_1}{2} \\ -\dfrac{k_2}{2} & -k_2 \end{pmatrix}.$$

[4]Notice that the output adjustment in the Theocharis model is given by

$$\dot{x}_j(t) = R_j(Q_{-j}(t)) \text{ for } j = 1, 2, ..., n,$$

as in the previous example

The characteristic polynomial is

$$\det(\boldsymbol{J_C} - \lambda \boldsymbol{I}) = \lambda^2 + (k_1 + k_2)\lambda + \frac{3k_1k_2}{4}.$$

So the system is always asymptotically stable since the positive coefficients satisfy the Routh-Hurwitz stability criterion with $n = 2$,

$$k_1 + k_2 > 0$$

and

$$k_1 k_2 > 0.$$

The discrete counterpart can be obtained when $\dot{x}_k(t)$ is replaced by $x_k(t+1) - x_k(t)$ resulting in the following discrte system:

$$\begin{aligned} x_1(t+1) &= (1-k_1)x_1(t) - \frac{k_1}{2}x_2(t) + k_1\frac{a-c_1}{2b} \\ x_2(t+1) &= (1-k_2)x_2(t) - \frac{k_2}{2}x_1(t) + k_2\frac{a-c_2}{2b} \end{aligned} \tag{23}$$

with cofficient matrix

$$\boldsymbol{J_D} = \begin{pmatrix} 1-k_1 & -\dfrac{k_1}{2} \\ -\dfrac{k_2}{2} & 1-k_2 \end{pmatrix}.$$

So we can apply the general argumnts in comparing the stability condition. The characteristic polynomial has the form

$$\det(\boldsymbol{J_D} - \lambda \boldsymbol{I}) = \lambda^2 + (-2 + k_1 + k_2)\lambda + \left(1 - k_1 - k_2 + \frac{3k_1k_2}{4}\right).$$

The roots are inside the unit circle if and only if

$$1 - k_1 - k_2 + \frac{3k_1k_2}{4} < 1$$

$$1 + (-2 + k_1 + k_2) + \left(1 - k_1 - k_2 + \frac{3k_1k_2}{4}\right) > 0$$

and

$$1 - (-2 + k_1 + k_2) + \left(1 - k_1 - k_2 + \frac{3kk_2}{4}\right) > 0$$

which can be simplified in the following way:

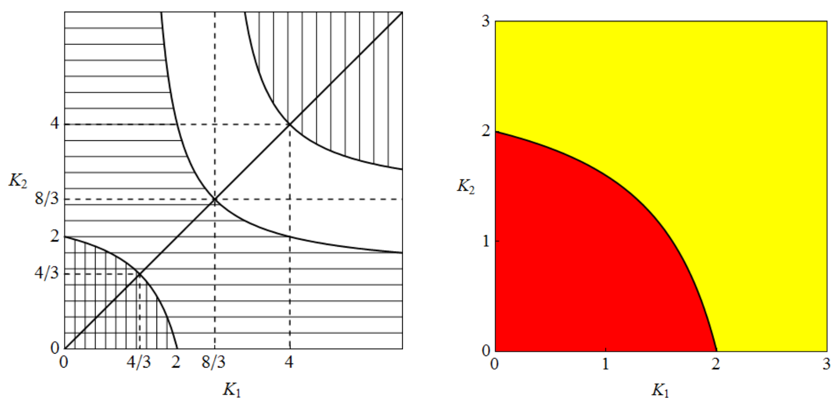
$$k_1 + k_2 - \frac{3k_1k_2}{4} > 0 \tag{24}$$

$$k_1k_2 > 0 \tag{25}$$

and

$$\frac{3k_1k_2}{4} - 2(k_1 + k_2) + 4 > 0. \tag{26}$$

Condition (25) is always satisfied. Conditions (24) and (25) are visualized in Figure 6(A). The middle hyperbola is the boundary of the horizontally-striped region in which condition (24) holds. The lower and higher hyperbolas are boundaries of the vertically-striped regions in which condition (25) holds. Apparently, the quarterly disk-shaped region in the lower-left corner is horizontally and vertically-striped and thus two conditions are satisfied there. It is the stability region of the discret system (23). The stability regions of the two systems in the (k_1, k_2) plane are shown in Figure 6(B). The red region is the stability region for the discret system while the union of the red and yellow regions is the stability region of the continuous system. The main result is graphically confirmed: if the discrete-time system is locally stable, then the corresponding continuous-time system is always stable but not *vice versa*.



(A) Stability conditions

(B) Stability regions

Figure 5. Stability conditions and region of th linear duopoly model

Example 3: Puu's nonlinear duopoly model

We consider a nonlinear two dimensional duopoly model proposed by Puu (2003). We retain the the linear cost function but replace the linear price function with an isoelastic price funciton,

$$p = \frac{1}{x_1 + x_2}.$$

The profit of firm j is

$$\pi_j = \frac{x_j}{x_j + x_{3-j}} - c_j x_j \text{ for } j = 1, 2.$$

Solving the first order condition of the profit maximization yields the best reply

$$R_j(x_{3-j}) = \sqrt{\frac{x_{3-j}}{c_j}} - x_{3-j} \text{ for } j = 1, 2.. \tag{27}$$

The positive equilibrium output at the Cournot point is

$$x_j^* = \frac{c_{3-j}}{(c_1 + c_2)^2} \text{ for } j = 1, 2.$$

To simplify the dynamic analysis, we adopt a naive expectation formation of output and construct the discrete-time output adjustment as

$$x_j(t+1) = \sqrt{\frac{x_{3-j}(t)}{c_j}} - x_{3-j}(t) \text{ for } j = 1, 2. \tag{28}$$

The coefficient matrix is

$$\boldsymbol{J_D} = \begin{pmatrix} 0 & \dfrac{c_2 - c_1}{2c_1} \\ \dfrac{c_1 - c_2}{2c_2} & 0 \end{pmatrix}$$

and the corresponding characteristic equation is $\det(\boldsymbol{J_D} - \lambda \boldsymbol{I}) = 0$ or

$$\lambda^2 = \frac{(c_1 - c_2)(c_2 - c_1)}{4c_1c_2}$$

where the stability conditions are

$$-1 < \frac{(c_1 - c_2)(c_2 - c_1)}{4c_1c_2} < 1.$$

The second inequality is always satisfied as the middle term is negative and the first inequality condition is rewritten as

$$c_1^2 - 6c_1c_2 + c_2^2 < 0$$

which can be solved for the ratio

$$3 - 2\sqrt{2}(\simeq 0.172) < \frac{c_2}{c_1} < 3 + 2\sqrt{2}(\simeq 5.828). \tag{29}$$

Therefore the discrete-time dynamic system (28) is locally asymptotically stable if the ratio of the marginal costs c_1 and c_2 satisfies the last two inequlities.

Let us convert the discrete-time system to the continuous-time system by subtracting $x_j(t)$ from the both sides of (28) and then replacing $x_j(t+1) - x_j(t)$ by $\dot{x}_j(t)$,

$$\dot{x}_j(t) = -x_j(t) - \sqrt{\frac{x_{3-j}(t)}{c_j}} - x_{3-j}(t) \text{ for } j = 1, 2. \tag{30}$$

The coefficient martix is

$$\boldsymbol{J_C} = \begin{pmatrix} -1 & \dfrac{c_2 - c_1}{2c_1} \\ \dfrac{c_1 - c_2}{2c_2} & -1 \end{pmatrix}.$$

The characteristic equation is $\det(\boldsymbol{J_C} - \lambda \boldsymbol{I}) = 0$ or

$$\lambda^2 + 2\lambda + \left(1 - \frac{(c_1 - c_2)(c_2 - c_1)}{4c_1 c_2}\right) = 0. \tag{31}$$

According to the Routh-Hurwitz stability criterion with $n = 2$, the real parts of the characteristic roots are negative if the sum and the product of the two coefficients are positive. Since the third term of equation (31) is positive,

$$1 - \frac{(c_1 - c_2)(c_2 - c_1)}{4c_1 c_2} = \frac{(c_1 + c_2)^2}{4c_1 c_2} > 0,$$

the continuous time dynamic system (30) is locally asymptotically stable for any $c_1 > 0$ and $c_2 > 0$. The stability regions of the two systems are illustrated as in Figure 7 in which the boundaries of the red and yellow regions are given by

$$c_2 = \left(3 + 2\sqrt{2}\right) c_1 \text{ and } c_2 = \left(3 - 2\sqrt{2}\right) c_1.$$

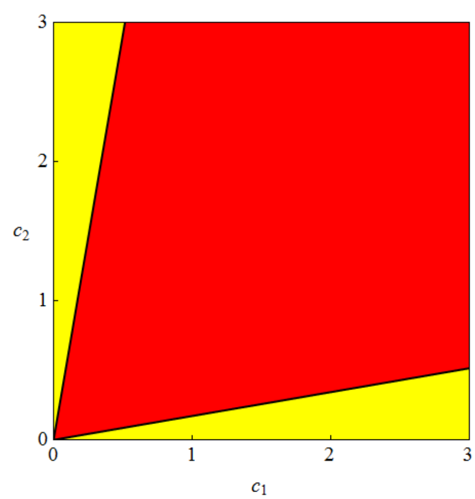


Figure 6. The stability regions in the nonlinear duopoly model

Example 4: Linear triopoly model

We now draw attention to a three dimensional linear model in which the best response of firm j is given by (18) with $n = 3$,

$$R_j(Q_{-j}) = \frac{a - c_j}{2b} - \frac{1}{2}Q_{-j}$$

where $Q = x_1 + x_2 + x_3$ and $Q_{-j} = Q - x_j$. The continuous dynamic system with adjustment toward best responses has the form

$$\dot{x}_j(t) = k_j \left(R_j(Q_{-j}) - x_j\right) \text{ for } k = 1, 2, 3 \tag{32}$$

with coefficient martrix

$$\boldsymbol{J_C} = \begin{pmatrix} -k_1 & -\frac{k_1}{2} & -\frac{k_1}{2} \\ -\frac{k_2}{2} & -k_2 & -\frac{k_2}{2} \\ -\frac{k_3}{2} & -\frac{k_3}{2} & -k_3 \end{pmatrix}.$$

The corresponding characteristic equation is $\det(\boldsymbol{J_C} - \lambda_C \boldsymbol{I}) = 0$ or

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where coefficients are given as

$$a_1 = k_1 + k_2 + k_3 > 0,$$

$$a_2 = \frac{3}{4}(k_1k_2 + k_2k_3 + k_1k_3) > 0$$

and

$$a_3 = \frac{1}{2}k_1k_2k_3 > 0.$$

Furthermore,

$$a_1a_2 - a_3 = 3\{(k_2 + k_3)k_1^2 + (k_1 + k_3)k_2^2 + (k_1 + k_2)k_3^2\} + 7k_1k_2k_3 > 0.$$

Hence the Routh-Hurwitz criterion with $n = 3$ is satisfied and the continuous system (32) is always locally asymptotically stable for $k_1 > 0$, $k_2 > 0$ and $k_3 > 0$.

The characteristic roots of the corresponding discrete-time system is givn by $\det(\boldsymbol{J_D} - \lambda_d \boldsymbol{I}) = 0$ where $\boldsymbol{J_D} = \boldsymbol{I} + \boldsymbol{J_C}$ or $\lambda^C = \lambda^D - 1$ so the characteristic polynomial has the form,

$$(\lambda - 1)^3 + a_1(\lambda - 1)^2 + a_2(\lambda - 1) + a_3 = \lambda^3 + b_1\lambda^2 + b_2\lambda + b_3$$

where

$$b_1 = a_1 - 3$$

$$b_2 = 3 - 2a_1 + a_2$$

and

$$b_3 = -(1 - a_1 + a_2 - a_3).$$

The stability conditions are given by

$$1 + b_1 + b_2 + b_3 > 0$$

$$1 - b_1 + b_2 - b_3 > 0$$

$$1 - b_2 + b_1 b_3 - b_3^2 > 0$$

$$3 - b_2 > 0.$$

The forms in terms of k_j of the other conditions are complicated and their signs may not be determined analytically. However it is graphically confirmed that the second condition is the stronger. Figure 7 shows the stability region in the discrte case as the red body.

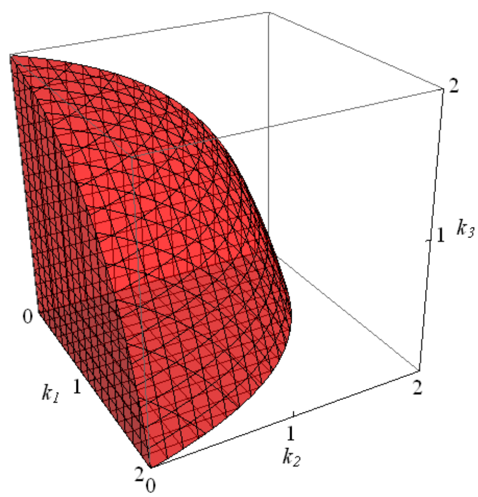


Figure 7.

Example 5: Nonlinear triopoly model

As is already discussed in Puu (2004), in a triopoly with isoelastic price function and linear cost functions, the best reply of firm j is given by

$$x_j = \sqrt{\frac{Q_{-j}}{c_j}} - Q_{-j} \text{ for } j = 1, 2, 3.$$

A discrete-time dynamic system with naive expectation is as follows:

$$x_1(t+1) = \sqrt{\frac{x_2(t)+x_3(t)}{c_1}} - x_2(t) - x_3(t),$$

$$x_2(t+1) = \sqrt{\frac{x_1(t)+x_3(t)}{c_2}} - x_1(t) - x_3(t),$$

$$x_3(t+1) = \sqrt{\frac{x_1(t)+x_2(t)}{c_3}} - x_1(t) - x_2(t).$$

where the stationary point $x_j^* = x_j(t) = x_j(t+1)$ is

$$x_1^* = \frac{2(c_2+c_3-c_1)}{(c_1+c_2+c_3)^2},$$

$$x_2^* = \frac{2(c_1+c_3-c_2)}{(c_1+c_2+c_3)^2},$$

$$x_3^* = \frac{2(c_1+c_2-c_3)}{(c_1+c_2+c_3)^2}.$$

To check the local stability of the nonlinear system, we linearize it around the stationary point and obtain the following form of the Jacobian matrix

$$\boldsymbol{J_D} = \begin{pmatrix} 0 & \dfrac{c_2+c_3-3c_1}{4c_1} & \dfrac{c_2+c_3-3c_1}{4c_1} \\ \dfrac{c_1+c_3-3c_2}{4c_2} & 0 & \dfrac{c_1+c_3-3c_2}{4c_2} \\ \dfrac{c_1+c_2-3c_3}{4c_3} & \dfrac{c_1+c_2-3c_3}{4c_3} & 0 \end{pmatrix}.$$

The characteristic equation is defined by $\det(\boldsymbol{J_D} - \lambda \boldsymbol{I}) = 0$ or

$$\lambda^3 - a_2\lambda - a_3 = 0$$

with

$$a_2 = -\frac{c_1^3+c_2^3+c_3^3 - 5\left[c_1^2(c_2+c_3)+c_2^2(c_1+c_3)+c_3^2(c_1+c_2)\right] + 30c_1c_2c_3}{16c_1c_2c_3}$$

and

$$a_3 = -\frac{(c_1+c_2-3c_3)(c_2+c_3-3c_1)(c_1+c_3-3c_2)}{32c_1c_2c_3}.$$

With new notations

$$\alpha = \frac{c_2}{c_1} \text{ and } \beta = \frac{c_3}{c_1},$$

the stability conditions can be written as

$$1 + a_2 + a_3 = \frac{(1+\alpha+\beta)^3}{32\alpha\beta} > 0,$$

$$1 + a_2 - a_3 = f_1(\alpha, \beta) > 0,$$
$$1 - a_2 - a_3^2 = f_2(\alpha, \beta) > 0,$$
$$3 - a_2 = f_3(\alpha, \beta) > 0.$$

Although the explicit forms of $f_i(\alpha, \beta)$ for $i = 1, 2, 3$ are long and complicated and thus not given, it is possible to check the stability conditions graphically. In Figure 8(A), $f_1(\alpha, \beta) \le 0$ in the blue regions and $f_3(\alpha, \beta) \le 0$ in the green regions while $f_2(\alpha, \beta) > 0$ in the red region. So in the white and red regions, $f_1(\alpha, \beta) > 0$ and $f_3(\alpha, \beta) > 0$. Consequently, the stability conditions are satisfied in the red region.

The Jacob matrix of the corresponding continuous system is obtained by $\boldsymbol{J_C} = \boldsymbol{J_D} - \boldsymbol{I}$ and then the characteristic equation is

$$\det(\boldsymbol{J_C} - \lambda \boldsymbol{I}) = \lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0$$

where

$$b_1 = 3$$
$$b_2 = \frac{(1 + \alpha + \beta)\left[6(\alpha + \beta + \alpha\beta) - (1 + \alpha^2 + \beta^2)\right]}{16\alpha\beta} = g_1(\alpha, \beta)$$
$$b_3 = \frac{(1 + \alpha + \beta)^3}{32\alpha\beta}.$$

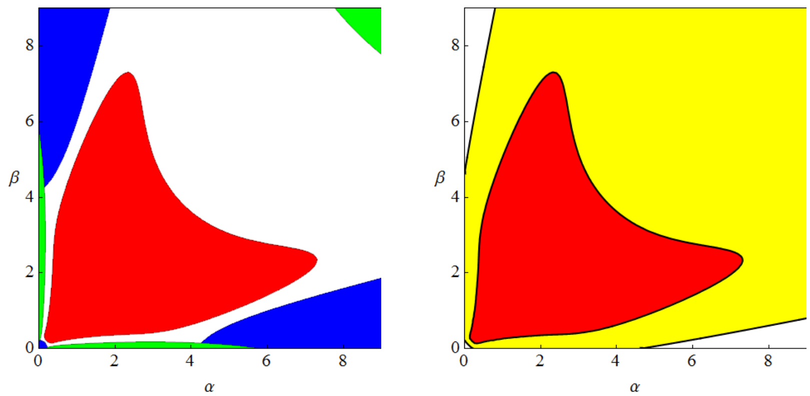
Since $b_1 > 0$ and $b_3 > 0$, the stability conditions of the continuous system is given by

$$g_1(\alpha, \beta) > 0$$

and

$$b_1 b_2 - b_3 = g_2(\alpha, \beta) > 0.$$

It is graphically confirmed that $g_2(\alpha, \beta) > 0$ implies $g_1(\alpha, \beta) > 0$ but not *vice versa*. So the boundary of the stability region is the locus of $g_2(\alpha, \beta) = 0$ that is described by the upward-sloping two black curves and the downward-sloping black curve depicted in a neighborhood of the origin in Figure 8(B). The stability region of the discrete system is also illustrated in red and located within the yellow region. Hence stability of the discrete system always implies the stability of the continuous system.



(A) (B)

Figure 8.

4 Delay in Continuous System

Delay has been thought to be one of the main ingredients for cyclic oscillations and thus delay economic dynamics is a relatively old research area. Haldane (1933) could be the first to examine economic dynamics in a delay differential equation. Since then, delay has been considered in various areas of economics, Kalecki (1935) and Goodwin (1950) for macroeconomic fluctuations, Howroyd and Russel (1984) for oligopoly dynamics, Mackey (1989) for price dynamics. Only recently Matsumoto and Szidarovszky (2014) formulate a delay monopoly as a model possesing the properties of the discrete system and the continuous system. To name a few, however it is true that the growth rate of delay study has been very slow. In this section we return to an n-dimensional model and consider how the delays in variables affect dynamics with adjustment toward best replies. To this end, we extend the duopoly model (22) in Example 2 to the n-dimensional oligopoly model,

$$\dot{x}_j(t) = k_j \left[-x_j(t) + R_j(Q_{-j}(t)) \right] \text{ for } j = 1, 2, ..., n. \tag{33}$$

To simplify the analysis, we impose the following assumption:

Assumption $k_j = k$ for all $j = 1, 2, ..., n$.

The coefficient matrix is

$$\boldsymbol{J_C} = \begin{pmatrix} -k & -k/2 & \cdots & -k/2 \\ -k/2 & -k & \cdots & -k/2 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ -k/2 & -k/2 & \cdots & -k \end{pmatrix} \tag{34}$$

and then the characteristic equation is given by $\det(\boldsymbol{J_C} - \lambda \boldsymbol{I}) = 0$ or

$$(-1)^n \left(\lambda + \frac{k}{2}\right)^{n-1} \left(\lambda + \frac{n+1}{2}k\right) = 0.$$

The characteristic roots are

$$\lambda_1^C = ... = \lambda_{n-1}^C = -\frac{1}{2}k < 0 \text{ and } \lambda_n^C = -\frac{n+1}{2}k < 0$$

and thus the n-dimensional model (33) is asymptotically stable. On the other hand, the characteristic roots of the corresponding discrete system are obtained by $\lambda^D = 1 + \lambda^C$,

$$\lambda_1^D = ... = \lambda_{n-1}^D = 1 - \frac{1}{2}k < 1 \text{ and } \lambda_n^C = 1 - \frac{n+1}{2}k < 1$$

which yield the stability conditions, $-1 < \lambda_{j-1}^D$ for $j = 1, 2, ..., n-1$ and $-1 < \lambda_n^C$ or simply

$$k < \frac{4}{n+1} \text{ for } n \geq 2. \tag{35}$$

Hence the stabilty depends on the value of the adjustment coefficient and the number of the firms. Theocharis problem is still alive in the discrete system. We now introduce delays into the continuous system (33) and see how the delay affect its dynamics. In the following, we examine the delay effects in three different ways: delays only in the comeptitors' variables, delays only in the own variable and delays in both variables.

4.1 Off-Diagonal Delays

We first consider the case in which the firms have delays for obtaining information about the comptitors' decisions which we call *information delays*. The dynamic system (33) with the same adjustment coefficients is modified as follows:

$$\dot{x}_j(t) = k_j \left[R_j(Q_{-j}(t - \tau)) - x_j(t)\right] \text{ for } j = 1, 2, ..., n \tag{36}$$

where $\tau > 0$ is the length of the information delay and is assumed to be identical for all firms for the sake of analytical simplicity. Assuming the exponential solutions, $x_j(t) = e^{\lambda t} u_j$ for $j = 1, 2, ..., n$ and substituting these into (36) give the coefficient matrix

$$\boldsymbol{J}_1 = \begin{pmatrix} -k & -\frac{k}{2}e^{-\lambda t} & \cdots & -\frac{k}{2}e^{-\lambda t} \\ -\frac{k}{2}e^{-\lambda t} & -k & \cdots & -\frac{k}{2}e^{-\lambda t} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ -\frac{k}{2}e^{-\lambda t} & -\frac{k}{2}e^{-\lambda t} & \cdots & -k \end{pmatrix} \tag{37}$$

and the corresponding characteristic equation is given by $\det(\boldsymbol{J}_1 - \lambda \boldsymbol{I}) = 0$ or

$$\left(\lambda + k - \frac{k}{2}e^{-\lambda t}\right)^{n-1}\left(\lambda + k + \frac{n-1}{2}ke^{-\lambda t}\right) = 0$$

that generates two independent equations,

$$\lambda + k - \frac{k}{2}e^{-\lambda t} = 0 \text{ and } \lambda + k + \frac{n-1}{2}ke^{-\lambda t} = 0. \tag{38}$$

Supposing that $\lambda = i\omega$ with $\omega > 0$ and substituting it into the first equation of (38), we have

$$i\omega + k - \frac{k}{2}\left(\cos\tau\omega - i\sin\tau\omega\right) = 0$$

which is divided into the real and imaginary parts,

$$k - \frac{k}{2}\cos\tau\omega = 0,$$

$$\omega - \frac{k}{2}\sin\tau\omega = 0.$$

Moving the constant terms to the right hand sides and adding the squares of these equations gives

$$\omega^2 = -\frac{3}{4}k^2 < 0$$

in which the inequality leads to the result that there is no $\omega > 0$. This implies that no stability switch occurs. In the same way, supposing that $\lambda = i\omega$ with $\omega > 0$, substituting it into the second equation of (38) and dividing the resultant expressions into the real and imaginary parts give

$$\begin{aligned} k + \frac{n-1}{2}k\cos\tau\omega &= 0, \\ \omega - \frac{n-1}{2}k\sin\tau\omega &= 0. \end{aligned} \tag{39}$$

Again, moving the constant terms to the right hand sides and adding the squares of these equations gives

$$\omega^2 = \frac{(n+1)(n-3)}{4}k^2$$

from which we derive the following two results,

(i) if $n \leq 3,$ then there is no $\omega > 0$, implying no stability switch;

(ii) if $n \geq 4,$ then there is $\omega^* = \frac{\sqrt{(n+1)(n-3)}}{2}k > 0,$ implying stability switch

Substituting ω^* into the first equatio of (39) and solving it for τ determine the threshold values of τ for which some of the characteristic roots are purely imaginary,[5]

$$\tau_m^* = \frac{1}{\omega^*}\left[\cos^{-1}\left(-\frac{2}{n-1}\right) + 2m\pi\right].$$

Since it is already shown that the system is asymptotically stable for $\tau = 0$, stability is switchd to instability when τ first arrives at the threshold value τ_0^*,

$$\tau_0^*(k,n) = \frac{2\cos^{-1}\left(-\frac{2}{n-1}\right)}{\sqrt{(n+1)(n-3}}\frac{1}{k} \tag{40}$$

which is a hyperbola with respect to k. The first factor of the right hand side expression approximately takes

$$2.06 \text{ if } n = 4,\ 1.21 \text{ if } n = 5 \text{ and } 0.87 \text{ if } n = 6.$$

So the hyperbolic curve shifts downward as the number of the firms in the market increases, that is, increasing n has a destabilizing effect in the sense that the stability region in the (k, τ) plane shrinks.

4.2 Diagonal Delays

We now examine the case in which the firms have delays for making decisins to change production and/or for putting these descisions into effects which we call *implementation delays*. The dynamic system (33) is modified as follows:

$$\dot{x}_j(t) = k_j\left[-x_j(t-\tau) + R_j(Q_{-j}(t))\right] \text{ for } j = 1, 2, ..., n \tag{41}$$

where $\tau > 0$ now denotes the length of the implementation delay and is assumed to be identical for all firms for the sake of analytical simplicity. Given $x_j = e^{-\lambda t}u_j$ for $j = 1, 2, ..., n$, the coefficient matrix is

$$\boldsymbol{J}_2 = \begin{pmatrix} -ke^{-\lambda t} & -\frac{k}{2} & \cdots & -\frac{k}{2} \\ -\frac{k}{2} & -ke^{-\lambda t} & \cdots & -\frac{k}{2} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ -\frac{k}{2} & -\frac{k}{2} & \cdots & -ke^{-\lambda t} \end{pmatrix} \tag{42}$$

and the corresponding characteristic equation is given by $\det(\boldsymbol{J}_2 - \lambda\boldsymbol{I}) = 0$ or

$$\left(\lambda - \frac{k}{2} + ke^{-\lambda t}\right)^{n-1}\left(\lambda + \frac{n-1}{2}k + ke^{-\lambda t}\right) = 0$$

[5] Substituting ω^* into the second equation and solving it for τ give the different form for the same value,

$$\tau_n^* = \frac{1}{\omega^*}\left[\pi - \sin^{-1}\left(\frac{2\omega^*}{(n-1)k}\right) + 2n\pi\right].$$

that generates two independent equations,

$$\lambda - \frac{k}{2} + ke^{-\lambda t} = 0 \text{ and } \lambda + \frac{n-1}{2}k + ke^{-\lambda t} = 0. \tag{43}$$

As in the same way as in the Section 4-1, we suppose that $\lambda = i\omega,\ \omega > 0$ and substitute it into the first equation of (43). Following the same procedure yields

$$\omega_a^* = \frac{\sqrt{3}}{2}k > 0$$

and

$$\tau_{a,m}^* = \frac{1}{\omega_a^*}\left[\cos^{-1}\left(\frac{1}{2}\right) + 2m\pi\right].$$

The threshold value $\tau_{a,0}^*$ is

$$\tau_{a,0}^*(k) = \frac{2\pi}{3\sqrt{3}}\frac{1}{k}. \tag{44}$$

Notice that this value is independent from the number of the firms.

Solving the scond equation of (43) with $\lambda = i\omega$ presents

$$\omega^2 = \frac{(n+1)(3-n)}{4}k^2$$

which can be positive only for $n = 2$,

$$\omega_b^* = \frac{\sqrt{3}}{2}k > 0$$

and

$$\tau_{b,m}^* = \frac{1}{\omega_b^*}\left[\cos^{-1}\left(-\frac{1}{2}\right) + 2m\pi\right].$$

The threhold value $\tau_{b,0}^*$ is

$$\tau_{b,0}^*(k) = \frac{4\pi}{3\sqrt{3}}\frac{1}{k} = 2\tau_{a,0}^*. \tag{45}$$

Equation (45) implies that the delay system (41) is stable for $\tau < \tau_{a,0}^*$ and loses stability for $\tau \geq \tau_{a,0}^*$. Hence equation (44) determines the stability switching curve even for $n = 2$.

4.3 Diagonal and Off-Diagonal Delays

In this subsection we deal with the case in which the implmentation and information delays coexist. However, for the sake of simplifity both delays are assumed to be idential. So the delay dynamic system is

$$\dot{x}_j(t) = k_j\left[-x_j(t-\tau) + R_j(Q_{-j}(t-\tau))\right] \text{ for } j = 1, 2, ..., n \tag{46}$$

where $\tau > 0$ now denotes the length of the implementation and infomation delay. The coefficient matrix is

$$\boldsymbol{J}_3 = \begin{pmatrix} -ke^{-\lambda t} & -\frac{k}{2}e^{-\lambda t} & \cdots & -\frac{k}{2}e^{-\lambda t} \\ -\frac{k}{2}e^{-\lambda t} & -ke^{-\lambda t} & \cdots & -\frac{k}{2}e^{-\lambda t} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ -\frac{k}{2}e^{-\lambda t} & -\frac{k}{2}e^{-\lambda t} & \cdots & -ke^{-\lambda t} \end{pmatrix} \tag{47}$$

and the corresponding characteristic equation is given by $\det(\boldsymbol{J}_3 - \lambda \boldsymbol{I}) = 0$ or

$$\left(\lambda + \frac{k}{2}e^{-\lambda t}\right)^{n-1} \left(\lambda + \frac{n+1}{2}ke^{-\lambda t}\right) = 0$$

that generates two independent equations,

$$\lambda + \frac{k}{2}e^{-\lambda t} = 0 \text{ and } \lambda + \frac{n+1}{2}ke^{-\lambda t} = 0. \tag{48}$$

As in the same way as in the Section 4-1, we suppose that $\lambda = i\omega,\ \omega > 0$ and substitute it into the first equation of (46) to obtain

$$\omega_A^* = \frac{k}{2} > 0$$

and

$$\tau_{A,m}^* = \frac{1}{\omega_A^*}\left(\frac{\pi}{2} + 2m\pi\right)$$

with

$$\tau_{A,0}^*(k) = \frac{\pi}{k}.$$

Solving the scond equation of (43) with $\lambda = i\omega$ for ω presents

$$\omega_B^* = \frac{n+1}{2}k > 0$$

and

$$\tau_{B,m}^* = \frac{1}{\omega_B^*}\left(\frac{\pi}{2} + 2m\pi\right)$$

with

$$\tau_{B,0}^*(k,n) = \frac{\pi}{n+1}\frac{1}{k} = \frac{1}{n+1}\tau_{A,0}^*(k). \tag{49}$$

Equation (49) implies that the delay system (46) is stable for $\tau < \tau_{b,0}^*$ and unstable for $\tau \geq \tau_{b,0}^*$. Hence $\tau_{b,0}^*$ determines the stability switching curve.

We now examine the locations of the stability switchi curves in three cases, (40), (44) and (**??**). First compare the first factors that depend only on the value of n as depicted in Figure 9(A). The blue, red and green curves are described, respectively, by

$$\frac{2\cos^{-1}\left(-\frac{2}{n-1}\right)}{\sqrt{(n+1)(n-3)}}, \ \frac{2\pi}{3\sqrt{3}} \text{ and } \frac{\pi}{n+1}.$$

It is seen that

$$\frac{2\cos^{-1}\left(-\frac{2}{n-1}\right)}{\sqrt{(n+1)(n-3)}} > \frac{\pi}{n+1} \text{ and } \frac{2\pi}{3\sqrt{3}} > \frac{\pi}{n+1} \text{ always for any } n > 3 \tag{50}$$

and

$$\frac{2\cos^{-1}\left(-\frac{2}{n-1}\right)}{\sqrt{(n+1)(n-3)}} \gtreqless \frac{2\pi}{3\sqrt{3}} \text{ if } n \lesseqgtr 5. \tag{51}$$

Figure 9(B) illustrates the stability switching curves in the (k,τ) regions in which the real curves have $n = 4$ and the dotted curves have $n = 6$. It is seen first that increasing n shifits the cuves downward except the $\tau = \tau_a^*(k)$, that is, it has a destabilizing effect. It is seen second that, as indicated by (51), the off-diagonal delay has stronger destabilizing effect than the diagonal delay if $n < 5$ and the relation is reversed if $n > 5$. It is lastly senn that the two delays has stronger destabilizing effect than the single delay.

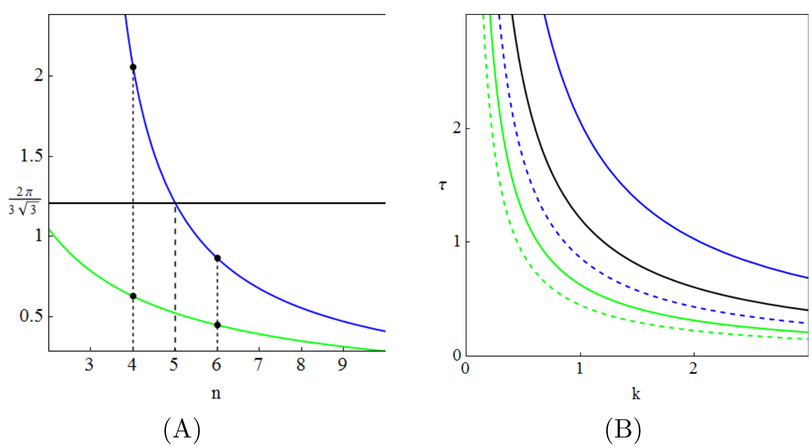


Figure 9.

We now turn attention to compariosn among the continous, descrete and delay systems. As before the stability region of the descrete sysems is coloured in red and the stability region of the continuous system is the uniton of the red and yellow regions. Notice that the boundary of the red region is described by the $k = 4/(n+1)$ curve. First we take $\tau = 0.7$ and illustrate the real curves of $\tau_0^*(k,n) = \tau$, $\tau_a^*(k) = \tau$ and $\tau_B^*(k,n) = \tau$ in blue, black and green, respectively.

It can be seen that all three curves are in the yellow region, implying that the delay system is more stable than the discrete system. We change the value of τ to 1.5 and illustrate the stability switching curve in the dotted curves in the same colour. It is seen that the dotted green curve is located in the red region while the some parts of the black and blue dotted curves are in the red region. It depends on the number of the firms involved in the market and the length of the delay whether the discrete system is more stable than the single delay system.

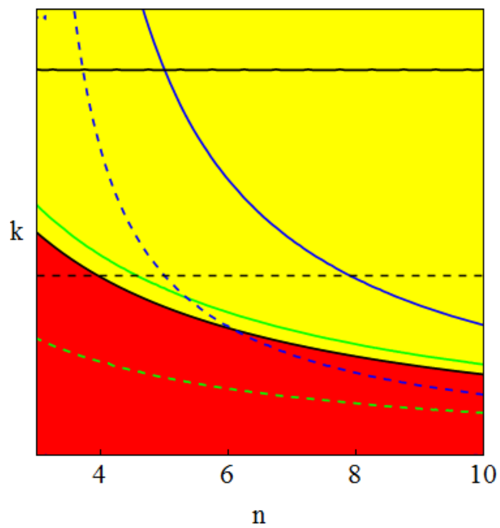


Figure 10.

5 Conclusions

In this brief note we presented illustrations an simple mathematical facts why continous dynamic system are more stable than their discrete counterparts. In the n-dimensional case the comparirion of the stability regions of the eigenvalues of the Jacobian showed the reason and in the two dimensional case the stability region s of the coefficients of the characteristic polynomials were compared to reach the same conclusion.

References

[1] Bischi, G-I., C. Chiarella, M. Kopel and F. Szidarovszky (2010), *Nonlinear Oligopolies: Stability and Bifurcation*, Springer-Verlag, Berlin/New York.

[2] Farebrother, R. (1973), Simplified Samuelson Conditions for Cubic and Quartic Equations, *The Manchester School of Economic and Social Studies*, 41, 395-400.

[3] Goodwin, R. (1951), The Nonlinear Accelerator and the Persistence of Business Cycles, *Econometrica*, 19, 1-17.

[4] Haldane, J. A. (1933), Contribution to the Theory of Price Fluctuations, *Review of Economic Studies*, 1, 186-195.

[5] Howroyd, T. and A. Russel (1984), Cournot Oligopoly Models with Time Delays, *Journal of Mathmatical Economics*, 13, 97-103.

[6] Kalecki, M. (1935), A Macrodynamic Theory of Business Cycle, *Econometrica*, 3, 327-344.

[7] Mackey, M. (1989), Commodity Price Fluctutaions: Price Dependent Delays and Nonlinearities as Explanatory Factors, *Journal of Economic Theory*, 48, 497-509.

[8] Matsumoto, A. and F. Szidarovszky (2014a), Theocharis Problem Reconsidered in Differentiated Oligopoly, 2014, Article ID 630351, 12 pages, *Economic Research International*.

[9] Matsumoto, A. and F. Szidarovszky (2014b), Discrete and Continuous Dynamics in Nonlinear Monopolies, *Applied Mathematics and Computations*, 232, 632-642.

[10] Puu, T. (2003), *Attractor, Bifurcation and Chaos*, 2nd edition, Springer-Verlag, Berlin/Heidelberg/New York.

[11] Theocharis, R. (1960), On the Stability of the Cournot Solution on the Oligopoly Problems, *Review of Economic Studies*, 27, 133-134.

[12] Szidarovszky, F. and T. Bahill. (1998) *Linear System Theory*, CRC Press, Boca Daton, Florida (2nd, edition)