Dynamics in International Subsidy Games with Unit-elasitic Demand*

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Abstract

In this paper, we construct a three-country model with two governments and two firms and consider dynamic behavior of the sequential subsidy game in which the governments determine their optimal trade policies and then the firms determine their optimal outputs. We first show the existence of an optimal trade policy under realistic conditions. In the case of symmetric firms, the governments adopt periodic mixed trade policy (i.e., one government gives subsidy and the other levies tax in one period and then the governments interchange their policies in the next period) if the adjustment is naive, and the governments adopt a stable mixed policy if adaptive. In the case of asymmetric firms, a firm receives subsidy if its cost is lower and pays tax if higher. If the Cournot output point under the optimal subsidy is locally stable, then its dynamics can be periodic which is synchronized with the periodic trade policy. If it is locally unstable, then complex dynamics involving chaos emerges regardless of the cost difference.

1 Introduction

Markets become imperfectly competitive due to many factors such as the small number of firms, the differentiated goods, the scale of economics, etc. In such an imperfectly competitive international market, the governments may be motivated to introduce trade policies like tariff, export subsidy and tax to increase national welfare of their countries. It has been demonstrated that an increase in a domestic export subsidy raises the domestic firm's output and its profit when the firms compete in a Cournot way (Brander and Spencer (1985)). It has been also demonstrated that an export tax can be optimal when the firms compete

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in a Bertrand way (Eaton and Grossman (1986)). It is now well-known that the source of this sharp contrast comes from the difference in the assumption on the firms' strategic behavior (that is, the goods are strategic substitutes or strategic complements). It is also well-known that this behavioral difference relates to the curvature and the elasticity of the demand function. Recently, constructing a simplified version of the three-country model with two firms and two governments, Bandyopadhyay (1997) shows, among others, the following clear-cut results when the demand is unit-elastic:

(1) When the foreign government is passive, the optimal trade policy of the active domestic government is free trade if the production costs of the two firms are identical, an export subsidy if the home firm has lower cost and an export tax otherwise.

(2) When both governments are active, a continuum of policy equilibria exist if the production costs are identical and no policy equilibrium exists if they are different.

As a consequence of the second result, no dynamic consideration has been provided yet. In particular, it is not known yet how the optimal policy as well as the optimal outputs change over time and what kinds of changes might occur. The main purpose of this paper is to consider dynamics of the optimal trade policy and the associated optimal outputs under unit-elastic demand and to show that rich dynamics can be born when natural constraints are imposed on the government's policy selections. This paper complements Bandyopadhyay (1997) from a dynamic point of view. It is a continuation of Matsumoto and Serizawa (2007) who focus mainly on the comparative statistic analysis of the similar model (i.e., the effects on the optimal outputs caused by a change in the trade policy of the domestic government). The dynamic model of outputs to be considered in this paper resembles nonlinear dynamic duopoly models, which have been extensively studies for the last twenty years. Comprehensive summary of the earlier work has been presented in Puu (2003). More recent developments on this field are given in Bischi, *et al.* (2009). This paper also aims to apply the theoretical results obtained there to the dynamic analysis in the framework of international economics.

The paper is organized as follows. Section 2 presents a variant of the three-country model in which both governments are active. Section 3 considers policy dynamics and Section 4 analyzes output dynamics with the optimal trade policy. Section 5 gives concluding remarks.

2 Model

The model presented below is a variant of the three-country model. There are two countries with one firm in each of them, and these firms export their product to a third country. The outputs of the firms are denoted by x and y, and constant marginal costs of the two firms are denoted by c_1 and c_2, respectively.

Competition in the third country is modeled through a two-stage game. At the first stage, the governments hosting their firms choose subsidy rates, s_i for $i = 1, 2$, in order to maximize their welfare, taking the optimal behavior of the firms as given. At the second stage, the firms employ the quantity competition in a Cournot way and choose profit maximizing outputs, taking their governments' trade policies as given. Optimal subsides and optimal outputs are backwardly determined. In particular, we solve the profit maximization problems of the firms, given the levels of the subsidy in Section 2.1, then examine the welfare maximization problems of the governments, given the optimal behavior of the firms and determine the optimal levels of the subsidy in Section 2.2. In order to get a complete description of the dynamics of the subsidy game in the latter part of the paper, we will specify the best reply functions of the firms and those of the governments in this section.

2.1 Profit Maximization

Let the inverse demand function be unit-elastic,

$$P = \frac{1}{Q},$$

where Q is the total output, $Q = x + y$.[1] At the second stage in which the governments' subsidies are given, firm x and firm y choose outputs to maximize their profits defined by

$$\pi_1 = (P - (c_1 - s_1))x,$$

and

$$\pi_2 = (P - (c_2 - s_2))y.$$

The first-order conditions of the profit maximization are given by

$$\frac{\partial \pi_1}{\partial x} = \frac{y}{(x+y)^2} - c_x = 0,$$

and

$$\frac{\partial \pi_2}{\partial y} = \frac{x}{(x+y)^2} - c_y = 0,$$

where $c_x = c_1 - s_1$ and $c_y = c_2 - s_2$ for notational simplicity.[2] We call the production cost including the subsidy an *actual cost.* Although we will formally show later that the actual costs are non-negative, we suppose for the time being that subsidies are given as $c_x > 0$ and $c_y > 0$. From the first-order conditions, the explicit forms of the firms' best reply functions are derived as

$$\bar{r}_1(y) = \sqrt{\frac{y}{c_x}} - y \tag{1}$$

[1] See Matsumoto and Szidarovszky (2009) that studies the same model with different demand, $P = Q^{-\lambda}$ and $\lambda \neq 1$.

[2] It can be checked that the second-order conditions are satisfied for any x and y that solve the first-order conditions.

and

$$\bar{r}_2(x) = \sqrt{\frac{x}{c_y}} - x. \tag{2}$$

An intersection of the best reply functions determines a Cournot equilibrium at which we can solve for the output quantities:

$$x^C(s_1, s_2) = \frac{c_y}{(c_x + c_y)^2} \tag{3}$$

and

$$y^C(s_1, s_2) = \frac{c_x}{(c_x + c_y)^2}, \tag{4}$$

where superscript C is attached to variables associated with the Cournot point.[3] The Cournot outputs in (3) and (4) are substituted into the profit functions to obtain the Cournot profits:

$$\pi_1^C(s_1, s_2) = \left(\frac{c_y}{c_x + c_y}\right)^2 \tag{5}$$

and

$$\pi_2^C(s_1, s_2) = \left(\frac{c_x}{c_x + c_y}\right)^2. \tag{6}$$

Dividing (3) by (4) and (5) by (6) yields, after arranging terms, the ratios of outputs and profits,

$$\frac{x^C}{y^C} = \frac{c_y}{c_x} \gtreqless 1 \text{ according to } c_y \gtreqless c_x,$$

and

$$\frac{\pi_x^C}{\pi_y^C} = \left(\frac{x^C}{y^C}\right)^2 \gtreqless 1 \text{ according to } x^C \gtreqless y^C.$$

These inequalities imply the following results on the optimal behavior of the firms: The firm with the lower actual cost produces more output and earns more profit than the firm with the higher actual cost.

2.2 Welfare Maximization

At the first stage of the sequential game, the governments determine the optimal levels of the subsidy so as to maximize the national welfare defined by

$$W_1(s_1, s_2) = \pi_1^C(s_1, s_2) - s_1 x^C(s_1, s_2), \tag{7}$$

and

$$W_2(s_2, s_1) = \pi_2^C(s_2, s_1) - s_2 y^C(s_2, s_1). \tag{8}$$

[3]Since $\bar{r}_1(y)$ and $\bar{r}_2(x)$ take mound-shaped curves starting at the origin, the curves intersects twice at $(0,0)$ and (x^C, y^C). The former is the trivial equilibrium point and the latter is the non-trivial equilibrium point. Our concern is on the non-trivial point and thus no further considerations are given to the trivial equilibrium point.

We derive the specific forms of the best reply functions of the governments and consider their characteristics in the policy space. Substituting x^C, $y^C, Q^C = x^C + y^C$ and $P^C = (Q^C)^{-1}$ into (7) and (8) yields the explicit forms of the welfare functions of the governments,

$$W_1(s_1, s_2) = \frac{(c_y - s_1)\, c_y}{(c_x + c_y)^2}, \tag{9}$$

and

$$W_2(s_2, s_1) = \frac{(c_x - s_2)\, c_x}{(c_x + c_y)^2}. \tag{10}$$

The government of country 1 maximizes $W_1(s_1, s_2)$ with respect to s_1 and the government of country 2 maximizes $W_2(s_2, s_1)$ with respect to s_2. We can solve the first-order conditions to obtain the best reply functions:

$$r_1(s_2) = -s_2 + (c_2 - c_1) \text{ and } r_2(s_1) = -s_1 + (c_1 - c_2), \tag{11}$$

where the second-order conditions are satisfied. These functions are essentially the same as those derived by Bandyopadhyay (1997). It is apparent that there is a continuum of optimal subsidies $s_1 + s_2 = 0$ for symmetric firms ($c_1 = c_2$) and no equilibrium exists for asymmetric firms ($c_1 \neq c_2$).

To avoid the indeterminacy of the optimal policy equilibrium in the case of asymmetric firms, we impose the following external upper and lower bound constraints on the levels of the optimal policy, s_i, taking account of the fact that the governments behave with control. The first constraint reflects the fact that the governments have the upper bound of the subsidy, due to their budget constraints. The second constraint takes account of the fact that the government does not levy such a strong export tax that might result in its firm to exit the market. Intuitively speaking, in choosing their policies, the governments neither take care of all the production costs nor take all of the profits.

Assumption 1. $s_i^L \leq s_i \leq s_i^U$ for $i = 1, 2$ where s_i^U is the upper bound of the subsidy level defined by $s_i^U = c_i$ and $s_i^L < 0$ is the lower bound of the subsidy level, which shows the upper bound of the export tax.

Under Assumption 1, the best reply function of the government of country 1 becomes piecewise linear with three segments:

$$\begin{cases} s_1^U & s_2 < s_2^u, \\ r_1(s_2) & s_2^u \leq s_2 \leq s_2^\ell, \\ s_1^L & s_2 > s_2^\ell, \end{cases}$$

where s_2^u and s_2^ℓ are defined by $r_1(s_2^u) = s_1^U$ and $r_1(s_2^\ell) = s_1^L$, respectively. In the same way, the best reply function of the government of country 2 is derived to be piecewise-linear with three segments:

$$\begin{cases} s_2^U & s_1 < s_1^u, \\ r_2(s_1) & s_1^u \leq s_1 \leq s_1^\ell, \\ s_2^L & s_1 > s_1^\ell, \end{cases}$$

where s_1^u and $s_1^\ell > 0$ are defined by $r_2(s_1^u) = s_2^U$ and $r_2(s_1^\ell) = s_2^L$, respectively.

An intersection of these modified best reply functions is a Nash equilibrium of the trade policy, (s_1^e, s_2^e). First of all, we should discuss the determination of the optimal trade policy in the case where $c_i < c_j$ and $s_j^L \leq s_j^u$. By definition of the piecewise linear best replies, it is clear that the optimal subsidy policy of firm i is $s_i^e = s_i^U$. From (3), (4) and Assumption 1 (i.e., $s_i^U = c_i$), the optimal output of firm 2 is zero if $c_1 < c_2$ and $s_2^L \leq s_2^u$ whereas the optimal output of firm 1 is zero if $c_2 < c_1$ and $s_1^L \leq s_1^u$. In both cases one of the firms will export nothing to the third country and the competition in the third country will be terminated. To confine our attention to the third-country model with active competition, we assume that s_j^u and s_j^L are given such that the following inequality holds.

Assumption 2. $s_j^L > s_j^u$ for $j = 1, 2$.

Notice that $s_j^u = c_j - 2c_i$ and $s_j^\ell = c_j - c_i - s_i^L$, so clearly $s_j^u < s_j^\ell$ for both firms. Assumption 2 requires that for $j = 1, 2$, $s_j^L > s_j^u$. In order to guarantee the existence of negative s_j^L bounds, we make the additional assumption:

Assumption 3. $c_j < 2c_i$ for $j = 1, 2$ and $i \neq j$.

Without losing generality, we can assume that in the case of asymmetric firms, $c_i < c_j$. First we show that $s_i^\ell < s_i^U$. Since $s_j^L > c_j - 2c_i > c_j - 2c_j = -c_j$, we have

$$s_i^\ell = c_i - c_j - s_j^L < c_i - c_j - (-c_j) = c_i = s_i^U.$$

However no such comparison can be made between s_i^L and s_i^ℓ. In order to guarantee that $s_i^L < s_i^\ell$, we make the following assumption:

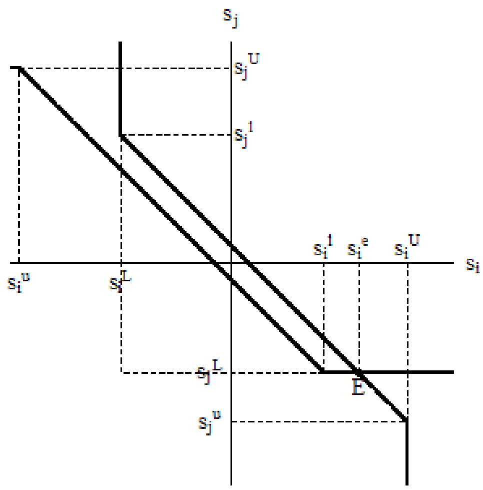
Assumption 4. $s_i^L + s_j^L < c_i - c_j$.

Notice that this assumption holds if the marginal costs c_i and c_j are close to each other. The best reply functions are shown in Figure 1, and from it we can conclude that the unique equilibrium is

$$s_i^e = c_j - c_i - s_j^L \text{ and } s_j^e = s_j^L \tag{12}$$

Hence we have the following:

Theorem 1 *Under Assumptions 1,2,3 and 4, if the firms are symmetric (i.e., $c_1 = c_2$), then there are infinitely many equilibria on the line $s_i = -s_j$ whereas if the firms are asymmetric with $c_i < c_j$, then $s_i^e = -s_j^L + (c_j - c_i) > 0$ and $s_j^e = s_j^L < 0$.*



Determination of the optimal policy under $c_i < c_j$

3 Dynamics Analysis of Trade Policy

Theorem 1 guarantees the existence of a unique optimal trade policy. In this section, we consider the dynamic behavior of the trade policy in the policy space (s_1, s_2). To this end, we assume the following discrete time dynamic process of the policy selection:

$$\begin{cases} s_1' = (1-\alpha_1)s_1 + \alpha_1 R_1(s_2), \\ \\ s_2' = (1-\alpha_2)s_2 + \alpha_2 R_2(s_1), \end{cases} \tag{13}$$

where $'$ denotes the unit-time advancement operator, α_i is the adjustment coefficient with $0 < \alpha_i \leq 1$ and $R_1(s_2)$ and $R_2(s_1)$ are the best reply functions restricted to their intervals, $[s_2^L, s_2^U]$ and $[s_1^L, s_1^U]$,

$$R_1(s_2) = \begin{cases} s_1^L \text{ for } s_2^\ell \leq s_2 \leq s_2^U, \\ \\ r_1(s_2) \text{ for } s_2^L \leq s_2 < s_2^\ell, \end{cases}$$

and

$$R_2(s_1) = \begin{cases} s_2^L \text{ for } s_1^\ell \leq s_1 \leq s_1^U, \\ \\ r_2(s_1) \text{ for } s_1^L \leq s_1 < s_1^\ell, \end{cases}$$

3.1 Symmetric firms: $c_1 = c_2$

In the case of identical costs, we first perform some numerical simulations to examine the dynamic behavior of the governments, second, confirm analytically that the numerical results are robust, and finally summarize these results in

Theorems 2 and 3. The numerical simulations are presented in Figure 2 where the policy is naively adjusted (i.e., $\alpha_i = 1$). The other case is given in Figure 3 where the policy is adaptively adjusted (i.e., $\alpha_i < 1$).

The feasible policy space is defined by the rectangle $S = [s_2^L, s_2^U] \times [s_1^L, s_1^U]$, which is divided into distinctive four parts by the horizontal and vertical lines, $s_1 = s_1^\ell$ and $s_2 = s_2^\ell$,

$$S_I = \{(s_1, s_2) \in S \mid s_1^\ell \leq s_1 \text{ and } s_2^\ell \leq s_2\},$$

$$S_{II} = \{(s_1, s_2) \in S \mid s_1 < s_1^\ell \text{ and } s_2^\ell \leq s_2\},$$

$$S_{III} = \{(s_1, s_2) \in S \mid s_1^\ell \leq s_1 \text{ and } s_2 < s_2^\ell\},$$

$$S_{IV} = \{(s_1, s_2) \in S \mid s_1 < s_1^\ell \text{ and } s_2 < s_2^\ell\}.$$

In Figures 2(A) and 3(A), we select three initial points denoted as I_1, I_2 and I_3 in S_I, S_{II} and S_{III}, respectively and depict the three trajectories starting from these points. In Figures 2(B) and 3(B), which are enlargements of the region S_{IV}, we also select three initial points denoted as i_1, i_2 and i_3 inside S_{IV} and depict three trajectories starting from these points. Simulations in Figure 2(A) indicate that the trajectories converge to period-2 cycles when the trade policies are naively adjusted. The two periodic points are symmetric with respect to the line $s_1 + s_2 = 0$. On the other hand, the simulations shown in Figure 3(A) indicate that the trajectories converge to stationary points on the line $s_1 + s_2 = 0$ when the trade policies are adaptively adjusted.

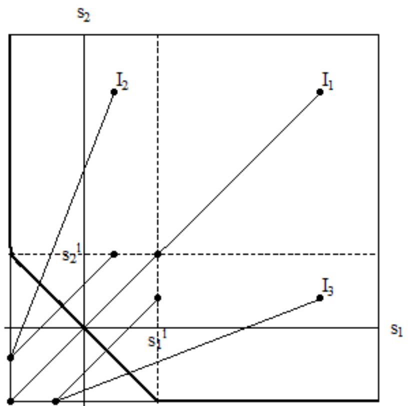
The first result on policy dynamics is summarized as follows:

Theorem 2 *If the firms are symmetric, then the naively adjusted process of the export trade policy (i.e., (13) with $\alpha_1 = \alpha_2 = 1$) gives rise to infinitely many stable period-2 cycles, and a trajectory starting from a point other than a stationary point converges to one of these cycles.*

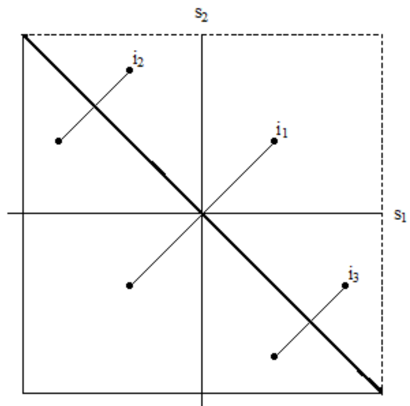
Proof. We prove this statement with four steps. (I): It can be seen that $R_1(s_2) = s_1^L$ and $R_2(s_1) = s_2^L$ for all $(s_1, s_2) \in S_I$. By the identical cost assumption, $R_1(s_2^L) = s_1^\ell$ and $R_2(s_1^L) = s_2^\ell$ whereas $R_1(s_2^\ell) = s_1^L$ and $R_2(s_1^\ell) = s_2^L$ by the definitions of s_2^ℓ and s_1^ℓ. Thus any trajectory starting at a point inside S_1 converges to the period-2 cycle with periodic points, (s_1^L, s_2^L) and (s_1^ℓ, s_2^ℓ). The trajectory with the initial point I_1 in Figure 3(A) is an example of this case. (II): Next take an initial point $(s_1^{II}, s_2^{II}) \in S_{II}$. Then the naive adjustment process conveys the point to $R_1(s_2^{II}) = s_1^L$ and $R_2(s_1^{II}) = -s_2^{II}$ and then $R_1(-s_2^{II}) = s_1^{II}$ and $R_2(s_1^L) = s_2^\ell$, that are bounced back to the point $(s_1^L, -s_2^{II})$. Thus any trajectory starting at a point $(s_1^{II}, s_2^{II}) \in S_{II}$ converges to the period-2 cycle with periodic points $(s_1^L, -s_2^{II})$ and (s_1^{II}, s_2^ℓ). The trajectory with the initial point I_2 is an example of this case. (III): In the same way, we can show that a trajectory starting at a point $(s_1^{III}, s_2^{III}) \in S_{III}$ converges to the period-2 cycle with periodic points, $(-s_2^{III}, s_2^L)$ and (s_1^ℓ, s_2^{III}). The trajectory with the initial point I_3 is an example of this case. (IV): We finally consider periodic behavior in Figure 3(B) where the initial points are selected inside of S_{IV}. Since the adjustment process in S_{IV} is given by

$$s_1' = -s_2 \text{ and } s_2' = s_1,$$

the governments expect the symmetric point $(-s_1^{IV}, -s_2^{IV})$ with respect to the $s_1+s_2=0$ locus for any initial point (s_1^{IV}, s_2^{IV}). Taking $(-s_1^{IV}, -s_2^{IV})$ as given, the government expect (s_1^{IV}, s_2^{IV}) in the next time period. Thus any point $(s_1^{IV}, s_2^{IV}) \in S_{IV}$ and its symmetric point $(-s_1^{IV}, -s_2^{IV}) \in S_{IV}$ are period-2 points. Three period-2 cycles depicted in Figure 2(B) are examples of this case. ■



(A) Convergence to S_{IV}



(B) Period-2 cycles in S_{IV}

Figure 2. Coexistence of period-2 cycles under naive expectation

We say that the governments take the *pure policy* if $s_1 s_2 > 0$, the *mixed policy* if $s_1 s_2 < 0$, and the *one-side free trade policy* if $s_1 s_2 = 0$ and $s_i \neq 0$ for $i = 1, 2$. Furthermore, we say that the governments take a pure subsidy policy if $s_1 > 0$ and $s_2 > 0$ and a pure tax policy if $s_1 < 0$ and $s_2 < 0$. In the mixed policy, one government pays subsidy and the other government charges tax. Figures 2(A) and 2(B) are divided into four rectangles by the horizontal and vertical axes. The rectangles on the top-right and the bottom-left represent the set of points which generate pure policy (i.e., the pure subsidy policy and the pure tax policy). On the other hand the rectangles on the top-left and on the bottom-right represent the set of points which generate mixed policy. If a point is on the either axis, it represents the one-side free trade in which one government gives no subsidy and charges no tax and the other government either gives subsidy or charges tax.

Let us denote the period-2 points of the trade policy by $S^A = (s_1^A, s_2^A)$ and $S^B = (s_1^B, s_2^B)$. The period-2 cycle means that if the governments expect S^A, then S^B is realized; and if the governments expect S^B in the next period, then S^A is realized. Let us take the case of the pure trade policy. If both governments expect that their competitors take the same policy, for example export subsidies $s_1^A > 0$ and $s_2^A > 0$, then in the next period the process becomes the opposite pure policy, that is, export taxes $s_1^B < 0$ and $s_2^B < 0$. So the governments alternate between the pure subsidy policy and the pure tax policy. In contrast to this, if both governments expect that their competitors take the mixed policy, for example, government 1 adopts the subsidy policy $s_1^A > 0$ while government 2 takes the tax policy $s_2^A < 0$, then, in the next time period the process becomes the subsidy with $s_1^B = -s_2^A$ for government 1 and the tax with $s_2^B = -s_1^A$ for government 2. The governments therefore alternate between the mixed policies in which the values and the signs of the subsidy and the tax are interchanged. We summarize these results as a corollary of Theorem 2:

Corollary 1 *The trade policy has an initial point dependency: (1) If both governments expect that their competitors adopt the subsidy policy, then the tax policy is realized, and vice versa. (2) If a mixed policy is expected, then the opposite mixed policy is realized where the realized point of the subsidy or the tax is mirror image of the expected point with respect to the -45° degree line.*

The optimal outputs associated to these periodic points are obtained by substituting the periodic points into (3) and (4): for $k = A, B$,

$$x^{Ck} = \frac{c_2 - s_2^k}{((c_1 - s_1^k) + (c_2 - s_2^k))^2} \text{ and } y^{Ck} = \frac{c_1 - s_1^k}{((c_1 - s_1^k) + (c_2 - s_2^k))^2}. \quad (14)$$

We now assume that the policy is adaptively adjusted (i.e., $\alpha_i < 1$) and the adjustment coefficients are the same (i.e., $\alpha_1 = \alpha_2 = \alpha$) for the sake of analytical simplicity. As can be seen in Figure 3(A), any trajectory with an initial point inside $S_I \cup S_{II} \cup S_{III}$ sooner or latter enters S_{IV}. It is therefore sufficient for our purpose to consider the dynamics observed within S_{IV}. Our second result on policy dynamics is summarized as follows:

Theorem 3 *If the firms are symmetric, then the symmetric adaptive adjustment process of the export trade policy (i.e., (13) with $\alpha_1 = \alpha_2 < 1$) is stable and converges to a point on the line $s_1 + s_2 = 0$.*

Proof. Let us start with an initial point $(s_1^0, s_2^0) \in S_{IV}$. The optimal policy at the next period is determined by the adaptively adjusted process,

$$s_1' = (1 - \alpha)s_1^0 + \alpha(-s_2^0),$$

$$s_2' = (1 - \alpha)s_2^0 + \alpha(-s_1^0).$$

The line passing through these two points, (s_1^0, s_2^0) and (s_1', s_2'), is written as $s_2 = as_1 + b$ where the slope a and the vertical intercept b are

$$a = \frac{s_2' - s_2^0}{s_1' - s_1^0} = 1 \text{ and } b = s_2^0 - s_1^0.$$

It is clear that the adaptive process maps the optimal policy (s_1', s_2') to a point on the $s_2 = as_1 + b$ locus. That is, the trajectories of the optimal policies are controlled by

$$s_1' = (1-\alpha)s_1 + \alpha(-s_1 - b),$$

$$s_2' = (1-\alpha)s_2 + \alpha(-s_2 + b).$$

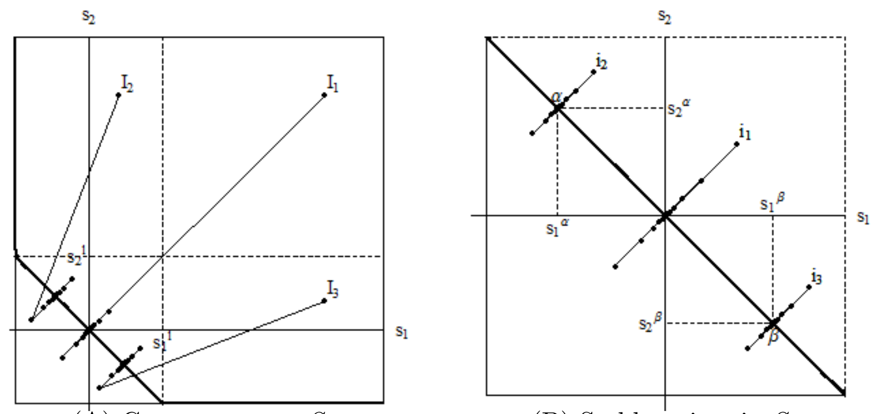
The adjusted processes are independent and governed by 1D difference equations with slopes less than unity in absolute value,

$$\left|\frac{ds_i'}{ds_i}\right| = |1 - 2\alpha| < 1.$$

Hence the adjustment process is stable and a trajectory converges oscillating to the stationary state associated with the initial point (s_1^0, s_2^0),

$$s_1^e = -\frac{s_2^0 - s_1^0}{2} \text{ and } s_2^e = \frac{s_2^0 - s_1^0}{2}$$

which is the intersection of the $s_2 = -s_1$ curve and the $s_2 = as_1 + b$ curve. It is clear that $s_1^e \gtreqless 0$ and $s_2^e \lesseqgtr 0$ if $s_1^0 \gtreqless 0$ and $s_2^0 \lesseqgtr 0$. ■



(A) Convergence to S_{IV} (B) Stable points in S_{IV}

Figure 3. Coexistence of stable stationary points under adaptive expectation

The stationary point (s_1^e, s_2^e) is on the line $s_1 + s_2 = 0$. Figure 3(B) shows that the trajectory starting from point i_1 converges to the origin, $s_1^e = s_2^e = 0$, the trajectory starting from point i_2 converges to point α with $s_1^\alpha < 0$, $s_2^\alpha > 0$ and $s_1^\alpha = -s_2^\alpha$, and the trajectory starting from i_3 converges to point β with $s_1^\beta > 0$, $s_2^\beta < 0$ and $s_1^\beta = -s_2^\beta$. This observation leads us to Corollary 2.

Corollary 2 *The optimal trade policy has an initial point dependency: if both governments start with the same initial expectations, then the free trade is* *materialized (i.e., $s_1^e = s_2^e = 0$); If they start with different initial expectations (i.e., $s_1^0 \neq s_2^0$), then the symmetric mixed trade policy is materialized:*

$$s_1^e = -\frac{s_2^0 - s_1^0}{2} \text{ and } s_2^e = -s_1^e.$$

The corresponding stationary output values are obtained by substituting s_1^e and s_2^e into (3) and (4),

$$x^C == \frac{c_2 - s_2^e}{((c_1 - s_1^e) + (c_2 - s_2^e))^2} \text{ and } y^C = \frac{c_1 - s_1^e}{((c_1 - s_1^e) + (c_2 - s_2^e))^2}. \tag{15}$$

3.2 Asymmetric firms: $c_1 \neq c_2$

It is clear from Figure 1 that the optimal policy is stable in the policy space if the firms are asymmetric, regardless of whether the policy is naively or adaptively adjusted. It is also clear that the optimal trade policy is mixed, which is summarized as follows:

Theorem 4 *If the firms are asymmetric, then the firm with lower production cost receives an export subsidy and the firm with higher production cost pays an export tax.*

The optimal output values are determined by substituting the optimal subsidies (12) into relations (3) and (4).

4 Dynamic Analysis of Output

In considering the output dynamics in the international subsidy game, we lag the output variables in (1) and (2) and construct the output adjustment process with adaptive expectations:

$$\begin{cases} x' = (1-\beta_1)x + \beta_1 \left(\sqrt{\dfrac{y}{c_1 - s_1}} - y \right), \\ \\ y' = (1-\beta_2)y + \beta_2 \left(\sqrt{\dfrac{x}{c_2 - s_2}} - x \right), \end{cases} \tag{16}$$

where β_i is the adjustment coefficient satisfying $0 < \beta_i \leq 1$, and s_1 and s_2 are governed by (13). As mentioned in the Introduction, the dynamic structure of system (16) resembles that of the nonlinear Cournot models extensively studied by Puu (2003) and Bischi *et al.* (2009) in which different subjects such as the emergence of complex dynamics involving chaos, multistability, the structure of the basin of attraction, delay dynamics, etc., are discussed. We skip the detailed examinations of system (16) and will apply these results to our dynamic analysis.

4.1 Symmetric firms: $c_1 = c_2$

As a benchmark, we take $\beta_1 = \beta_2 = 1$ and consider the output dynamics under naive expectation. Let us denote the actual cost ratio by

$$k = \frac{c_1 - s_1}{c_2 - s_2}.$$

It is already shown in Puu (2003) that loss of stability occurs when the actual cost ratio satisfies the following equation,[4]

$$\frac{(k-1)^2}{4k} = 1 \tag{17}$$

where the smaller solution is $3 - 2\sqrt{2}(\simeq 0.172)$ and the larger solution is $3 + 2\sqrt{2}(\simeq 5.828)$. It follows that if the actual cost ratio stays within the interval bounded by the smaller and larger solutions, then the Cournot point is stable. In the same way if the actual cost ratio falls outside the interval, it becomes locally unstable. We assume the stability of the Cournot point and examine the effects caused by the policy on output dynamics for a while. It is also shown in Puu (2003) that the dynamics is symmetric with respect to $k = 1$. In order to get new results, we confine our consideration to the case of $c_1 < c_2$ and assume that $c_1 = 1$ throughout the analysis for the sake of analytical convenience. It can be checked that the nonnegativity of the output trajectories is guaranteed when the actual cost ratio is at least $4/25(= 0.16)$.

If there is no policy lag in the sense that the firms receive the subsidies from the governments without any time delays, and the policy is naively adjusted, then the dynamics of the outputs and the subsidies are controlled by the dynamic equations

$$\begin{cases} x' = \sqrt{\dfrac{y}{c_1 - s_1}} - y, \\[2ex] y' = \sqrt{\dfrac{x}{c_2 - s_2}} - x, \\[2ex] s_1' = -s_2, \\[1ex] s_2' = -s_1, \end{cases} \tag{18}$$

where the domain of the policy dynamics is restricted to the rectangle, $[s_1^L, s_1^\ell] \times [s_2^L, s_2^\ell]$ for analytical simplicity.[5] According to Theorem 2, the optimal trade policy oscillates between two values, $S^A = (s_1^A, s_2^A)$ and $S^B = (s_1^B, s_2^B)$ where $s_1^A = -s_2^B$ and $s_2^A = -s_1^B$. We have already solved for the output quantities at

[4] This equation is obtained by setting the product of the derivatives of the best reply functions, $\bar{r}_1(y)$ and $\bar{r}_2(x)$, evaluated at the Cournot point equal to -1.

[5] As can be seen in Figure 2(A), any trajectory starting at a point outside S_{IV} will enter S_{IV} after several iterations.

each periodic point given in (14). Indeed, the Cournot outputs at point S^A are

$$x^{CA} = \frac{c_2 - s_2^A}{((c_1 - s_1^A) + (c_2 - s_2^A))^2} \text{ and } y^{CA} = \frac{c_1 - s_1^A}{((c_1 - s_1^A) + (c_2 - s_2^A))^2}$$

and the Cournot outputs at point S^B are

$$x^{CB} = \frac{c_2 - s_2^B}{((c_1 - s_1^B) + (c_2 - s_2^B))^2} \text{ and } y^{CB} = \frac{c_1 - s_1^B}{((c_1 - s_1^B) + (c_2 - s_2^B))^2}.$$

Since the output dynamics depends on the policy dynamics but not vice versa, we can be fairly certain that the output dynamics gives rise to a periodic cycle when the trade policy has a period-2 cycle. We specify the parameter values as $c_1 = c_2 = 1$, $S^A = (0.64, -0.4)$ and $S^B = (0.4, -0.64)$ and perform simulations. First of all, it can be pointed out that the output equilibria, $C^A = (x^{CA}, y^{CA})$ and $C^B = (x^{CB}, y^{CB})$, are locally asymptotically stable under these specifications if the trade policy is fixed since the cost ratios are greater than $3 - 2\sqrt{2}$,

$$k^A = \frac{c_1 - s_1^A}{c_2 - s_2^A} \simeq 0.366 \text{ and } k^B = \frac{c_1 - s_1^B}{c_2 - s_2^B} \simeq 0.257.$$

However, the trade policy is not fixed, it is switched from one periodic point to the other in every period. Figure 4 reveals that the output dynamics is represented by a period-2 cycle, which is synchronized with the period-2 cycle of the optimal subsidy. Figure 4(A) shows a return map. The best reply functions of firm 2 are illustrated as mound-shaped curves and the inner curve is shifted to the outer curve when the policy is switched from S^A to S^B. In the same way, the best reply functions of firm 1 are illustrated as two upward sloping curves and the shift from the left curve to the right is caused by the policy switching from S^A to S^B. Here C^a and C^b are the two periodic points of the output cycle. Figure 4(B) depicts the time trajectory of output y. Contrary to our intuition, the periodic points of the output cycle are not the Cournot points denoted by C^A and C^B in Figure 4(A). The reason is that dynamic equations of the outputs are switched from $(x', y') = (\bar{r}_1^A(y), \bar{r}_2^A(x))$ with S^A to $(x', y') = (\bar{r}_1^B(y), \bar{r}_2^B(x))$ with S^B at every iteration step where

$$\bar{r}_1^A(y) = \sqrt{\frac{y}{c_1 - s_1^A}} - y \text{ and } \bar{r}_2^A(x) = \sqrt{\frac{x}{c_2 - s_2^A}} - x$$

and

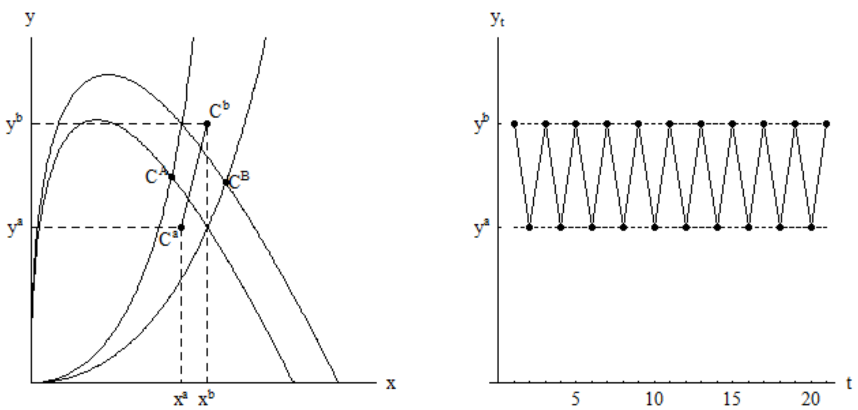
$$\bar{r}_1^B(y) = \sqrt{\frac{y}{c_1 - s_1^B}} - y \text{ and } \bar{r}_2^B(x) = \sqrt{\frac{x}{c_2 - s_2^B}} - x.$$

The periodic points $C^a = (x^a, y^a)$ and $C^b = (x^b, y^b)$ are the fixed points of the composite functions of $\bar{r}_i^A$ and $\bar{r}_i^B$ for $i = 1, 2$,

$$x^a = \bar{r}_1^A(\bar{r}_2^B(x^a)),\ y^a = \bar{r}_2^A(\bar{r}_1^B(y^a)),\ x^b = \bar{r}_1^B(\bar{r}_2^A(x^b)) \text{ and } y^b = \bar{r}_2^B(\bar{r}_1^A(y^b)).$$

We summarize these results as follows:

Theorem 5 *If the firms are symmetric, the trade policies and the outputs are naively adjusted and there is no policy lag, then the 4D dynamic system (18) gives rise to a period-2 cycle of the outputs which is synchronized with the period-2 cycle of the trade policies.*

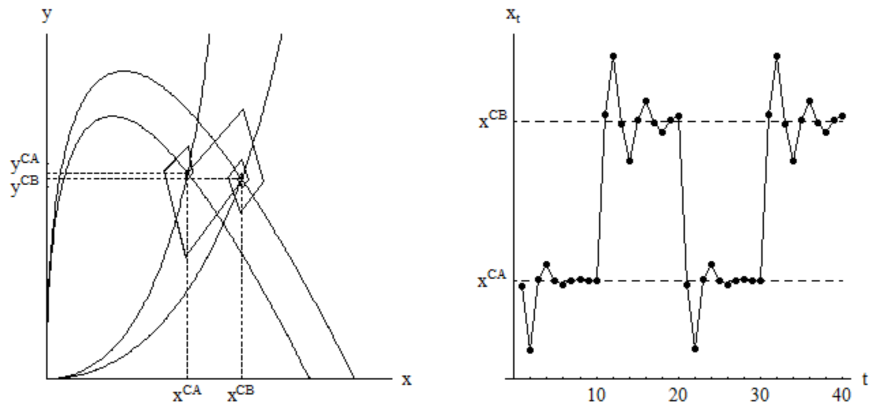


(A) Return map (B) Time trajectory

Figure 4. Birth of a period-2 cycle

We has assumed so far that there is no time lag in implementing the trade policy. However, the government policy usually works with a time lag since there is an inevitable delay between the subsidy decision and the actual payment which is the result of the political process. To examine the effect caused by the policy delay, a 10-period lag is introduced such that the trade policy is switched at every 10 periods. Simulation results are illustrated in Figure 5 where we use the same parameter specifications with the only difference that the length of lag is changed to 10 from zero. In Figures 5(A) and 5(B), the output dynamics shows the cyclic behavior in the following way: it fluctuates around x^{CA} for 10 periods and then around x^{CB} for the next 10 periods, after which it jumps back to the original cyclic behavior. We have already seen that x^{CA} and x^{CB} are locally stable if the trade policy is fixed. The cyclic behavior around each stationary point is a dumping oscillation. When the policy is changed in the middle of the converging process, the trajectory changes its direction and starts approaching a new equilibrium. As a result, a new dumping oscillation is born, which is again interrupted before arriving at the equilibrium by a change of the policy. A n-period time lag of the trade policy creates a period-$2n$ cycle. It fluctuates around one stationary point for n periods and then jumps to a neighborhood of the other stationary point when the policy is changed. Then it fluctuates around the new stationary state for the next n periods and jumps back to the previous neighborhood when the policy is changed again. This recursive process repeats itself. Dynamics with time lag can be summarized as follows:

Theorem 6 *If the firms are symmetric, the policies and the outputs are naively* *adjusted and there exists a n-period policy lag, then the 4D dynamic system gives rise to a period-2 cycle of the trade policy and a period-2n cycle of the output.*



(A) Return map (B) Time trajectory

Figure 5. Birth of a period-20 cycle

If the policy is naively adjusted and the output is adaptively adjusted, then essentially the same dynamics will be observed. In other words, as far as the policy is periodically changed, the output dynamics is affected by these policy switching and exhibits periodic behavior no matter which formation of the expectations is selected.

If the policy is adaptively adjusted, then the policy adjustment process converges to the stationary point according to Theorem 2. We have, however, qualitatively different output dynamics, as will be seen shortly. In this case we can suppose without loss of generality that the dynamic process of the trade policy is rapid and the firms receive the stationary values of the trade policy from the beginning of the output dynamic process. This assumption reduces the 4D dynamic system to the 2D output dynamic system (16) with $s_1 = s_1^e$, $s_2 = s_2^e$ and $s_1^e = -s_2^e$:

$$\begin{cases} x' = \sqrt{\dfrac{y}{c + s_2^e}} - y, \\ \\ y' = \sqrt{\dfrac{x}{c - s_2^e}} - x, \end{cases} \tag{19}$$

where $c = c_1 = c_2$.[6] The stability of system (19) depends on the actual cost ratio,

$$k = \frac{c + s_2^e}{c - s_2^e} \tag{20}$$

[6] It is possible to construct the dynamic system in terms of s_1^e. However, the results become qualitatively the same.

Given c, k increases from zero to infinity as s_2^e increases from $-c$ to c. In the linear model, local instability implies global instability. However this is not necessarily the case with nonlinear models because the nonlinearities may prevent unstable trajectories from globally diverging. We restrict our analysis to the unstable case, henceforth, to examine what dynamics the nonlinear output system can generate. The stability of the output is violated if $k \leq 3-2\sqrt{2}$ and the output trajectories are non-negative as far as $k \geq 4/25$. Therefore, the output stationary state becomes locally unstable but the trajectories are nonnegative if

$$-\frac{21}{29}c < s_2^e < -\frac{\sqrt{2}}{2}c. \tag{21}$$

Here $k = 3-2\sqrt{2}$ if s_2^e is equal to this upper bound which is the threshold of the loss of stability and will be called the *instability value* and $k = 4/25$ if s_2^e is equal to this lower bound, which is the threshold of loss of nonnegativity and will be called the *nonnegativity value.* When s_2^e decreases from the instability value to the nonnegativity value in the interval, the stationary state is destabilized and goes to chaos through a period-doubling cascade as shown in Figure 6. We summarize this result as follows:

Lemma 1. *If the firms are symmetric and the trade policy is adaptively adjusted, then the naively adjusted output exhibits various dynamics ranging from a period-2 cycle to chaotic fluctuations, depending on the value of* s_2^e *in the interval bounded by the instability value and the nonnegativity value.*

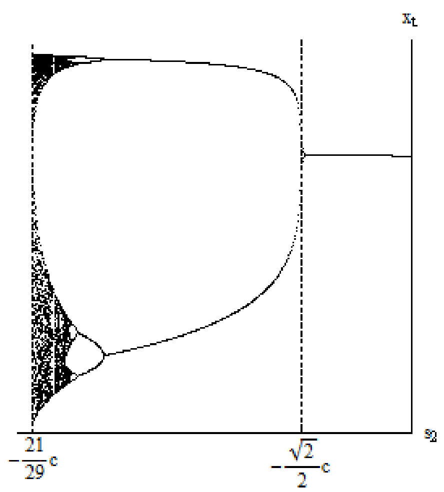


Figure 6. One-parameter bifurcation diagram with respect to s_2^e.

Now we replace the naive expectation formation with the adaptive expectation formation by taking $\beta_1 < 1$ and $\beta_2 < 1$. The output dynamic system is

$$\begin{cases} x' = (1-\beta_1)x + \beta_1 \left(\sqrt{\dfrac{y}{c+s_2^e}} - y \right), \\ \\ y' = (1-\beta_2)y + \beta_2 \left(\sqrt{\dfrac{x}{c-s_2^e}} - x \right). \end{cases} \tag{22}$$

It is also shown in Puu (2003) that loss of stability under the adaptive expectation occurs when the cost ratio and the speeds of adjustment satisfy the following relation:

$$\frac{(k-1)^2}{4k} = \frac{1}{\beta_1} + \frac{1}{\beta_2} - 1 \tag{23}$$

under which the determinant of the Jacobian matrix of the dynamic system (22) becomes unity. To examine what dynamics can be generated, we simulate system (22) when the adjustment speeds are identical ($\beta_1 = \beta_2 = \beta$). Taking the same parameter specifications as in Figure 6, Figure 7 shows the two-parameter bifurcation diagram in which the horizontal coordinate is the level of the optimal subsidy to firm 2 (i.e., s_2^e) and the vertical one is the identical adjustment speed (i.e., β). Solving (23) for β and substituting (20) yields the partition curve of the parameter space (s_2^e, β),

$$\beta = \frac{2(c-s_2^e)(c+s_2^e)}{c^2}. \tag{24}$$

Given c, for all (s_1^e, β) under the partition curve, the stability condition is satisfied and this stable region is colored in red in Figure 7. For all (s_1^e, β) above the curve, the stability condition is violated but the nonlinearities of the dynamics system prevent diverging trajectories. The different colors of the regions correspond to the different periods of periodic cycles up to period 16. The gray regions indicate that the period of the cycle is larger than 16 or chaos emerges. This is summarize as:

Lemma 2. *If the firms are symmetric and the trade policy is adaptively and rapidly adjusted, then the adaptively adjusted output exhibits various dynamics ranging from a period-2 cycle to chaotic fluctuations, depending on the values of* (s_2^e, β).

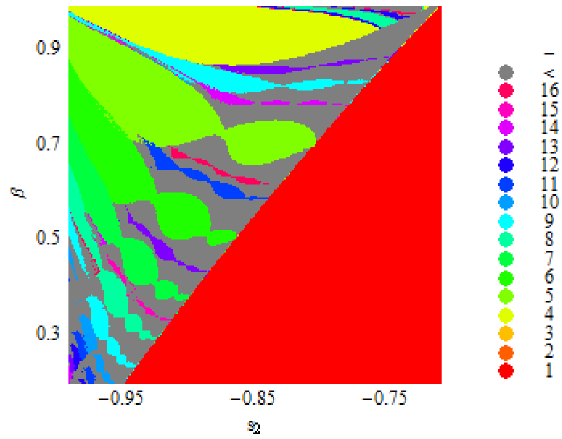


Figure 7. Two-parameter bifurcation diagaram

4.2 Asymmetric firms: $c_1 \neq c_2$

We now turn our attention to the asymmetric firms with $c_1 < c_2$. We have already seen that the dynamic process of the trade policy is stable and converges to the optimal points given in (12),

$$s_1^e = -s_2^L + (c_2 - c_1) \text{ and } s_2^e = s_2^L,$$

where $0 > s_2^L > c_2 - 2c_1$ due to Assumption 2. The corresponding optimal outputs are obtained by substituting these optimal subsidies into the expressions in (15),

$$x^C = \frac{c_2 - s_2^L}{4c_1^2} \text{ and } y^C = \frac{(2c_1 - c_2) + s_2^L}{4c_1^2}.$$

If we assume that the implementation of the trade policy has no time lags, then the output dynamic system with $\beta_1 = \beta_2 = 1$ is

$$\begin{cases} x' = \sqrt{\dfrac{y}{c_1 - (-s_2^L + c_2 - c_1)}} - y, \\ \\ y' = \sqrt{\dfrac{x}{c_2 - s_2^L}} - x. \end{cases} \tag{25}$$

Substituting s_1^e and s_2^e into the actual cost ratio yields

$$k = \frac{2c_1 - c_2 + s_2^L}{c_2 - s_2^L}. \tag{26}$$

It follows that, given c_1 and c_2, the actual cost ratio decreases to zero from $(2c_1 - c_2)/c_2 > 0$ if s_2^L decreases to its lower bound $s_2^u = -(2c_1 - c_2)$ from zero. Depending on the value of s_2^L, the output dynamics can be destabilized. Furthermore, due to the nonlinearities of (16), the output dynamics can exhibit a rich dynamics if s_2^L is in the interval

$$c_2 - \frac{50}{29}c_1 < s_2^L < c_2 - \frac{1}{2-\sqrt{2}}c_1. \tag{27}$$

The upper bound value and the lower bound value of s_2^L make the actual cost ratio equal to $3 - 2\sqrt{2}$ and $4/25$, respectively.

When the symmetric adaptive expectation formation is adopted (i.e., $\beta_1 = \beta_2 = \beta < 1$), the output dynamic system is

$$\begin{cases} x' = (1-\beta)x + \beta\left(\sqrt{\dfrac{y}{c_1 - (-s_2^L + c_2 - c_1)}} - y\right), \\ \\ y' = (1-\beta)y + \beta\left(\sqrt{\dfrac{x}{c_2 - s_2^L}} - x\right). \end{cases} \tag{28}$$

Solving (23) for β and substituting (26) yield the partition curve

$$\beta = \frac{2(c_2 - s_2^L)(2c_1 - c_2 + s_2^L)}{c_1^2} \tag{29}$$

which divides the (s_2^L, β)-space into two parts: stable region in the right to the curve and unstable region left.

It can be shown that dynamics generated by (19), respectively (22), are essentially the same as dynamics generated by (25), respectively (28). Introducing the new variables $C = c_1$ and $S = c_1 - c_2 + s_2^L$ reduces (25) and (28) to (19) and (22), respectively. One system can be transformed to the other through variable changes. Thus both systems are topologically conjugate to each other and generate qualitatively the same dynamics. In particular, the instability value and the nonnegativity value of (25) can be obtained from those values of (19) with C and S.

$$S = -\frac{21}{29}C \Longrightarrow s_2^L = c_2 - \frac{50}{29}c_1$$

and

$$S = -\frac{\sqrt{2}}{2}C \Longrightarrow s_2^L = c_2 - \frac{1}{2-\sqrt{2}}c_1.$$

Furthermore the partition line, (24) with C and S, can be transformed to (29).

$$\beta = \frac{2(C-S)(C+S)}{C^2} \Longrightarrow \beta = \frac{2(c_2 - s_2^L)(2c_1 - c_2 + s_2^L)}{c_1^2}.$$

The equivalence of the dynamic systems implies that (25) generates the same dynamics as illustrated in Figure 6 with replacing the interval (21) with (27). Similarly, the output dynamics by (28) is the same as illustrated in Figure 7 with replacing the partition curve (24) with (29). We can summarize these results as follows:

Lemma 3. *If the firms are asymmetric, then the naively adjusted dynamic system (25) starts the period-doubling bifurcation leading to chaos if* s_2^L *decreases from the instability value of the interval (27) to the nonnegative value whereas the adaptively adjusted dynamic system (28) generates complex dynamics involving chaos for* (s_2^L, β) *such as*

$$\beta > \frac{2(c_2 - s_2^L)(2c_1 - c_2 + s_2^L)}{c_1^2}.$$

Lemmas 1 and 2 are concerned with the output dynamics of the symmetric firms while Lemma 3 is concerned with the output dynamics of the asymmetric firms. Notice that the results are essentially the same, hence, the production cost differences do not affect the asymptotic behavior of the unstable output dynamics if the trade policy is adaptively adjusted. These results can be summarized as follows:

Theorem 7 *If the optimal trade policy is asymptotically stable, then the output dynamic system generates the same dynamics regardless of the symmetry or asymmetry of the firms.*

The output dynamic system with adaptive expectation might lead to negative quantities in numerical simulations. To avoid such economically unfavorable phenomena, we repeated the simulations with the modified output dynamic equations

$$\begin{cases} x' = (1-\beta)x + \beta Max\left[0, \left(\sqrt{\dfrac{y}{c_1 + s_2^e}} - y\right)\right], \\ \\ y' = (1-\beta)y + \beta Max\left[0, \left(\sqrt{\dfrac{x}{c_2 - s_2^e}} - x\right)\right]. \end{cases} \tag{30}$$

The results are shown in Figure 7. This is not the only way to prevent negative quantities. Following Yousefi (2002), we can also use the alternative formulation

$$\begin{cases} x^{'} = Max\left[0, (1-\beta)x + \beta\left(\sqrt{\dfrac{y}{c_1 + s_2^e}} - y\right)\right], \\ \\ y^{'} = Max\left[0, (1-\beta)y + \beta\left(\sqrt{\dfrac{x}{c_2 - s_2^e}} - x\right)\right]. \end{cases} \tag{31}$$

It is clear that the asymptotic behavior of (30) is different from the asymptotic behavior of (31). Figure 8 shows the two-parameter bifurcation diagram generated by (31). Apparently there are many differences between the bifurcation diagrams of Figures 7 and 8. However, our main finding that the adaptive systems can generate rich dynamics still can hold in the bifurcation diagram shown in Figure 8.

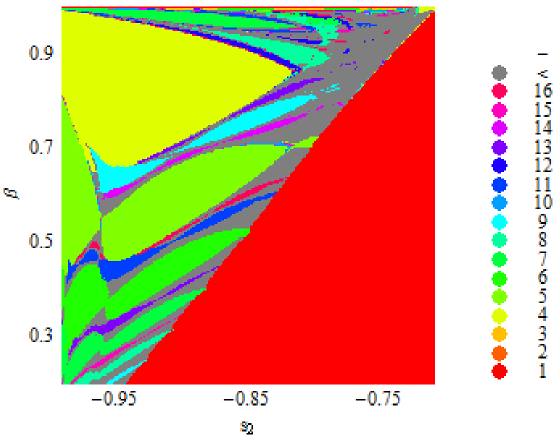


Figure 8. The two-parameter bifurcation diagram generated by (**??**)

5 Conclusion

In this paper, we construct a three-country model with two governments and two firms and consider the dynamic behavior of the sequential subsidy game in which the governments determine their optimal trade policies and then the firms determine their optimal outputs. We first deal with the governments' decision process from static and dynamic points of view. We find that the cost difference and the expectation formation of the governments are crucial in characterizing the optimal trade policy. In short, if the firms are symmetric, then there are infinitely many optimal policies (Theorem 1). A symmetric period-2 cycle of the trade policy emerges if naive adjustment process is adopted (Theorem 2) and a trajectory converges to one of the optimal policies if adaptive adjustment process is used (Theorem 3). If the firms are asymmetric, then a unique optimal policy exits and is asymptotically stable regardless of the expectation formations (Theorem 4). We then deal with the output dynamics and demonstrate that the expectation formation of the government matters but the cost difference does not matter. If the trade policy is adaptively adjusted, then an output trajectory exhibits periodic cycle which is synchronized with the period-2 cycle of the optimal trade policy even if the Cournot output equilibrium is locally stable (Theorems 5 and 6). If the trade policy is adaptively adjusted, then complex output dynamics involving chaos emerges regardless of the expectation formation (Theorem 7). Finally it is worth mentioning that complex dynamics can be born under a small or even zero difference of the production costs in our model while much larger difference is required to generate chaotic dynamics in nonlinear duopoly models with isoelastic price function.

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