Mixed Cournot-Bertrand Competition in N-firm Differentiated Oligopolies*

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Abstract

An n-firm mixed oligopoly is examined with product differentiation, in which quantity-adjusting firms and price-adjusting firms coexist and compete with each other. In a mixed duopoly framework, Singh and Vives (1984) show that a dominant strategy for a firm is a quantity-strategy if the goods are substitutes and it is a price-strategy if the goods are complements. It is demonstrated that this clear-cut result in the duopoly framewor is not necessarily valid in an n-firm oligopoly framework, and a choice of the dominant strategy depends on the average quality ratio of the goods produced by the quantity-adjusting and by the price-adjusting firms.

key words: product differentiation, n-firm mixed oligopoly, dominant stragery, Cournot-Bertrand competition, Bertrand-Cournot competition

1 Introduction

In their seminal paper, Singh and Vives (1984) examine a duopoly with product differentiation and show, among others, the following clear-cut results:

(i) profits are higher (lower) under Cournot competition than under Bertrand competition if the goods are substitutes (complements);

(ii) a quantity-adjusting strategy is more (less) profitable than a price-adjusting strategy if the goods are substitutes (complements).

Under the pure competition in which all firms are either quantity-adjusting (i.e., Cournot competition) or price-adjusting (i.e., Bertrand competition), Häckner (2000) constructs an n-firm differentiated oligopoly model with $n > 2$ showing that result (1) is sensitive to the duopoly assumption. In particular, it is

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demonstrated that low-quality firms charge higher price under price competition than under quantity competition if the goods are complement, which is not true in the duopoly framework. Recently Matsumoto and Szidarovszky (2008) complements the results developed by Häckner (2000) and show that there is a case in which high-qualified firms can charge higher price under the same circumstance. In those studies, however, the focuses are put only on the pure competition. As a result, it has not yet been analyzed whether or not result (2) is true in an n-firm mixed competition in which quantity-adjusting firms and price-adjusting firms coexist and compete with each other.[1]

The main purpose of this study is to demonstrate that result (2) is also sensitive to the duopoly assumption and that the dominant strategy under mixed competition depends on the ratio of the average qualities of the goods produced by the quantity-adjusting and by the price-adjusting firms.

In what follows, various forms of demands are introduced in Section 2. The dominance of a quantity strategy over a price strategy in a mixed duopoly are reviewed in Section 3. Optimal behavior of the firms under mixed competitions are considered in Section 4. Our main results on behavioral comparison are presented in Section 5. Concluding remarks are given in Section 6.

2 Demands

As in Singh and Vives (1984), there is a continuum of consumers of the same type. Following Häckner (2000), the utility function of the representative consumer is given by

$$U(\mathbf{q},\mathbf{p}) = \sum_{i=1}^{n} A_i q_i - \frac{1}{2}\left(\sum_{i=1}^{n} q_i^2 + 2\gamma \sum_{j\neq i}^{n} q_i q_j\right) - \sum_{i=1}^{n} p_i q_i \tag{1}$$

where $\mathbf{q} = (q_i)$ is the quantity vector, $\mathbf{p} = (p_i)$ is the price vector, A_i measures the quality of good i and $\gamma \in [-1, 1]$ measures the substitutability between the goods: $\gamma > 0$, $\gamma < 0$ or $\gamma = 0$ imply that the goods are substitutes, complements or independent. Moreover, the goods are perfect substitutes if $\gamma = 1$ and perfect complements if $\gamma = -1$. In this study, we confine our analysis to the case in which the goods are imperfect substitutes or complements and not independent, by assuming that $|\gamma| < 1$ and $\gamma \neq 0$.

Solving the first-order conditions of the optimal consumption of good k, we

[1] The mixed Cournot-Bertrand oligopoly was first considered by Bylka and Komar (1976) and then extended in various directions. For static study, Szidarovszky and Molnár (1992) show the equivalence of the equilibrium to the non-linear complementarity problem and also prove the existence and uniqueness of the Nash equilibrium in an n-firm oligopoly. For dynamic study, Matsumot and Onozaki (2005) and Yousefi and Szidarovszky (2005) model the dynamic process of mixed duopolies with nonlinear demands and show a birth of complicated fluctuations involving chaos if nonlinearities become stronger.

obtain its inverse demand function,

$$p_k = A_k - q_k - \gamma \sum_{i \neq k}^{n} q_i. \tag{2}$$

That is, the price vector is a linear function of the output vector:

$$\mathbf{p} = \mathbf{A} - \mathbf{B}\mathbf{q} \tag{3}$$

where $\mathbf{A} = (A_k)$, $\mathbf{B} = (B_{ki})$ with $B_{kk} = 1$ and $B_{ki} = \gamma$ $(i \neq k)$.

We consider an n-firm mixed oligopoly with product differentiation. The n firms are divided into two groups. Let $K = \{1, 2, ..., m\}$ and $\bar{K} = \{m+1, ..., n\}$ be the sets of quantity and price setting firms, respectively. If $\bar{K}$ is empty, then the firms are in Cournot competition. If K is empty, then the firms are in Bertrand competition. In this study we assume $n \geq m+1$ and $m \geq 1$ to draw attention to the mixed oligopoly. In this case, we say that the firms enter a *Cournot-Bertrand* (CB) competition. On the other hand, if the strategies are interchanged, that is, the firms in K are price-adjusting and the firms in $\bar{K}$ are quantity-adjusting, then we say that the firms enter a *Bertrand-Cournot* (BC) competition. This mixed oligopoly generalizes Cournot and Bertrand oligopolies in the sense that firms having different strategies coexist.

We start deriving demands that the firms perceive in CB competition. If we introduce the notation,

$$\mathbf{q}^K = (q_i)_{i \in K},\ \mathbf{q}^{\bar{K}} = (q_{\bar{\imath}})_{\bar{\imath} \in \bar{K}},\ \mathbf{p}^K = (p_i)_{i \in K},\ \mathbf{p}^{\bar{K}} = (p_{\bar{\imath}})_{\bar{\imath} \in \bar{K}},$$

then the inverse demand equation (3) can be rewritten as

$$\begin{pmatrix} \mathbf{p}^K \\ \mathbf{p}^{\bar{K}} \end{pmatrix} = \begin{pmatrix} \mathbf{A}^K \\ \mathbf{A}^{\bar{K}} \end{pmatrix} - \begin{pmatrix} \mathbf{B}^{KK} & \mathbf{B}^{K\bar{K}} \\ \mathbf{B}^{\bar{K}K} & \mathbf{B}^{\bar{K}\bar{K}} \end{pmatrix} \begin{pmatrix} \mathbf{q}^K \\ \mathbf{q}^{\bar{K}} \end{pmatrix}$$

where vector $\mathbf{A}$ and matrix $\mathbf{B}$ are decomposed accordingly, or as

$$\mathbf{p}^K = \mathbf{A}^K - \mathbf{B}^{KK}\mathbf{q}^K - \mathbf{B}^{K\bar{K}}\mathbf{q}^{\bar{K}} \tag{4}$$

and

$$\mathbf{p}^{\bar{K}} = \mathbf{A}^{\bar{K}} - \mathbf{B}^{\bar{K}K}\mathbf{q}^K - \mathbf{B}^{\bar{K}\bar{K}}\mathbf{q}^{\bar{K}}. \tag{5}$$

In CB competition, the strategy of firm $k \in K$ is its output q_k and the strategy of firm $\bar{k} \in \bar{K}$ is its price $p_{\bar{k}}$.

We now rewrite the inverse demands relations (4) and (5) in terms of the strategic variables. Assuming that $\mathbf{B}^{\bar{K}\bar{K}}$ is invertible[2] and solving (5) for $\mathbf{q}^{\bar{k}}$ yields

$$\mathbf{q}^{\bar{K}} = \left(\mathbf{B}^{\bar{K}\bar{K}}\right)^{-1} \mathbf{A}^{\bar{K}} - \left(\mathbf{B}^{\bar{K}\bar{K}}\right)^{-1} \mathbf{B}^{\bar{K}K}\mathbf{q}^K - \left(\mathbf{B}^{\bar{K}\bar{K}}\right)^{-1} \mathbf{p}^{\bar{K}}$$

[2] $\mathbf{B}^{\bar{K}\bar{K}}$ is the $n-m$ by $n-m$ matrix and has 1 on the diagonal and γ on the off-diagonal. It is invertible if $\det \mathbf{B}^{\bar{K}\bar{K}} = (1-\gamma)^{n-m-1}(1+(n-m-1)\gamma) \neq 0$. The diagonal element of

which is then substituted into (4) to obtain

$$\begin{aligned}\mathbf{p}^K &= \mathbf{A}^K - \mathbf{B}^{K\bar{K}}\left(\mathbf{B}^{\bar{K}\bar{K}}\right)^{-1}\mathbf{A}^{\bar{K}} - \left[\mathbf{B}^{KK} - \mathbf{B}^{K\bar{K}}\left(\mathbf{B}^{\bar{K}\bar{K}}\right)^{-1}\mathbf{B}^{\bar{K}K}\right]\mathbf{q}^K \\ &+ \mathbf{B}^{K\bar{K}}\left(\mathbf{B}^{\bar{K}\bar{K}}\right)^{-1}\mathbf{p}^{\bar{K}}.\end{aligned}$$

If we further introduce the following notation,

$$\boldsymbol{\alpha}^{\bar{K}} = \left(\mathbf{B}^{\bar{K}\bar{K}}\right)^{-1}\mathbf{A}^{\bar{K}},\ \boldsymbol{\beta}^{\bar{K}K} = \left(\mathbf{B}^{\bar{K}\bar{K}}\right)^{-1}\mathbf{B}^{\bar{K}K},\ \boldsymbol{\beta}^{\bar{K}\bar{K}} = \left(\mathbf{B}^{\bar{K}\bar{K}}\right)^{-1},$$

$$\boldsymbol{\alpha}^{K} = \mathbf{A}^K - \mathbf{B}^{K\bar{K}}\left(\mathbf{B}^{\bar{K}\bar{K}}\right)^{-1}\mathbf{A}^{\bar{K}},$$

$$\boldsymbol{\beta}^{KK} = \mathbf{B}^{KK} - \mathbf{B}^{K\bar{K}}\left(\mathbf{B}^{\bar{K}\bar{K}}\right)^{-1}\mathbf{B}^{\bar{K}K} \text{ and } \boldsymbol{\beta}^{K\bar{K}} = -\mathbf{B}^{K\bar{K}}\left(\mathbf{B}^{\bar{K}\bar{K}}\right)^{-1},$$

then $\mathbf{p}^K$ and $\mathbf{q}^{\bar{K}}$ are written as linear functions of strategic vectors $\mathbf{q}^K$ and $\mathbf{p}^{\bar{K}}$:

$$\mathbf{p}^K = \boldsymbol{\alpha}^K - \boldsymbol{\beta}^{KK}\mathbf{q}^K - \boldsymbol{\beta}^{K\bar{K}}\mathbf{p}^{\bar{K}}$$

and

$$\mathbf{q}^{\bar{K}} = \boldsymbol{\alpha}^{\bar{K}} - \boldsymbol{\beta}^{\bar{K}K}\mathbf{q}^K - \boldsymbol{\beta}^{\bar{K}\bar{K}}\mathbf{p}^{\bar{K}}.$$

The k^{th} components of $\mathbf{p}^K$ and $\mathbf{q}^{\bar{K}}$, p_k and $q_{\bar{k}}$, are written as

$$p_k = \frac{(1+(n-m-1)\gamma)A_k - \gamma\sum_{\bar{\imath}=m+1}^{n} A_{\bar{\imath}} - (1-\gamma)(1+(n-m)\gamma)q_k + \gamma\left(\sum_{\bar{\imath}=m+1}^{n} p_{\bar{\imath}} - (1-\gamma)\sum_{i=1, i\neq k}^{m} q_i\right)}{1+(n-m-1)\gamma} \tag{6}$$

and

$$q_{\bar{k}} = \frac{(1+(n-m-1)\gamma)A_{\bar{k}} - \gamma\sum_{\bar{\imath}=m+1}^{n} A_{\bar{\imath}} - (1+(n-m-2)\gamma)p_{\bar{k}} + \gamma\left(\sum_{\bar{\imath}=m+1, \bar{\imath}\neq\bar{k}}^{n} p_{\bar{\imath}} - (1-\gamma)\sum_{k=1}^{m} q_k\right)}{(1-\gamma)(1+(n-m-1)\gamma)}. \tag{7}$$

We now turn our attention to the case of BC competition in which p_k is the strategic variable for firm k and so is $q_{\bar{k}}$ for firm $\bar{k}$. Since BC competition is dual to CB competition, interchanging sets K and $\bar{K}$ generates the k^{th} components of $\mathbf{q}^k$ and $\mathbf{p}^{\bar{k}}$, q_k and $p_{\bar{k}}$, as functions of the strategic variables:

$$q_k = \frac{(1+(m-1)\gamma)A_k - \gamma\sum_{i=1}^{m} A_i - (1+(m-2)\gamma)p_k + \gamma\left(\sum_{i=1, i\neq k}^{m} p_i - (1-\gamma)\sum_{\bar{\imath}=m+1}^{n} q_{\bar{\imath}}\right)}{(1-\gamma)(1+(m-1)\gamma)} \tag{8}$$

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its inverse matrix is

$$\frac{1+(n-m-2)\gamma}{(1-\gamma)(1+(n-m-1)\gamma)}$$

and the off-diagonal element is

$$-\frac{\gamma}{(1-\gamma)(1+(n-m-1)\gamma)}.$$

The invertibility of the matirx B^{KK} is defined in the same way. This issue is discussed in more detail in Szidarovszky and Yakowitz (1978).

and

$$p_{\bar{k}} = \frac{(1+(m-1)\gamma)A_{\bar{k}} - \gamma \sum_{i=1}^{m} A_i - (1-\gamma)(1+m\gamma)q_{\bar{k}} + \gamma \left(\sum_{i=1}^{m} p_i - (1-\gamma) \sum_{\bar{\imath}=m+1, \bar{\imath}\neq \bar{k}}^{n} q_{\bar{\imath}} \right)}{(1+(m-1)\gamma)} \tag{9}$$

Firm k chooses optimal strategy to maximize its profit $\pi_k = (p_k - c_k)x_k$ where c_k denotes the constant marginal cost. In a linear demand oligopoly, changes in the production cost do not affect the optimal behavior of the firms qualitatively. Thus the zero-cost assumption is usually adopted for the sake of analytical simplicity, and we follow this tradition in the following discussions.

3 Review: Mixed Duopoly

Before proceeding to the general n-firm mixed oligopoly, we briefly review the results on the behavioral comparison in a duopoly framework. To this end, we take $n = 2$ and $m = 1$. Let firm 1 be quantity-adjusting and firm 2 price-adjusting under CB competition. The usual procedure for solving the profit maximization problems leads to the optimal behavior of the firms as follows:

$$q_1^{CB} = \frac{2A_1 - \gamma A_2}{4 - 3\gamma^2}, \ p_1^{CB} = (1-\gamma^2)q_1^{CB} \text{ and } \pi_1^{CB} = (1-\gamma^2)\left(q_1^{CB}\right)^2$$

and

$$p_2^{CB} = \frac{-\gamma A_1 + (2-\gamma^2)A_2}{4 - 3\gamma^2}, \ q_2^{CB} = p_2^{CB} \text{ and } \pi_2^{CB} = \left(q_2^{CB}\right)^2.$$

Subscript "CB" over variables indicates that the corresponding variables are evaluated at a CB equilibrium.

Firm 2 is quantity-adjusting under BC competition and its optimal behavior is dual to the optimal behavior of quantity-adjusting firm 1 under CB competition. We get its optimal behavior by replacing A_i by A_j, q_i by q_j and p_i by p_j for $i \neq j$, $i, j = 1, 2$:

$$q_2^{BC} = \frac{-\gamma A_1 + 2A_2}{4 - 3\gamma^2}, \ p_2^{BC} = (1-\gamma^2)q_2^{BC} \text{ and } \pi_2^{BC} = (1-\gamma^2)\left(q_2^{BC}\right)^2.$$

In the same way, duality transforms the optimal behavior of firm 2 under CB competition into the optimal behavior of firm 1 under BC competition,

$$p_1^{BC} = \frac{(2-\gamma^2)A_1 - \gamma A_2}{4 - 3\gamma^2}, \ q_1^{BC} = p_1^{BC} \text{ and } \pi_1^{BC} = \left(q_1^{BC}\right)^2.$$

Let us compare the profits of firms 1 and 2 under CB competition with those under BC competition. Ratios of the profits are

$$\frac{\pi_1^{CB}}{\pi_1^{BC}} = (1-\gamma^2)\left(\frac{2-\alpha\gamma}{2-\gamma^2-\alpha\gamma}\right)^2 \text{ and } \frac{\pi_2^{CB}}{\pi_2^{BC}} = \frac{1}{1-\gamma^2}\left(\frac{(2-\gamma^2)\alpha-\gamma}{2\alpha-\gamma}\right)^2$$

where α is the ratio of A_2 over A_1. Differences between the numerator and the denominator of the profit ratios are

$$(1-\gamma^2)(2-\alpha\gamma)^2 - (2-\gamma^2-\alpha\gamma)^2 = -\gamma^3(\alpha^2\gamma - 2\alpha + \gamma)$$

and

$$\big((2-\gamma^2)\alpha - \gamma\big)^2 - (1-\gamma^2)(2\alpha-\gamma)^2 = \gamma^3(\alpha^2\gamma - 2\alpha + \gamma).$$

The right hand sides of these two equations are negative and positive, respectively, if $\gamma < 0$. Hence

$$\pi_1^{CB} < \pi_1^{BC} \text{ and } \pi_2^{CB} > \pi_2^{BC} \text{ if } \gamma < 0. \tag{10}$$

Assume next that $\gamma > 0$. Then it can be shown that $\alpha^2\gamma - 2\alpha + \gamma < 0$ under the non-negativity conditions for the optimal prices and outputs,

$$\frac{2-\gamma^2}{\gamma} \geq \alpha \geq \frac{\gamma}{2-\gamma^2}$$

implying that

$$\pi_1^{CB} > \pi_1^{BC} \text{ and } \pi_2^{CB} < \pi_2^{BC} \text{ if } \gamma > 0. \tag{11}$$

The results (10) and (11) in the mixed duopoly agree with those obtained by Singh and Vives (1984) and, as a benchmark for measuring firms' profitability, they can be summarized in the following way.

Theorem 1 *In a mixed duopoly, a quantity-adjusting strategy is more profitable than a price-adjusting strategy if the goods are substitutes whereas the price-adjusting strategy is more profitable than the quantity-adjusting strategy if the goods are complements.*

4 Mixed N-Firm Oligopoly

4.1 CB Competition

We will begin our discussion on the n-firm mixed oligopoly by considering the optimal behavior of the firms under CB competition. We first derive the best responses and then obtain equilibrium outputs, prices and profits.

4.1.1 Best Responses

Since firm $k \in K$ is quantity-adjusting, it chooses quantity of good k so as to maximize its profit subject to (6), taking the other firms' quantities and prices as given. Solving the first-order condition of the profit maximizing problem yields the best response

$$q_k = \frac{(1+(n-m-1)\gamma)A_k - \gamma \sum_{\bar{\imath}=m+1}^{n} A_{\bar{\imath}} + \gamma\left(\sum_{\bar{\imath}=m+1}^{n} p_{\bar{\imath}} - (1-\gamma)\sum_{i\neq k}^{m} q_i\right)}{2(1-\gamma)(1+(n-m)\gamma)}. \tag{12}$$

Using the simple fact that $\sum_{i=1}^{m} q_i = q_k + \sum_{i \neq k}^{m} q_i$, the best response can be rewritten as

$$q_k = \frac{(1+(n-m-1)\gamma)A_k - \gamma \sum_{\bar{\imath}=m+1}^{n} A_{\bar{\imath}} + \gamma \left(\sum_{\bar{\imath}=m+1}^{n} p_{\bar{\imath}} - (1-\gamma) \sum_{i=1}^{m} q_i \right)}{(1-\gamma)(2+(2(n-m)-1)\gamma)}. \tag{13}$$

The second-order condition (SOC) for the profit maximization is satisfied if

$$\frac{\partial^2 \pi_k}{\partial q_k^2} = -2 \frac{(1-\gamma)(1+(n-m)\gamma)}{1+(n-m-1)\gamma} < 0. \tag{14}$$

If $0 < \gamma < 1$, then the terms on the right hand side of (14) are positive and thus this SOC is always satisfied. If $-1 < \gamma < 0$, the signs of $1 + (n - m)\gamma$ and $1+(n-m-1)\gamma$ are ambiguous and then the sign of the SOC is also ambiguous, so we need additional conditions to fulfill the SOC. Since

$$1 + (n-m)\gamma < 1 + (n-m-1)\gamma, \tag{15}$$

the required condition is $1 + (n - m)\gamma > 0$ or $1 + (n - m - 1)\gamma < 0$.

Since firm $\bar{k} \in \bar{K}$ is price-adjusting, it chooses a price of good $\bar{k}$ so as to maximize its profit subject to (7), taking the other firms' quantities and prices as given. Solving the first-order condition of the profit maximizing problem yields the best response

$$p_{\bar{k}} = \frac{(1+(n-m-1)\gamma)A_{\bar{k}} - \gamma \sum_{\bar{\imath}=m+1}^{n} A_{\bar{\imath}} + \gamma \left(\sum_{\bar{\imath} \neq \bar{k}}^{n} p_{\bar{\imath}} - (1-\gamma) \sum_{i=1}^{m} q_i \right)}{2(1+(n-m-2)\gamma)}. \tag{16}$$

Using the simple fact that $\sum_{\bar{\imath}=m+1}^{n} p_{\bar{\imath}} = p_{\bar{k}} + \sum_{\bar{\imath} \neq \bar{k}}^{n} p_{\bar{\imath}}$, the best response can be rewritten as

$$p_{\bar{k}} = \frac{(1+(n-m-1)\gamma)A_{\bar{k}} - \gamma \sum_{\bar{\imath}=m+1}^{n} A_{\bar{\imath}} + \gamma \left(\sum_{\bar{\imath}=m+1}^{n} p_{\bar{\imath}} - (1-\gamma) \sum_{i=1}^{m} q_i \right)}{2+(2(n-m)-3)\gamma}. \tag{17}$$

The SOC for the profit maximization is satisfied if

$$\frac{\partial^2 \pi_{\bar{k}}}{\partial p_{\bar{k}}^2} = -2 \frac{(1+(n-m-2)\gamma)}{(1-\gamma)(1+(n-m-1)\gamma)} < 0.$$

If $0 < \gamma < 1$, then this SOC is always satisfied. If $-1 < \gamma < 0$, then by the same reason above, we need additional conditions to satisfy the SOC. Since

$$1 + (n-m-1)\gamma < 1 + (n-m-2)\gamma, \tag{18}$$

the required condition is $1+(n-m-1)\gamma > 0$ or $1+(n-m-2)\gamma < 0$. From (15) and (18), in order to fulfill the SOCs in the optimizing problems under both CB and BC competitions, we assume the following:

Assumption 1. If $\gamma < 0$, either $1 + (n - m)\gamma > 0$ or $1 + (n - m - 2)\gamma < 0$ holds.

4.1.2 CB Equilibrium

We will next show that there is a unique CB equilibrium. Adding (13) over all quantity-adjusting firms and (17) over all price-adjusting firms gives

$$\sum_{k=1}^{m} q_k = \frac{(1+(n-m-1)\gamma)\sum_{k=1}^{m} A_k - m\gamma \sum_{\bar{k}=m+1}^{n} A_{\bar{k}} + m\gamma \sum_{\bar{k}=m+1}^{n} p_{\bar{k}}}{(1-\gamma)(2+(2n-m-1)\gamma)}$$

and

$$\sum_{\bar{k}=m+1}^{n} p_{\bar{k}} = \frac{(1-\gamma)\sum_{\bar{k}=m+1}^{n} A_{\bar{k}} - (n-m)(1-\gamma)\gamma \sum_{i=1}^{m} q_i}{2+(n-m-3)\gamma}.$$

If we introduce the notation,

$$Q = (1-\gamma)\sum_{k=1}^{m} q_k,\ P = \sum_{\bar{k}=m+1}^{n} p_{\bar{k}},\ B_k = \sum_{k=1}^{m} A_k \text{ and } B_{\bar{k}} = \sum_{\bar{k}=m+1}^{n} A_{\bar{k}},$$

the above two equations can be rewritten as the simultaneous equations for Q and P

$$\begin{cases} (2+(2n-m-1)\gamma)Q - m\gamma P = (1+(n-m-1)\gamma)B_k - m\gamma B_{\bar{k}}, \\ (n-m)\gamma Q + (2+(n-m-3)\gamma)P = (1-\gamma)B_{\bar{k}}, \end{cases} \tag{19}$$

where we assume

$$\Delta^{CB} = (2+(2n-m-1)\gamma)(2+(n-m-3)\gamma) + m(n-m)\gamma^2 \neq 0 \tag{20}$$

to have a non-trivial solution. From the solution of (19) we obtain

$$\begin{aligned} P - Q &= \frac{1}{\Delta^{CB}}\{-(1+(n-m-1)\gamma)(2+(2(n-m)-3)\gamma)B_k \\ &\quad + [2+(2n-3)\gamma + (2n(m-1)-(2m-1)(m+1))\gamma^2]B_{\bar{k}}\}. \end{aligned}$$

$P-Q$ is substituted into (13) and (17) to obtain the Cournot-Bertrand equilibrium output and price,[3]

$$q_k^{CB} = \frac{a_k^{CB} A_k - b_k^{CB} B_k - b_{\bar{k}}^{CB} B_{\bar{k}}}{\Delta^{CB}(1-\gamma)(2+(2(n-m)-1)\gamma)} \tag{21}$$

and

$$p_{\bar{k}}^{CB} = \frac{a_k^{CB} A_k - b_k^{CB} B_k - b_{\bar{k}}^{CB} B_{\bar{k}}}{\Delta^{CB}(2+(2(n-m)-3)\gamma)} \tag{22}$$

[3] The CB output and price as well as the CB equilibrium values of another variables to be obtained below in the n-firm oligopoly can be reduced to those in the duopoly obtained above if we take $n=2$ and $m=1$.

where

$$a_k^{CB} = \Delta^{CB}(1 + (n - m - 1)\gamma),$$

$$b_k^{CB} = \gamma(1 + (n - m - 1)\gamma)(2 + (2(n - m) - 3)\gamma)$$

and

$$b_{\bar{k}}^{CB} = \gamma(2 + (2(n - m) - 1)\gamma)(1 + ((n - m) - 2)\gamma).$$

From the perceived demand (6) of firm k and the best response (12), we can obtain the price at which firm k sells its output as a function of its optimal quantity,

$$p_k^{CB} = \frac{(1 - \gamma)(1 + (n - m)\gamma)}{1 + (n - m - 1)\gamma} q_k^{CB}. \tag{23}$$

In the same way, solving (7) and (16) for $q_{\bar{k}}$ yields the output that firm $\bar{k}$ sells as a function of its optimal price

$$q_{\bar{k}}^{CB} = \frac{1 + (n - m - 2)\gamma}{(1 - \gamma)(1 + (n - m - 1)\gamma)} p_{\bar{k}}^{CB}. \tag{24}$$

Using (23) and (24), the Cournot-Bertrand profits of firms k and $\bar{k}$ are written as

$$\pi_k^{CB} = \frac{(1 - \gamma)(1 + (n - m)\gamma)}{1 + (n - m - 1)\gamma} \left(q_k^{CB}\right)^2 \tag{25}$$

and

$$\pi_{\bar{k}}^{CB} = \frac{(1 - \gamma)(1 + (n - m - 1)\gamma)}{1 + (n - m - 2)\gamma} \left(q_{\bar{k}}^{CB}\right)^2 \tag{26}$$

4.2 BC Competition

In BC competition, the strategies of firms are interchanged. Namely, the first m firms in K are price-adjusting and the last $(n - m)$ firms in $\bar{K}$ are quantity-adjusting. By duality, we can obtain the optimal behavior under BC competition from the optimal behavior under CB competition. In particular, the best responses of firms k and $\bar{k}$ are

$$p_k = \frac{(1 + (m - 1)\gamma)A_k - \gamma \sum_{i=1}^{m} A_i + \gamma \left(\sum_{i=1,\ i \neq k}^{m} p_i - (1 - \gamma) \sum_{\bar{k}=m+1}^{n} q_{\bar{k}} \right)}{2(1 + (m - 2)\gamma)} \tag{27}$$

and

$$q_{\bar{k}} = \frac{(1 + (m - 1)\gamma)A_{\bar{k}} - \gamma \sum_{k=1}^{m} A_k + \gamma \left(\sum_{k=1}^{m} p_k - (1 - \gamma) \sum_{\bar{\imath}=m+1,\ \bar{\imath} \neq \bar{k}}^{n} q_{\bar{\imath}} \right)}{2(1 - \gamma)(1 + m\gamma)} \tag{28}$$

It is easy to see that the second-order conditions for the profit maximizing problems are always satisfied if the goods are substitutes. If the goods are complements, then the following additional conditions are necessary.

Assumption 2. If $\gamma < 0$, then either $1 + m\gamma > 0$ or $1 + (m-2)\gamma < 0$.

Duality also gives BC equilibrium price and output from (21) and (22):

$$p_k^{BC} = \frac{a_k^{BC} A_k - b_k^{BC} B_k - b_{\bar{k}}^{BC} B_{\bar{k}}}{\Delta^{BC}(2 + (2m-3)\gamma)} \tag{29}$$

and

$$q_{\bar{k}}^{BC} = \frac{a_k^{BC} A_{\bar{k}} - b_k^{BC} B_k - b_{\bar{k}}^{BC} A_{\bar{k}}}{\Delta^{BC}(1-\gamma)(2 + (2m-1)\gamma)} \tag{30}$$

where

$$\begin{aligned} \Delta^{BC} &= (2 + (m-3)\gamma)(2 + (n+m-1)\gamma) + m(n-m)\gamma^2 \\ a_k^{BC} &= \Delta^{BC}(1 + (m-1)\gamma) \\ b_k^{BC} &= \gamma(1 + (m-2)\gamma)(2 + (2m-1)\gamma) \end{aligned}$$

and

$$b_{\bar{k}}^{BC} = \gamma(1 + (m-1)\gamma)(2 + (2m-3)\gamma)$$

Superscript "BC" over variables has the similar meaning as "CB."

Using the demands and the best responses, the output of the price-adjusting firm k and the price of quantity-adjusting firm $\bar{k}$ can be expressed as linear functions of the strategic variables, p_k^{BC} and $q_{\bar{k}}^{BC}$:

$$q_k^{BC} = \frac{1 + (m-2)\gamma}{(1-\gamma)(1 + (m-1)\gamma)} p_k^{BC} \tag{31}$$

and

$$p_{\bar{k}}^{BC} = \frac{(1-\gamma)(1 + m\gamma)}{1 + (m-1)\gamma} q_{\bar{k}}^{BC} \tag{32}$$

From (31) and (32), the profits under BC competition have the forms

$$\pi_k^{BC} = \frac{(1-\gamma)(1 + (m-1)\gamma)}{1 + (m-2)\gamma} \left(q_k^{BC}\right)^2 \tag{33}$$

and

$$\pi_{\bar{k}}^{BC} = \frac{(1-\gamma)(1 + m\gamma)}{1 + (m-1)\gamma} \left(q_{\bar{k}}^{BC}\right)^2. \tag{34}$$

5 Comparison of the optimal behavior

We compare the optimal behavior under CB competition with that under BC competition to find out which competition is more profitable. It, however, requires tedious calculations to make the comparison under general values of m and n. To simplify the analysis, we focus on the special case in which the following condition is satisfied:

Assumption 3. $n = 2m$.

Notice that Assumptions 1 and 2 become identical under Assumption 3. In the following analysis, we focus our attention to the optimal behavior of firm k. This treatment will suffice for our purpose to show that Theorem 1 does not necessarily hold in the general n-firm mixed oligopoly.

5.1 Ratios of optimal values

Replacing n with $2m$, we can verify that

$$\Delta^{CB} = \Delta^{BC} = (2 + (3m-1)\gamma)(2 + (m-3)\gamma) + m^2\gamma^2,$$

$$b_k^{CB} = b_{\bar{k}}^{BC} = \gamma(1 + (m-1)\gamma)(2 + (2m-3)\gamma)$$

and

$$b_{\bar{k}}^{CB} = b_k^{BC} = \gamma(1 + (m-2)\gamma)(2 + (2m-1)\gamma).$$

Let Δ^{CB} and Δ^{BC} be denoted as Δ. Then we can also verify that

$$a_k^{CB} = a_k^{BC} = \Delta(1 + (m-1)\gamma).$$

To simplify the notation, we denote k by 1, $\bar{k}$ by 2, a_k^{CB} and a_k^{BC} by a_1, b_k^{CB} and $b_{\bar{k}}^{BC}$ by b_1 and $b_{\bar{k}}^{CB}$ and b_k^{BC} by b_2, respectively. With this new notation, CB price (23) and BC price (29) are rewritten as

$$p_1^{CB} = \frac{(1 + m\gamma)mb_2A_1}{\Delta(1 + (m-1)\gamma)(2 + (2m-1)\gamma)}\left(\varphi^{CB}(\alpha_1) - \alpha_2\right) \tag{35}$$

and

$$p_1^{BC} = \frac{mb_1A_1}{\Delta(2 + (2m-3)\gamma)}\left(\varphi^{BC}(\alpha_1) - \alpha_2\right) \tag{36}$$

where α_1, respectively α_2, denotes the ratio between the average quality offered by the firms in K, respectively $\bar{K}$, and the quality offered by firm k,

$$\alpha_1 = \frac{B_1/m}{A_1}, \text{ respectively } \alpha_2 = \frac{B_2/m}{A_1} \tag{37}$$

and

$$\varphi^{CB}(\alpha_1) = \frac{a_1}{mb_2} - \frac{b_1}{b_2}\alpha_1 \text{ and } \varphi^{BC}(\alpha_1) = \frac{a_1}{mb_1} - \frac{b_2}{b_1}\alpha_1. \tag{38}$$

Both $\varphi^{CB}(\alpha_1)$ and $\varphi^{BC}(\alpha_1)$ are linear in α_1. If $b_1 \neq b_2$, then they intersect once at point (α_1^*, α_2^*) where

$$\alpha_1^* = \alpha_2^* = \frac{a_1}{(b_1 + b_2)m} \text{ and } \varphi^{CB}(\alpha_1^*) = \varphi^{BC}(\alpha_1^*) = \alpha_2^*. \tag{39}$$

Taking the ratio of (35) over (36), then

$$\frac{p_1^{CB}}{p_1^{BC}} - 1 = \frac{1 - \varepsilon^2}{\varphi^{BC}(\alpha_1) - \alpha_2}\left(\alpha_2 - \varphi_p(\alpha_1)\right) \tag{40}$$

where

$$\varphi_p(\alpha_1) = \frac{\varphi^{BC}(\alpha_1) - \varepsilon^2\varphi^{CB}(\alpha_1)}{1 - \varepsilon^2} \tag{41}$$

and

$$\varepsilon^2 = \frac{(1 + m\gamma)(1 + (m-2)\gamma)}{(1 + (m-1)\gamma)^2}.$$

It is easy to confirm the following properties of $\varphi_p(\alpha_1)$. The definition of $\varphi_p(\alpha_1)$ in (41) indicates that $\varphi_p(\alpha_1^*) = \alpha_2^*$, that is, the $\alpha_2 = \varphi_p(\alpha_1)$ curve passes through the point (α_1^*, α_2^*). For $\alpha_1 = 0$

$$\varphi_p(0) = \frac{a_1(b_2 - \varepsilon^2 b_1)}{m(1-\varepsilon^2)b_1 b_2} = \frac{a_1}{m(1-\varepsilon^2)b_1 b_2}\frac{\gamma^3(1+(m-2)\gamma)}{1+(m-1)\gamma} \tag{42}$$

and differentiating $\varphi_p(\alpha_1)$ with respect to α_1 gives

$$\frac{d\varphi_p(\alpha_1)}{d\alpha_1} = \frac{\varepsilon^2 b_1^2 - b_2^2}{(1-\varepsilon^2)b_1 b_2} = \frac{2\gamma^5(1+(m-2)\gamma)}{(1-\varepsilon^2)b_1 b_2}, \tag{43}$$

where the definitions of b_1, b_2 and ε^2 are substituted into the denominators of the middle expressions in (42) and (43) to obtain the last expressions.

With the new notation, CB quantity (21) and BC quantity (31) are rewritten as

$$q_1^{CB} = \frac{mb_2 A_1}{\Delta(1-\gamma)(2+(2m-1)\gamma)}\left(\varphi^{CB}(\alpha_1) - \alpha_2\right)$$

and

$$q_1^{BC} = \frac{(1+(m-2)\gamma)mb_1 A_1}{\Delta(1-\gamma)(1+(m-1)\gamma)(2+(2m-3)\gamma)}\left(\varphi^{BC}(\alpha_1) - \alpha_2\right).$$

Taking the ratio of q_1^{CB} over q_1^{BC}, we have

$$\frac{q_1^{CB}}{q_1^{BC}} - 1 = \frac{(b_1-b_2)(b_1+b_2)}{(\varphi^{BC}(\alpha_1)-\alpha_2)b_1 b_2}(\alpha_1^* - \alpha_1). \tag{44}$$

Finally, from the ratio of (25) over (33), we get

$$\frac{\pi_1^{CB}}{\pi_1^{BC}} - 1 = \frac{(1+\varepsilon)(1-\varepsilon)(\varphi_\pi^+(\alpha_1)-\alpha_2)\left(\alpha_2 - \varphi_\pi^-(\alpha_1)\right)}{(\varphi^{BC}(\alpha_1)-\alpha_2)^2}. \tag{45}$$

where

$$\varphi_\pi^+(\alpha_1) = \frac{\varphi^{BC}(\alpha_1) + \varepsilon\varphi^{CB}(\alpha_1)}{1+\varepsilon} \tag{46}$$

and

$$\varphi_\pi^-(\alpha_1) = \frac{\varphi^{BC}(\alpha_1) - \varepsilon\varphi^{CB}(\alpha_1)}{1-\varepsilon} \tag{47}$$

with

$$\frac{d\varphi_\pi^-(\alpha_1)}{d\alpha_1} = \frac{\varepsilon b_1^2 - b_2^2}{(1-\varepsilon^2)b_1 b_2}. \tag{48}$$

Subtracting (43) from (48) gives

$$\frac{d\varphi_\pi^-(\alpha_1)}{d\alpha_1} - \frac{d\varphi_p(\alpha_1)}{d\alpha_1} = \frac{\varepsilon(b_1^2 - b_2^2)}{(1-\varepsilon^2)b_1 b_2}. \tag{49}$$

5.2 $\gamma > 0$: the goods are substitutes

We first assume that the goods are substitutes (i.e., $\gamma > 0$). Let us start with the price comparison. If $\gamma > 0$, then $\Delta > 0$, $a_1 > 0$, $b_1 > 0$, $b_2 > 0$ and $b_1 - b_2 = \gamma^3 > 0$. It also implies that the first factor in each of (35) and (36) is positive. The non-negativity conditions for the prices are

$$\alpha_2 \leq \varphi^{CB}(\alpha_1) \text{ and } \alpha_2 \leq \varphi^{BC}(\alpha_1).$$

From (38) and (39), it can be examined that the boundary curves, $\alpha_2 = \varphi^{CB}(\alpha_1)$ and $\alpha_2 = \varphi^{BC}(\alpha_1)$, slope downwards, have positive intercepts and intersect once at point (α_1^*, α_2^*) with $\alpha_1^* = \alpha_2^* > 0$. We denote a feasible region of α_1 and α_2 by Ω which is defined by

$$\Omega = \left\{(\alpha_1, \alpha_2) \mid \varphi^{CB}(\alpha_1) \geq \alpha_2,\ \varphi^{BC}(\alpha_1) \geq \alpha_2,\ \alpha_1 > 0 \text{ and } \alpha_2 > 0\right\}.$$

Since

$$\varphi^{CB}(\alpha_1) - \varphi^{BC}(\alpha_1) = \frac{b_1^2 - b_2^2}{b_1 b_2}(\alpha_1^* - \alpha_1) \text{ and } b_1^2 - b_2^2 > 0,$$

the upper bound of Ω for $\alpha_1 < \alpha_1^*$ is the $\alpha_2 = \varphi^{BC}(\alpha_1)$ curve and the upper bound of Ω for $\alpha_1 > \alpha_1^*$ is the $\alpha_2 = \varphi^{CB}(\alpha_1)$ curve. Relations (42) and (43) indicate that the $\alpha_2 = \varphi_p(\alpha_1)$ curve slopes upwards and has a positive intercept. Since the numerator of the first factor in the right hand side of (40) is positive as ε^2 is positive and less than unity under $\gamma > 0$[4] and the denominator is also positive by the non-negativity condition of p_1^{BC}, we arrive at the following condition for the price difference:

$$sign\left[p_1^{CB} - p_1^{BC}\right] = sign\left[\alpha_2 - \varphi_p(\alpha_1)\right]. \tag{50}$$

Hence we have the following results on the price difference:

Proposition 1. *If $\gamma > 0$, then the $\alpha_2 = \varphi_p(\alpha_1)$ curve divides the feasible region Ω into two subregions: $p_1^{CB} > p_1^{BC}$ in the subregion above the curve and $p_1^{CB} < p_1^{BC}$ below.*

Before making output comparison, we notice that $p_1^{CB} \geq 0$ implies $q_1^{CB} \geq 0$ due to (23) and $p_1^{BC} > 0$ implies $q_1^{BC} \geq 0$ due to (31). We then notice that the first factor of (44) is positive. In consequence, we arrive at the following condition for the quantity difference,

$$sign\left[q_1^{CB} - q_1^{BC}\right] = sign[\alpha_1^* - \alpha_1]. \tag{51}$$

The sign of the quantity difference is determined by the difference between α_1^* and α_1, so we have the following result.

Proposition 2. *If $\gamma > 0$, then $q_1^{CB} > q_1^{BC}$ for $\alpha_1 < \alpha_1^*$ and $q_1^{CB} < q_1^{BC}$ for $\alpha_1 > \alpha_1^*$.*

[4] $(1 + m\gamma)(1 + (m-2)\gamma) - (1 + (m-1)\gamma)^2 = -\gamma^2 < 0$ implies $\varepsilon^2 < 1$ and $\varepsilon^2 > 0$ is clear.

Finally we draw our attention to profit comparison. Returning to the right hand side of (45), we see that its denominator is positive. The first two factors of the numerator are positive under $0 < \varepsilon^2 < 1$. Since the non-negativity conditions of the prices imply

$$\varphi_\pi^+(\alpha_1) = \frac{\varphi^{BC}(\alpha_1) + \varepsilon\varphi^{CB}(\alpha_1)}{1+\varepsilon} \geq \alpha_2,$$

the third factor is also non-negative. Hence we have

$$sign\left[\pi_1^{CB} - \pi_1^{BC}\right] = sign\left[\alpha_2 - \varphi_\pi^-(\alpha_1)\right] \tag{52}$$

Equation (49) indicates that the slope of $\varphi_\pi^-(\alpha_1)$ is steeper than the slope of $\varphi_p(\alpha_1)$. In Figure 1,[5] the division of the feasible region by the equal-profit curve is depicted. From (52) we conclude the following result:

Theorem 2 *If $\gamma > 0$, then the $\alpha_2 = \varphi_\pi^-(\alpha_1)$ curve divides the feasible region into two subregions: $\pi_1^{CB} > \pi_1^{BC}$ in the darker-gray region above the curve in Figure 1 and $\pi_1^{CB} < \pi_1^{BC}$ in the lighter-gray region below.*

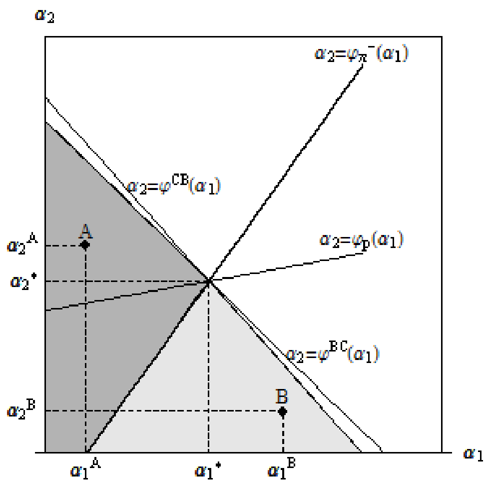

Figure 1. $\gamma = 0.975$ and $m = 3$

In the mixed duopoly with product differentiation, as summarized in the first half of Theorem 1, a quantity strategy is always more profitable than a price strategy if the goods are substitutes. So the firm takes the quantity strategy in a circumstance in which it can choose its behavioral strategy. However, Theorem 2 implies that this profit dominance of the quantity strategy with $\gamma > 0$ is not necessarily true in the n-firm mixed oligopoly framework. In particular, Propositions 1, 2 and Theorem 2 imply the following facts:

[5] In Figure 1, $m = 3$ and $\gamma = 0.975$ are taken to visualize the difference between the two curves, $\alpha_1 = \varphi^{CB}(\alpha_1)$ and $\alpha_2 = \varphi^{BC}(\alpha_1)$. Although it is analytically confirmed that the two curves are different for any other combinations of parameters. The difference, however, becomes small and invisible if a larger value of m and/or a smaller value of γ are taken.

(i) the quantity strategy dominates the price strategy at point A of Figure 1 where $\alpha_1^A < \alpha_2^A$, that is, the quantity-adjusting strategy makes firm 1 produce more output, charge a higher price and earn more profit;

(ii) the price strategy dominates the quantity strategy at point B of Figure 1 where $\alpha_1^B > \alpha_2^B$, that is, the price-adjusting strategy makes firm 1 produce more output, charge a higher price and earn more profit.

From (37), $\alpha_1 > \alpha_2$ (respectively, $\alpha_1 < \alpha_2$) means that the average quality of the goods produced by the quantity-adjusting firms is higher (respectively, lower) than the average quality of the goods produced by the price-adjusting firms. If we observe the division of the feasible region in Figure 1 carefully, it can be seen that $\alpha_1 < \alpha_2$ is a sufficient condition that the quantity-strategy is more profitable than the price-strategy. Since $\varphi_\pi^-(0) < 0$,[6] firm 1 produces larger output, charges a lower price and earns more profit under CB competition than under BC competition for α_1 and α_2 in the region standing between the $\alpha_2 = \varphi_\pi^-(\alpha_1)$ curve and the $\alpha_2 = \alpha_1$ curve in which $\alpha_2 < \alpha_1$. Hence $\alpha_1 > \alpha_2$ is a necessary condition that the price strategy is more profitable than the quantity strategy.

5.3 $\gamma < 0$: the goods are complements

We next assume that the goods are complements (i.e., $\gamma < 0$) and will compare optimal prices, outputs and profits under CB competition with those under BC competition. To fulfill the second-order condition, either $1 + m\gamma > 0$ or $1 + (m-2)\gamma < 0$ is assumed by Assumption 1 or 2. We call the first condition SOC_1 and the second SOC_2. In Figure 2 in which m is taken to be 20, SOC_1 holds in the darker-gray region while SOC_2 holds in the lighter-gray region. The $1 + m\gamma = 0$ curve is the upper boundary of the darker-gray region whereas the $1 + (m-2)\gamma = 0$ curve is the lower boundary of the ligther-gray region. The dashed curves are the $\Delta = 0$ loci. $\Delta < 0$ in the region between the dashed curves and $\Delta > 0$ otherwise. We refer to four points denoted by A, a, B and b shortly after. We will start with considering the behavioral comparisons under SOC_1 and then proceed to those under SOC_2.

[6] By definition

$$\varphi_\pi^-(0) = \frac{a_1(b_2 - \varepsilon b_1)}{m(1-\varepsilon)b_1 b_2}.$$

In the case of $\gamma > 0$, $\varepsilon^2 b_1^2 - b_2^2 = (\varepsilon b_1 - b_2)(\varepsilon b_1 + b_2) > 0$ where the inequality is due to (43). This inequality with $\varepsilon b_1 + b_2 > 0$ leads to $\varepsilon b_1 - b_2 > 0$ implying $\varphi_\pi^-(0) < 0$.

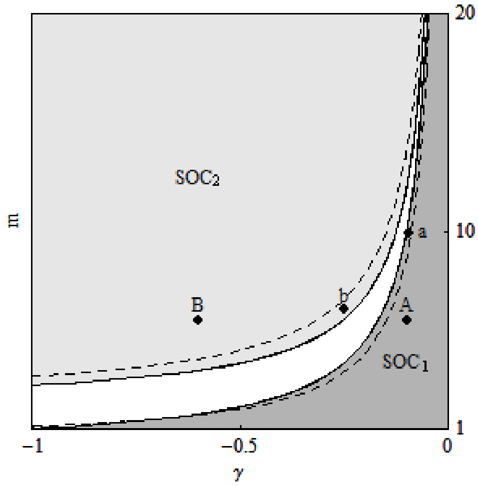

Figure 2 Regions of SOC_1 and SOC_2

5.3.1 The case of SOC_1

If $\gamma < 0$ and $1+m\gamma > 0$, then $1+(m-1)\gamma > 0$, $1+(m-2)\gamma > 0$, $2+(2m-1)\gamma > 0$ and $2+(2m-3)\gamma > 0$ which, in turn, imply that $b_1 < 0$, $b_2 < 0$ and $b_1 - b_2 = \gamma^3 < 0$. Although $\Delta a_1 > 0$, the sign of Δ is undetermined in general and so is the sign of a_1. We identity two cases: one is the case where Δ is positive and the other is the case where Δ is negative.[7]

If $\Delta > 0$ is further assumed, then $a_1 > 0$. Since the first factors in (35) and (36) are negative, the non-negativity conditions of the prices are given by

$$\varphi^{CB}(\alpha_1) - \alpha_2 \leq 0 \text{ and } \varphi^{BC}(\alpha_1) - \alpha_2 \leq 0.$$

$a_1 > 0$ implies that $\alpha_1^* < 0$ and $\alpha_2^* < 0$. It follows that the prices and outputs are positive for $\alpha_1 > 0$ and $\alpha_2 > 0$. Returning to (40), we see that its first factor in the right hand side is negative. Hence, the price difference is determined by the following condition,

$$sign\left[p_1^{CB} - p_1^{BC}\right] = sign\left[\varphi_p(\alpha_1) - \alpha_2\right], \tag{53}$$

where $\varphi_p(\alpha_1)$ slopes downward by (43). Consequently $\varphi_p(\alpha_1) < 0$ for all $\alpha_1 > 0$ and hence $p_1^{CB} < p_1^{BC}$ always. Retuning to (44), we notice that its first factor in the right hand side is negative where $b_1 - b_2 < 0$. Hence the quantity difference is determined by the following condition,

$$sign[q_1^{CB} - q_1^{BC}] = sign[\alpha_1 - \alpha_1^*] \tag{54}$$

where $\alpha_1 - \alpha_1^* > 0$ for $\alpha_1 > 0$. We therefore have the following result:

[7] The third case of $\Delta = 0$ is eliminated from considerations due to (20) with $n = 2m$.

Proposition 3. *If* $\gamma < 0$, $1 + m\gamma > 0$ and $\Delta > 0$, *then* $p_1^{CB} < p_1^{BC}$ *and* $q_1^{CB} > q_1^{BC}$.

We now return to (45) and examine the profit difference. It can be verified that $0 < \varepsilon^2 < 1$ under $\gamma < 0$ and $1+m\gamma > 0$. The first and second factors of the numerator of (45) are positive. Since $\varphi_\pi^+(\alpha_1)$ is the weighted average of $\varphi^{CB}(\alpha_1)$ and $\varphi^{BC}(\alpha_1)$, the non-negativity of the prices implies that $\varphi_\pi^+(\alpha_1) < \alpha_2$, so the third factor is negative. Hence the profit difference is determined by the sign of the fourth factor

$$sign\left[\pi_1^{CB} - \pi_1^{BC}\right] = sign\left[\varphi_\pi^-(\alpha_1) - \alpha_2\right] \tag{55}$$

where (48) implies that the slope of $\varphi_\pi^-(\alpha_1)$ is ambiguous. We take $\gamma = -0.1$ and $m = 6$, which correspond to the point A in Figure 2, and make the equal-profit curve, $\alpha_2 = \varphi_\pi^-(\alpha_1)$, be positive sloping, as shown in Figure 3. It can be seen that the first quadrant of (α_1, α_2) is divided into two parts: $\pi_1^{CB} < \pi_1^{BC}$ in the lighter-gray region above the curve and the inequality is reversed in the darker-gray region below. If the $\alpha_2 = \varphi_\pi^-(\alpha_1)$ curve slopes downwards, it does not intersect the first quadrant, which graphically means the disappearance of the darker-gray region from Figure 3 and economically means that $\pi_1^{CB} < \pi_1^{BC}$ always, which is the same as the latter half of Theorem 1. Summing up, we have the following results:

Theorem 3 *Assume that* $\gamma < 0$, $1 + m\gamma > 0$ *and* $\Delta > 0$. *(i) If* $\sqrt{\varepsilon}b_1 - b_2 < 0$, *then the positive-sloping* $\alpha_2 = \varphi_\pi^-(\alpha_1)$ *curve divides the first quadrant of the* (α_1, α_2) *space into two parts:* $\pi_1^{CB} < \pi_1^{BC}$ *in the divided region above the curve and* $\pi_1^{CB} > \pi_1^{BC}$ *below. (2) If* $\sqrt{\varepsilon}b_1 - b_2 > 0$, *then the* $\alpha_2 = \varphi_\pi^-(\alpha_1)$ *curve slopes downwards and* $\pi_1^{CB} < \pi_1^{BC}$ *always holds in the first quadrant.*

When we graphically show the profit comparison, the colors are selected in such a way that darker-gray means $\pi_1^{CB} > \pi_1^{BC}$ and lighter-gray means $\pi_1^{CB} < \pi_1^{BC}$. The second half of Theorem 1 indicates that in a mixed duopoly, a price strategy is always more profitable than a quantity strategy if the goods are complements. Theorem 3 implies that this profit dominance with $\gamma < 0$ in a duopoly framework is not necessarily true in the n-firm mixed oligopoly framework if ε is large enough to make $\varepsilon b_1 - b_2 < 0$ whereas the profit dominance is still true if ε is small enough to make $\varepsilon b_1 - b_2 > 0$.

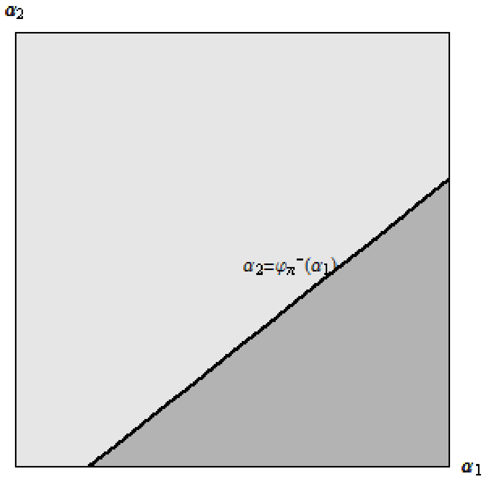

Figure 3. $1 + m\gamma > 0$ and $\Delta > 0$ where $\gamma = -0.1$ and $m = 6$.

We now assume $\Delta < 0$ that leads to $a_1 < 0$. Since the first factors in (35) and (36) are positive, the non-negativity conditions of the prices are

$$\varphi^{CB}(\alpha_1) - \alpha_2 \geq 0 \text{ and } \varphi^{BC}(\alpha_1) - \alpha_2 \geq 0$$

which is the same as the conditions in the case in which the goods are substitutes. In the same way, we obtain

$$sign[p_1^{CB} - p_1^{BC}] = sign[\alpha_2 - \varphi_p(\alpha_1)]$$

$$sign[q_1^{CB} - q_1^{BC}] = sign[\alpha_1^* - \alpha_1]$$

and

$$sign[\pi_1^{CB} - \pi_1^{BC}] = sign[\alpha_2 - \varphi_\pi^-(\alpha_1)],$$

which are the same as (50), (51) and (52). There are, however, differences: The equal-price curve, $\varphi_p(\alpha_1) = \alpha_2$, is negative sloping and the slope of the profit-equal curve $\varphi_\pi^-(\alpha_1) = \alpha_2$, can be of either sign. The division of the feasible region is depicted in Figure 4 in which we take point a (i.e., $\gamma = -0.095$ and $m = 10$). Despite of these differences, the main results is the same as Theorem 2, and summarized as follows.

Theorem 4 *Assume that $\gamma < 0$, $1 + m\gamma > 0$ and $\Delta < 0$, then the $\alpha_2 = \varphi_\pi^-(\alpha_1)$ curve divides the feasible region Ω into two parts: $\pi_1^{CB} > \pi_1^{BC}$ in the divided region above the curve and $\pi_1^{CB} < \pi_1^{BC}$ below.*

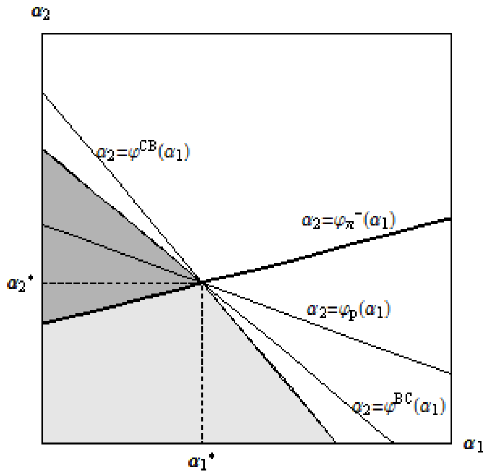

Figure 4. $1 + m\gamma > 0$ and $\Delta < 0$ where $\gamma = -0.095$ and $m = 10$

5.3.2 The case of SOC_2

If $\gamma < 0$ and $1 + (m - 2)\gamma < 0$, then $1 + (m - 1)\gamma < 0$, $2 + (2m - 1)\gamma < 0$ and $2+(2m-3)\gamma < 0$, which, in turn, imply that $b_1 < 0$, $b_2 < 0$ and $b_1 - b_2 = \gamma^3 < 0$. Although $\Delta\alpha_1 < 0$, the sign of Δ is undetermined and so is the sign of α_1. We identity two cases as in the case of SOC$_1$: one is the case where Δ is positive and the other is the case where Δ is negative.

Let us suppose that $\Delta > 0$, then $a_1 < 0$ which leads to $\alpha_1^* > 0$ and $\alpha_2^* > 0$. (35) and (36) imply that the non-negativity conditions for the prices and outputs are

$$\varphi^{CB}(\alpha_1) \geq \alpha_2 \text{ and} \varphi^{BC}(\alpha_1) \geq \alpha_2.$$

The feasible region for the price and outputs are Ω. $\varphi_p(\alpha_1)$ slopes upwards and $\varphi_\pi^+(\alpha_1) > \alpha_2$. It is clear that $\varphi_\pi^-(\alpha_1)$ slopes upwards and is steeper than $\varphi_p(\alpha_1)$. From (40), (44) and (45), we have

$$sign[p_1^{CB} - p_1^{BC}] = sign[\alpha_2 - \varphi_p(\alpha_1)],$$

$$sign[q_1^{CB} - q_1^{BC}] = sign[\alpha_1^* - \alpha_1]$$

and

$$sign[\pi_1^{CB} - \pi_1^{BC}] = sign[\alpha_2 - \varphi_\pi^-(\alpha_1)].$$

These three conditions are the same as (50), (51) and (52), respectively. In consequence the results obtained here are the same as the results obtained in the case of $\gamma > 0$. The profit differences are summarized as follows:

Theorem 5 *If $\gamma < 0$, $1 + (m - 2)\gamma < 0$ and $\Delta > 0$, then the $\alpha_2 = \varphi_\pi^-(\alpha_1)$ curve divides the feasible region Ω into two parts: $\pi_1^{CB} > \pi_1^{BC}$ in the divided region above the curve and $\pi_1^{CB} < \pi_1^{BC}$ below.*

We take $\gamma = -0.6$ and $m = 6$, which correspond to the point B in Figure 2 and depict the division of the feasible region Ω in Figure 4. The similarities between Figure 1 and Figure 5 are clear. $\pi_1^{CB} > \pi_1^{BC}$ in the darker-gray region and $\pi_1^{CB} < \pi_1^{BC}$ in the lighter-gray region. Hence the profit dominance with $\gamma < 0$ obtained in the duopoly framework is not true in the n-firm oligopoly framework.

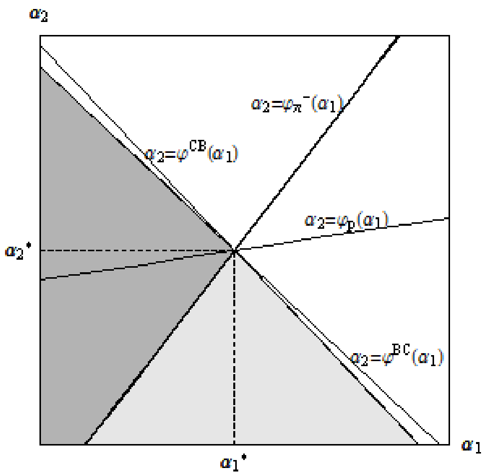

Figure 5. $1 + (m-2)\gamma < 0$ and $\Delta > 0$ where $\gamma = -0.6$ and $m = 6$

In the case of negative Δ, $a_1 > 0$ which leads to $\alpha_1^* < 0$ and $\alpha_2^* < 0$. The non-negativity conditions for the prices and outputs are

$$\varphi^{CB}(\alpha_1) \leq \alpha_2 \text{ and} \varphi^{BC}(\alpha_1) \leq \alpha_2.$$

The first quadrant of the (α_1^*, α_2^*) space is the feasible region for the price and outputs. $\varphi_p(\alpha_1)$ slopes upwards. $\varphi_\pi^+(\alpha_1) < \alpha_2$ for $\alpha_1 > 0$ and $\alpha_2 > 0$. It is clear that $\varphi_\pi^-(\alpha_1)$ slopes upwards and is steeper than $\varphi_p(\alpha_1)$. From (40), (44) and (45), we have

$$sign[p_1^{CB} - p_1^{BC}] = sign[\varphi_p(\alpha_1) - \alpha_2],$$

$$sign[q_1^{CB} - q_1^{BC}] = sign[\alpha_1 - \alpha_1^*]$$

and

$$sign[\pi_1^{CB} - \pi_1^{BC}] = sign[\varphi_\pi^-(\alpha_1) - \alpha_2].$$

The three conditions are the same as (53), (54) and (55), respectively. The results obtained under SOC_1, $\Delta > 0$ and $\sqrt{\varepsilon}b_1 - b_2 < 0$ are the same as the results obtained under SOC_2 and $\Delta < 0$. The profit difference can be summarized as follows:

Theorem 6 *If* $\gamma < 0$, $1 + (m-2)\gamma < 0$ *and* $\Delta < 0$, *then the* $\alpha_2 = \varphi_\pi^-(\alpha_1)$ *locus divides the feasible region* R_+ *into two parts:* $\pi_1^{CB} < \pi_1^{BC}$ *in the divided region above the locus and* $\pi_1^{CB} > \pi_1^{BC}$ *below.*

We take $\gamma = -0.25$ and $m = 6$, which correspond to point b in Figure 2 and depict the division of the feasible region in Figure 4. As far as profit comparison is our concern, the similarity between Figures 3 and 5 is clear: $\pi_1^{CB} < \pi_1^{BC}$ in the lighter-gray region and $\pi_1^{CB} > \pi_1^{BC}$ in the darker-gray region.

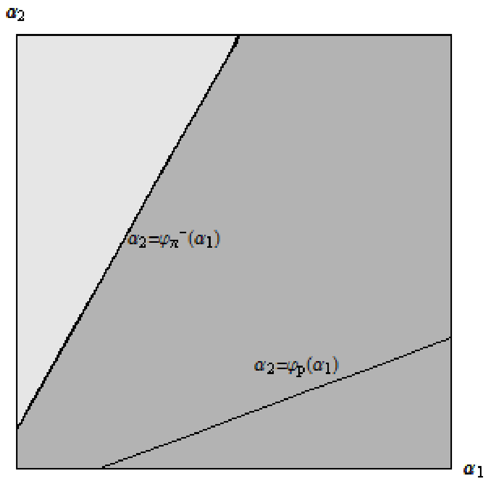

Figure 6. $1 + (m - 2)\gamma < 0$ and $\Delta < 0$ where $\gamma = -0.25$ and $m = 6$.

6 Concluding Remark

In a mixed duopoly with product differentiation, Theorem 1 indicates that a quantity-strategy is more profitable than a price-strategy if the goods are substitutes and the profit dominance is reversed if the goods are complements. Our main aim is to reconsider whether these clear-cut results still hold in a general (n-firm) oligopoly framework or not. Theorems 2, 3, 4, 5 and 6 lead us to the conclusion that the results obtained in the duopoly framework are no longer valid in general oligopoly framework. If the firms can precommit to quantity or price strategy, the dominant strategy depends on the average quality ratio of the goods produced by the quantity-adjusting firms and the goods produced by the price-adjusting firms.

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