Controlling Cournot-Nash Chaos

Akio Matsumoto

Department of Economics
Chuo University

742-1, Higashi-Nakano
Hachioji, Tokyo,192-0393
Japan

Tel: 81-(0)426-74-3351; Fax: 81-(0)426-74-3425
E-mail: akiom@tamacc.chuo-u.ac.jp

June 23, 2004

Abstract

The recently developing theory of non-linear dynamics shows that any economic model can generate complex dynamics involving chaos if its nonlinearities become strong enough. This study constructs a nonlinear Cournot duopoly model, reveals conditions for generating chaos and then considers controlling chaos. The main purpose of this paper is to demonstrate that chaos generated in Cournot competition is in a double bind from the long-run perspective: a firm with a lower marginal production cost prefers a stable (i.e., controlled) market to a chaotic (i.e., uncontrolled) market while a firm with a higher marginal cost prefers the chaotic market.

Acknowledgement

Financial supports from Chuo University (Joint Research Grant 0382) and from Ministry of Education, Culture, Sports, Science and Technology (Grand-in-aid for Scientific Research (B)15330037) are highly appreciated.

1 Introduction

In his seminal paper, Day [1982] has introduced nonlinear dynamics into economics and showed that even traditional economic dynamics models can exhibit complex dynamics involving chaos when nonlinearities become strong enough. Since then, a lot of efforts have been devoted to investigate conditions for the existence of chaotic dynamics in economic dynamic models and to provide fruitful theoretical insights. Among others, it is demonstrated that competitive markets can be chaotic, see Boldrin and Woodford [1990] and Majumdar et al. [2000]. This finding implies that the standard assumptions of economic theory such as convex preferences and production technologies are compatible with chaotic dynamics. In the recent literature, it is also demonstrated that even oligopolistic markets may become chaotic; see Rand [1978], Kopel [1995], Puu [2000] and studies included in Puu and Sushko [2002]. These works indicate that the higher-dimensional deterministic model of a dynamic economy can explain various complex dynamic behavior of the economic variables. More recently, Matsumoto and Nonaka [2004] reveal an interesting characteristics of economic chaos by showing that chaotic dynamics may be profitable from the long-run perspective in two-dimensional output adjustment process of complementary goods.

In spite of these developments, unstable fluctuations have been regarded as unfavorable phenomena in traditional economics that has mainly focused on a stationary state. It is therefore a natural question of whether chaotic behavior can be stabilized or controlled to make the state converging. In fact, the research for such a direction has already begun and there already exist several studies on controlling chaos, see Kopel [1997], Kass [1998], Bala et al. [1998], Mendes and Mendes [2000], to name a few. In the existing literature, however, not much has yet been revealed with respect to the economic implications of generating chaos and controlling chaos. In particular, it has not been determined which is preferable, controlling chaos or generating chaos.

The main purpose of this paper is to consider economic implications of generating chaos as well as controlling chaos in the nonlinear duopoly model developed by Puu [2000]. An equivocal characteristics of economic chaos will be demonstrated. From the long-run point of view, one of the duopolists can be beneficial and the other harmful in the chaotic market. Putting control reverses the situation: the beneficial duopolist in the chaotic market becomes disadvantageous and the harmful firm advantageous in the controlled market. This implies that either way of generating chaos or controlling chaos is unable to make both duopolists happy together.

The paper is organized as follows. Section 2 constructs a simple nonlinear duopoly model. Section 3 examines two statistical properties of chaotic duopoly map; the long-run average behavior and autocorrelation between outputs. Section 4 considers controlling chaos. Section 5 provides summary and concluding remarks.

2 Puu's Model

Rand [1978] shows that a Cournot duopoly model with unimodal reaction function can give rise to chaotic dynamics. Puu [2000] presents a possible economic underpinning which supports the unimodal reaction function. Since we consider controlling chaos in Puu's setting, we recapitulate the fundamental structure of Puu's model in this section.

The market demand is assumed to be isoelastic such that price p is reciprocal to the total demand Q,

$$p = \frac{1}{Q}. \tag{1}$$

There are two firms, denoted by X and Y, producing the amounts of goods x and y with constant marginal costs a and b, respectively. Goods are perfect substitutes so that, provided demand equals supply, the total demand equals the total supplies, $Q = x + y$. Their expected profits become accordingly,

$$\begin{cases} \Pi_x = \dfrac{x}{x + y^e} - ax, \\ \\ \Pi_y = \dfrac{y}{x^e + y} - by, \end{cases} \tag{2}$$

where "e" denotes an expected value. The usual procedure to maximize profit leads to unimodal reaction functions which construct the following dynamic process under the naive expectation formation (i.e., $x^e_{t+1} = x_t$ and $y^e_{t+1} = y_t$),

$$\begin{cases} x_{t+1} = f(y_t), \\ \\ y_{t+1} = g(x_t), \end{cases} \tag{3}$$

where $f(y)$ is the reaction function of firm X and $g(x)$ is the reaction function of firm Y. Both are specified as

$$\begin{aligned} f(y_t) &= \sqrt{\frac{y_t}{a}} - y_t, \\ \\ g(x_t) &= \sqrt{\frac{x_t}{b}} - x_t. \end{aligned} \tag{4}$$

As can be seen in Figure 1 below, $f(y)$ has its maximum value $\frac{1}{4a}$ at $y = \frac{1}{4a}$, and its domain should be restricted to the interval $[0, \frac{1}{a}]$ for nonnegative values of output, and so does $g(x)$ with replacing a with b.[1] When the dynamic process is designed to map the maximum point to an interior point of the interval, it can generate positive productions to both firms. Solving $g(\frac{1}{4a}) \leq \frac{1}{a}$ and $f(\frac{1}{4b}) \leq \frac{1}{b}$ gives the upper and lower bounds of the marginal cost b in terms of the marginal cost a,

$$\frac{4}{25}a \leq b \leq \frac{25}{4}a. \tag{5}$$

If these inequalities are violated, the dynamic process (3) induces negative output values.

As denoted as "C" in Figure 1, the fixed point of this dynamic process, which we call the Cournot point, is the intersection of the reaction curves,

$$x^c = \frac{b}{(a+b)^2} \text{ and } y^c = \frac{a}{(a+b)^2}, \tag{6}$$

where superscript "c" is attached to variables to indicate the one at the Cournot point.[2] Substituting these Cournot outputs in the profit functions (2) gives the profits earned at the Cournot point,

$$\Pi_x^c = \left(\frac{b}{a+b}\right)^2 \text{ and } \Pi_y^c = \left(\frac{a}{a+b}\right)^2. \tag{7}$$

Taking ratios of Cournot outputs (6) and Cournot profits (7) gives

$$\frac{x^c}{y^c} = \frac{b}{a} \text{ and } \frac{\Pi_x^c}{\Pi_y^c} = \left(\frac{b}{a}\right)^2, \tag{8}$$

which we summarize in

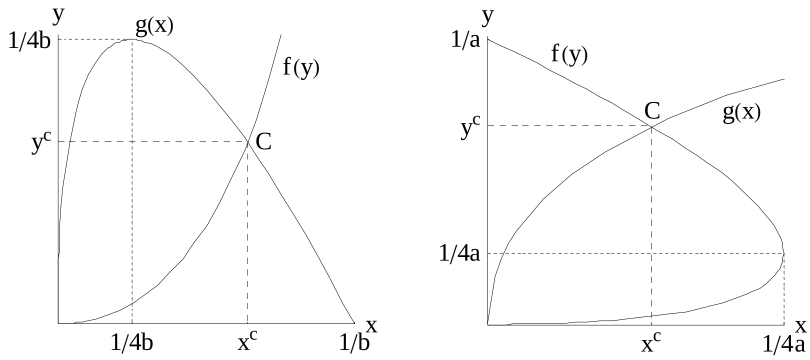
Theorem 1 *At the Cournot point, a firm with the lower marginal cost produces more output and makes more profit than a firm with the higher marginal cost,*

$$a \ \mathsf{T} \ b \textit{ implies } x^c \ \mathsf{S} \ y^c \textit{ and } \Pi_x^c \ \mathsf{S} \ \Pi_y^c.$$

[1] The ratio of the horizontal axis and the vertical axis in Figure 1 is appropriately adjusted in order to emphasize the mound-shape of reaction functions.

[2] The dynamic process has two fixed points: one is the trivial point, $(0, 0)$, and the other is a non-trivial point, (x^c, y^c). Our concern is on the nontrival point and thus no further consideration is given to the trival point.

We call a firm with the lower marginal cost an *efficient* firm and one with the higher marginal cost an *inefficient* firm. Theorem 1 implies the natural result that an efficient firm dominates the market and is more profitable than an inefficient firm at the Cournot point. Firm X is efficient and firm Y is inefficient in Figure 1(A) where $x^c > y^c$ and $\Pi_x^c > \Pi_y^c$ while the relative efficiency is reversed in Figure 1(B) where $x^c < y^c$ and $\Pi_x^c < \Pi_y^c$.



(A) $a = 1,\ b = 4$. (B) $a = 1,\ b = 0.2$.

Figure 1. Reaction functions and Cournot points.

To examine the stability of the Cournot point, we make a linear approximation of the dynamic process at the Cournot point and construct the Jacobi matrix,

$$J_C = \begin{pmatrix} 0 & \dfrac{b-a}{2a} \\ \dfrac{a-b}{2b} & 0 \end{pmatrix}, \tag{9}$$

where the trace and determinant of J_C are

$$tr\, J_C = 0 \text{ and } det\, J_C = \frac{(a-b)^2}{4ab}.$$

It can be checked that the conditions (S2) and (S3) of Stability Result presented in Appendix A are satisfied. Thus loss of stability occurs when the absolute value of a eigenvalue becomes unity, that is, when $det\, J_C = 1$ or $(a-b)^2 = 4ab$ holds. The dynamic process, therefore, is stable if the marginal cost of firm Y falls inside the interval bounded by the two roots of the last equation,

$$(3 - 2\sqrt{2})a < b < (3 + 2\sqrt{2})a. \tag{10}$$

Since we are interested in controlling chaotic oscillations, we first derive the condition under which the Cournot point is locally unstable. Notice that if $a = b$, then both eigenvalues equal zero, so stability necessarily holds. So we might assume that $a \neq b$. Taking into account the upper or lower bound for economically feasible (that is, non-negative) production level given in (5) and the stable interval defined in (10), we find that the unstable condition can be expressed by the ratio of the marginal production costs and is split into two cases according to whether a is larger or smaller than b. Since one condition is derived from the other (that is, two are reciprocal), we can make, without a loss of generality, the following assumption:

Assumption. $b > a$ and $(3 + 2\sqrt{2}) < \dfrac{b}{a} \leq \dfrac{25}{4}$.

Under Assumption, the Cournot point is unstable so that trajectories starting from any points of a neighborhood of the Cournot point move away. However the nonlinearity of the dynamic process prevents trajectories from globally diverging from the Cournot point. Trajectories are bounced back to the neighborhood soon or later but move away again. This process is repeated again and again. That is, the dynamic process does not converge to the Cournot point but keeps fluctuating within a limited region. Figure 2 depicts a bifurcation diagram with respect to the cost ratio under the condition that the $b/a > 1$ as set in Assumption. It can be seen that as the ratio increases from $3 + 2\sqrt{2}$, the Cournot point becomes unstable and then bifurcates to a stable period-2 cycle, generates the period doubling sequence and leads to a two-piece chaotic attractor, which finally merges to a one-piece attractor as the ratio approaches its upper bound, $\frac{25}{4}$.

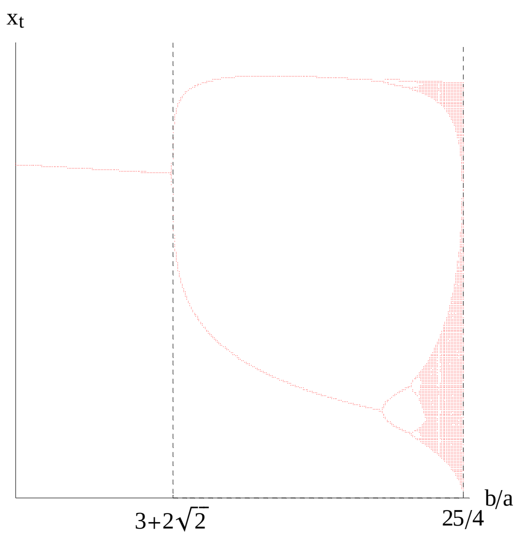


Figure 2. Bifurcation Diagram.

3 Statistical Properties

It is well-known that chaotic dynamics has two salient features: sensitivity to the initial conditions and irregularity of trajectory. We examine these features of chaotic duopolies in this section. The former feature implies that even a slightly different choice of initial conditions can drastically alter the whole future behavior of the trajectories. Therefore it is meaningless to investigate an individual chaotic trajectory. Instead, we examine the long-run average behavior of chaotic trajectories in the first half of this section. The latter feature implies that it is difficult to make precise expectations about future values of variables along chaotic trajectories. Alternatively, forecasting errors are inevitable in every period. A natural consequence is to reject the current predictor which shows systematic errors and then to alter the formation of expectations. Nevertheless such alternations are not adequately explored in Puu's setting. We thus consider "consistency" of the naive expectations in the second half of this section.

3.1 Long-run Average Return

A consequence of the sensitivity is that even if two trajectories start from similar initial conditions, they will be apart sooner or later and move in such a different and complicated way that it is difficult to predict their long-run behavior. One way to characterize such chaotic dynamics is to turn our attention to statistical or long-run behavior. Under rather weak mathematical condition, frequencies of chaotic trajectories may converge to a stable density function. Once the explicit form of the density function is constructed, it can be shown that for any continuous function of a variable, its time average along the chaotic trajectory is equal to its space average.[3] This implies that it is possible to analytically calculate the long-run average behavior which is independent from a choice of initial points. However, it is, in general, difficult to construct such an explicit form of the density function. We thus numerically calculate the long-run average behavior over sufficiently long period of

[3] Let $x_{t+1} = \theta(x_t)$ be a dynamical process where x_t is a variable at time t. Suppose the frequencies of the trajectory $\{x_t\}_{t=0}^{\infty}$ converges to a density function φ. Then for any continuous function f and for any initial point x_0,

$$\lim_{T->\infty} \frac{1}{T} \sum_{t=0}^{T-1} f(\theta^t(x_0)) = \int f(x)\varphi(x)dx,$$

where $\theta^t = \theta^{t-1} \cdot \theta$ and $\theta^0 = 1$. See, for example, Chapter 8 of Day [1994] for more details.

time as a proxy for the analytical value of the long-run average behavior.

Given the dynamic process (3) with Assumption, the average profit of time-serise with T periods is defined,

$$\bar{\Pi}_i = \frac{1}{T}\sum_{t=0}^{T-1} \Pi_i(x_t, y_t), \quad i = x, y.$$

Figure 3 illustrates the results of the numerical simulations of the long-run average profits of firm X (left) and firm Y (right) (denoted as $\bar{\pi}_x$ and $\bar{\pi}_y$) against variations of the marginal cost ratio. For comparison, graphs of the Cournot profits are also depicted there. In each illustration, the ratio is increased in steps of 0.002 from 5.75 to 6.25, and for each of these ratio values, the average is calculated from the last $1,000$ out of $5,000$ iterations. It can first be observed that the average profit for each firm is identical to its Cournot profit when the ratio is less than $3 + 2\sqrt{2}(\simeq 5.8)$ (that is, the Cournot point is stable). It is also clear that the average profit is less than the Cournot profit for firm X (i.e., the efficient firm), and greater for firm Y (i.e., the inefficient firm) when the ratio is greater than $3 + 2\sqrt{2}$ but less than 6.25. The numerical simulations indicate the following results:

Theorem 2 *When the Cournot point is unstable, the long-run average profit of the efficient firm is less than the Cournot profit while the long-run average profit of the inefficient firm is more than the Cournot profit,*

$$\frac{b}{a} \geq 3 + 2\sqrt{2} \textit{ implies } \bar{\Pi}_x \leq \Pi_x^c \textit{ and } \bar{\Pi}_y \geq \Pi_y^c.$$

Theorem 1 and Theorem 2 imply that efficient firm X dominates the market but is harmful in the unstable market in the sense that its long-run average profit is less than its Cournot profit. On the other hand, inefficient firm Y has only a small share of the market but is beneficial in the sense that its long-run average profit is higher than its Cournot profit.

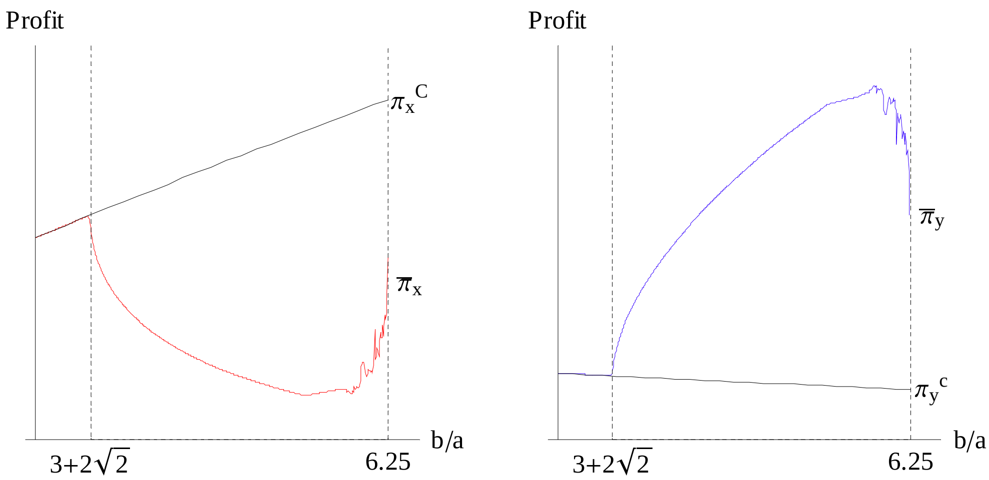


Figure 3. Long-run average profits and Cournot profits.

3.2 AutoCorrelations

We apply a formation of naive expectations to build dynamic process (3). The formation is given by $x^e_{t+1} = x_t$ and $y^e_{t+1} = y_t$, that is, today's expected value is equal to yesterday's realized value. It is the simplest formation of more general backward-looking expectations where the expected price is a function of the past prices. A conceptual problem with native expectations is that it may lead to systematic forecasting errors, that is, firms are making wrong expectations along chaotic trajectories. On the other hand, it has been shown that irregularity of chaotic trajectories is indistinguishable from randomness of a stochastic process if autocorrelations between outputs are zero at all lags (see, for example, Sakai and Tokumaru [1980]). In this case, there is no reason to switch to another formation of expectations. In the simple setting of a cobweb economy, Hommes [1996] explores whether a chaotic cobweb model with the backward-looking expectations can have such statistical property, and numerically shows that under the naive expectations, prices are correlated with past prices if the supply function is monotonic and are uncorrelated if the supply function is non-monotonic (e.g., piecewise linear). On the contrary, we show that the autocorrelation coefficients of the naive expectations are significantly different from zero even if the dynamic process is nonlinear enough to produce chaotic fluctuations

Following Hommes [1996], we define *consistency* of expectations by means of the autocorrelation function (ACF) of expectation errors. Expectation errors are given by

$$e_t = x_t - x^e_t.$$

The (empirical) autocorrelation coefficients ρ_k of expectation errors are defined as

$$\rho_k = \frac{c_k}{c_0}, \quad -1 \leq \rho_k \leq 1,$$

with

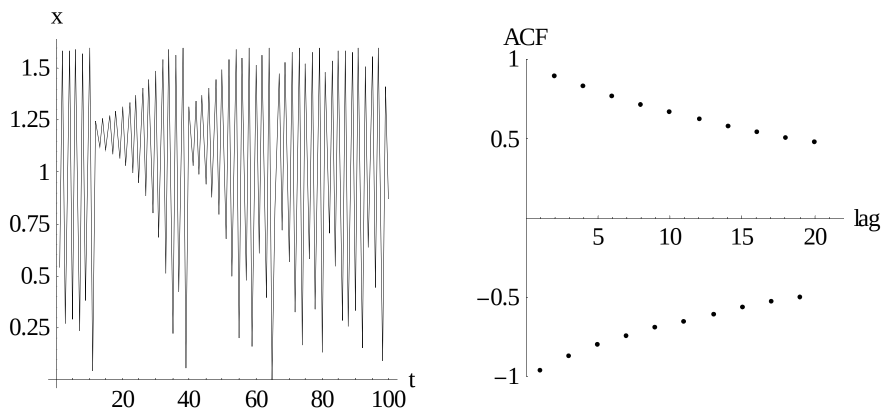
$$\bar{e} = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} e_t,$$

$$c_k = \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N} (e_t - \bar{e})(e_{t+k} - \bar{e}), \quad k \geq 0.$$

An expectation formation is called *consistent* if the autocorrelation coefficients of the expectation errors are zero for all $k \geq 1$; *weakly consistent* if there exists a $K \geq 2$, such that the autocorrelation coefficients of the expectation errors are zero for all $k \geq K$; *inconsistent* if it is not weakly consistent. Consistent expectations mean that firms are not able to distinguish chaotic fluctuations from random fluctuations so that they have no reason to change their beliefs. On the other hand, inconsistent expectations mean that they can make distinction and therefore change expectation formations accordingly.

The dynamic process with the naive expectations (3), which shows periodic cycles and irregular behavior as seen in Figure 2, consists of two independent iterative process, $f(y_t)$ and $g(x_t)$. Iterations produced by the two-dimensional map are essentially the same as the ones by the combined one-dimensional map derived by substituting one into the other. Only difference is that one period of the combined map is equivalent to two periods in terms of the originally introduced lag. We investigate forecast errors generated by the combined one-dimensional map $f(g(x))$ with the initial condition $x_0 = 0.1$ and parameters $a = 0.1$ and $b = 0.625$. Figure 4 gives the simulation results. Chaotic time series of output x is depicted in the left and its sample autocorrelation coefficients at the first 20 lags in the right. A period-doubling bifurcation diagram depicted in Figure 2 implies that output trajectories converge to a 2-piece chaotic attractor. The chaotic time series jumps back and forth between two intervals. Accordingly, the corresponding ACF of expectation errors has "period" 2, with strong negative autocorrelations at odd lags and strong positive autocorrelations at even lags. Therefore the numerical simulations indicate the following:

Theorem 3 *Naive expectations are inconsistent so that there exists systematic forecasting errors in the dynamics process (3).*



(A) Chaotic Trajectory (B) Auto Correlation

Figure 4. Output fluctuations and corresponding expectation errors.

4 Controlling Chaos

In this section, we apply two distinctive control methods, *the adaptive control method* advocated by Huang [2002] and a generalized version of *the pole-placement method* developed by Ueta and Kawakami [1995], for the output adjustment process to stabilize the Cournot point embedded in the chaotic attractor. Both methods are known for a long time in applied mathematics The adaptive control method, as called the Mann's iteration method, has been investigated very intensively in the numerical analysis literature. The pole-placment method is also a definite stabilizing approach in system design, and it has many important applications such as constructing observers.

4.1 Adaptive Control Method

Naive expectations are inconsistent as it has been shown in the last section. Sooner or later, each of duopolists might discover significant autocorrelation coefficients by observing time series data and may change his beliefs on the opponent behavior. In this subsection, we suppose that duopolists change their beliefs to adaptive expectations from naive expectations. Optimal output in the next period is determined as a weighted average of current period expectation of the rival firm and current period's actual output. The dynamic process (3) turns to be

$$\begin{cases} x_{t+1} = (1-\lambda_x)x_t + \lambda_x f(y_t), \\ y_{t+1} = \lambda_y g(x_t) + (1-\lambda_y)y_t, \end{cases} \tag{11}$$

where λ_x and λ_y are adjustment parameters of firms X and Y. Since the modified (adaptive) process (11) is back to the original (naive) process (3) for $\lambda_x = \lambda_y = 1$ and gives rise to no dynamics for $\lambda_x = \lambda_y = 0$, the adjustment parameters are assumed to take intermediate values between zero and unity.

It is well-known that adaptive expectations are an another example of backward-looking expectations and useful to stabilize a unstable economic process. It can be shown that adaptive expectations by the firms can be described by a four-dimensional system, where the eigenvalues of the Jacobian are zero with multiplicity 2, and the eigenvalue of the Jacobian of (11). In this sense the two processes can be considered equivalent to each other as shown in Appendix B. Puu [2000] also extends his naive expectations model to the adaptive expectation model. However, he draws his attentions to a possibility that the adaptive process can have fractal attractors in two dimensional embedding parameter space. On the other hand, Huang [2001] generalizes the concept of adaptive expectations to adaptive adjustment mechanism in such a way that economic variables are directly adjusted adaptively to equilibrium point. Following his formulation, we consider an adaptive feedback mechanism as a means of controlling the unstable chaotic process.

The goal of implementation of the adaptive adjustment is to stabilize the chaotic dynamic process directly through the adjustment parameter of each duopolist. We retain Assumption so that the Cournot point is unstable in the naive process (3). It can be checked that the adaptive process (11) has exactly the same fixed points as the naive process (3). In the following, we are interested in stabilizing the unstable dynamic process by using perturbations of control parameters.

The Jacobi matrix of the adaptive process (11) evaluated at the Cournot point is

$$J_A = \begin{pmatrix} 1-\lambda_x & \dfrac{\lambda_x(b-a)}{2a} \\ \dfrac{\lambda_y(a-b)}{2b} & 1-\lambda_y \end{pmatrix}.$$

where a trace and determinant of J_A are

$$trJ_A = 2 - (\lambda_x + \lambda_y),$$

$$det\mathbf{J}_A = (1-\lambda_x)(1-\lambda_y) + \frac{\lambda_x\lambda_y(a-b)^2}{4ab}.$$

According to Stability Result in Appendix A, the domain of stability of equilibria is determined by the following three conditions,

$$\begin{cases} det\mathbf{J}_A < 1 & \Rightarrow \lambda_y > \dfrac{4c\lambda_x}{(1+c)^2\lambda_x - 4c}, \\ det\mathbf{J}_A > trJ_A - 1 & \Rightarrow \dfrac{(1+c)^2}{4c}\lambda_x\lambda_y > 0, \\ det\mathbf{J}_A > -trJ_A - 1 & \Rightarrow 2(2-\lambda_x-\lambda_y) + \dfrac{(1+c)^2}{4c}\lambda_x\lambda_y > 0, \end{cases} \tag{12}$$

where $c = \frac{b}{a}$ is the marginal production cost ratio. Since we assume that the marginal costs are positive, and the adjustment parameters are between zero and unity, the second and third inequalities of (12) are always true. That is, there are no situations in which iterative dynamics either diverges or generates the period-doubilng bifurcation. Thus the domain of stability of the Cournot point is defined only by the flutter boundary, $\det \mathbf{J}_A = 1$, and depicted as a gray-coloured region in Figure 5. It is divided into three parts, a light-gray region and two dark-gray regions by the steeper real line (i.e., $\lambda_y = c\lambda_x$) and flatter real line (i.e., i.e., $\lambda_y = \frac{1}{c}\lambda_x$) on which the discriminant of the characteristic equation is zero. Eigenvalues of J_A are real for parameters in the dark-gray regions and complex for the light-gray region. Thus a trajectory monotonically (oscillatory) converges to the Cournot point when a pair of adjustment coefficients are in the dark (light)-gray region. The naive process, which is unstable under Assumption, can be stabilized by appropriate choice of the adjustment coefficients.

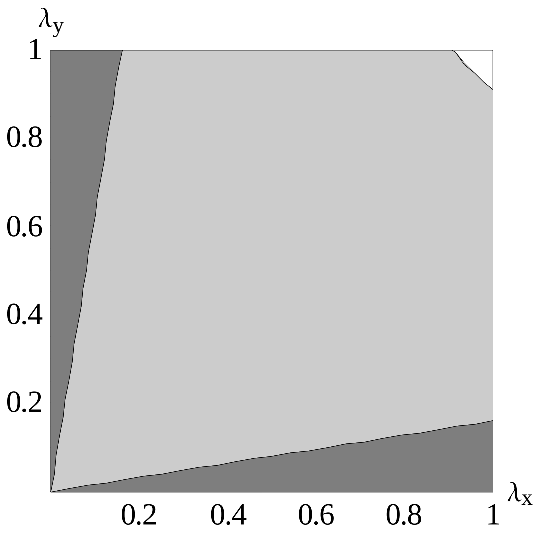


Figure 5. Domain of Stability with $c = \frac{25}{4}$.

The adaptive adjustment cannot fully control chaotic trajectories when the adaptive coefficient are close to 1. In fact, the white region located at the north-east part of Figure 5 is the unstable region which is separated from the stable region by the $\det J_A = 1$ locus. The locus passes through the north-east corner $(1, 1)$ when $c = 3 + 2\sqrt{2}$ for which the loss of stability occurs, shifts upward from the corner when $c < 3+2\sqrt{2}$ (alternatively, the unstable region disappears) and shifts downward when $c > 3 + 2\sqrt{2}$ (the unstable region becomes larger). Since $c = 6.25$ is the maximum value for the original process under economic feasibility, the white region in Figure 5 reaches the maximum size. Although it is only a small part of the whole region, its existence indicates incapability or limitation of the adaptive adjustment mechanism.

We performed two numerical simulations to see how adaptive adjustment controls the unstable Cournot point. Parameter values were fixed as $a = 1$ and $b = 6.25$ (i.e., $c = 6.25$) henceforth. We then considered different values of the adjustment coefficients. It was shown that a chaotic trajectory could be stabilized in the first simulation and could not in the second simulation. In the following figures, "red" means a trajectory generated by the original (naive) process and "blue" a trajectory by the controlled (adaptive) process.

We selected $\lambda_x = \lambda_y = 0.8$ in the first simulation. In the left part of Figure 6 in which return maps are depicted, we observe that two trajectories start at the same initial point, $x_0 = 0.01$ and $y_0 = g(x_0)$, and the blue trajectory converges oscillatory to the Cournot point while the red trajectory keeps fluctuating. In the right part of Figure 6 in which the corresponding time series are presented, we see that the blue trajectory almost reaches the Cournot point a little bit more than 20 periods of time.

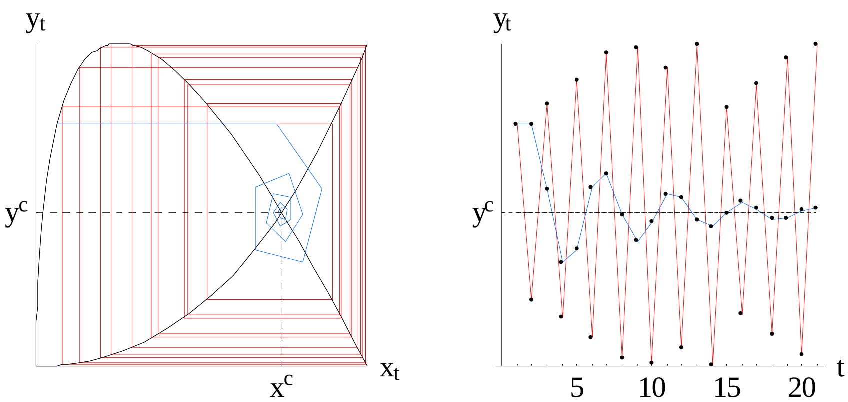


Figure 6. Example of controlled trajectory.

We increased the value of the adjustment parameters to $\lambda_x = \lambda_y = 0.96$ in the second simulation. In Figure 7 in which two return maps are illustrated, we can see that both trajectories start at the same initial point as in the first simulation and the red trajectory on the right part of the figure keeps fluctuating while the blue trajectory does not converge to the Cournot point either but approaches a quasi-periodic cycle. This is an example of uncontrollable chaotic behavior.

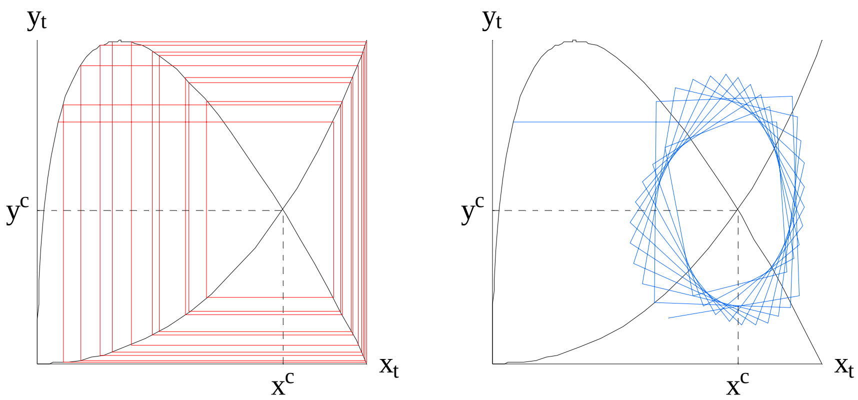


Figure 7. Example of uncontrollable trajectory.

The adaptive control can be considered as a "private" or "individual" method to control a chaotic market. Each firm has a wide spectrum of the control power. A larger adjustment coefficient means to place an emphasis on the current profit maximizing output in order to determine the output in the next period, while a smaller one on the current realized output. Depending on the degree of emphasis, firms can gradually approach the Cournot point in either a monotonic or oscillatory way. As shown in the previous section, firms are able to learn by observing actual output time series that they make forecasting errors and thus have incentives to change their expectation formations. The shape of the stability domain illustrated in Figure 5 implies that the chaotic market can become controllable through individual efforts by changing naive expectations to adaptive expectations. As it was also shown in the previous section, the efficient firm is harmful and the inefficient firm is beneficial in the chaotic market. Since it prefers the chaotic market, the inefficient firm possibly continues to form naive expectations (that is, $\lambda_y = 1$). Even so, the efficient firm alone can stabilize the chaotic market if it selects the adjustment coefficient sufficiently small. In particular, the market is stabilized for any pair of the adjustment coefficients such that $\lambda_y = 1$ and $\lambda_x < \frac{4c}{(1-c)^2}$ for which the upper horizontal line crosses the $detJ = 1$ locus shown in Figure 5.

4.2 Pole Placement Method

In this subsection, we apply the Ueta-Kawakami method for fully stabilizing the unstable Cournot point. It is a generalized version of the RGOD method,[4] a conventional state feedback method to control chaos. A general idea is to put control when a trajectory wandering in the chaotic attractor gets close to a Cournot point. In particular, we stabilize the Cournot point by using perturbations of parameters as the control input.

In order to apply linear control theory, we linearize the dynamic process (3) in a neighborhood of the Cournot point with respect to its variables and parameters:

$$\begin{pmatrix} \Delta x_{t+1} \\ \Delta y_{t+1} \end{pmatrix} = A \begin{pmatrix} \Delta x_t \\ \Delta y_t \end{pmatrix} + B \begin{pmatrix} \Delta a \\ \Delta b \end{pmatrix}, \tag{13}$$

where Δ indicates a difference of a variable or parameter from the corresponding stationary value. In (13), A is the Jacobi matrix, which is identical to J_C introduced in Section 2,

$$A = \begin{pmatrix} 0 & \dfrac{b-a}{2a} \\ \dfrac{a-b}{2b} & 0 \end{pmatrix}, \tag{14}$$

and $B = (B_a\ B_b)$ shows the effect of the variations of parameters a and b on the state variables, which is given by

$$B_a = \begin{pmatrix} -\dfrac{1}{2a(a+b)} \\ 0 \end{pmatrix} \text{ and } B_b = \begin{pmatrix} 0 \\ -\dfrac{1}{2b(a+b)} \end{pmatrix}. \tag{15}$$

For stabilizing the unstable Cournot point, we construct the state feedback

$$\begin{pmatrix} \Delta a \\ \Delta b \end{pmatrix} = C \begin{pmatrix} \Delta x_t \\ \Delta y_t \end{pmatrix}, \tag{16}$$

where $C = (c_{ij})$ $(i = a, b,\ j = 1, 2)$ is a constant matrix to be determined later. Substituting (16) into (13) yields the following homogeneous system

$$\begin{pmatrix} \Delta x_{t+1} \\ \Delta y_{t+1} \end{pmatrix} = J_P \begin{pmatrix} \Delta x_t \\ \Delta y_t \end{pmatrix} \text{ with } J_P = A + BC. \tag{17}$$

Consequently, the corresponding characteristic equation is

$$|J_P - \mu I| = 0 \tag{18}$$

[4]RGOD stands for Romeiras, Grebogi, Ott and Dayawansa. See their paper, Romeiras et al. [1992].

It is now clear that the system can be stabilized if a suitable gain matrix C can be found such that the Jacobi matrix, J_P, has eigenvalues inside the unit circle. Thus the stabilization problem of an unstable Cournot point of system (3) is reduced to the well-known pole assignment problem of linear control theory.

We first check whether the system is controllable (i.e., whether such C can be found), otherwise choosing control gain is meaningless. The eigenvalue placement theorem of linear control theory implies that if, in our case, $rank(B_a \mid AB_a) = 2$ or $rank(B_b \mid AB_b) = 2$, then (3) is stabilizable.[5] It follows from (14) and (15) that

$$\begin{aligned} rank\,(B_a \mid AB_a) &= rank \begin{pmatrix} -\dfrac{1}{2a(a+b)} & 0 \\ 0 & -\dfrac{a-b}{4ab(a+b)} \end{pmatrix} \\ &= 2, \end{aligned}$$

and

$$\begin{aligned} rank\,(B_b \mid AB_b) &= rank \begin{pmatrix} 0 & \dfrac{a-b}{4ab(a+b)} \\ -\dfrac{1}{2b(a+b)} & 0 \end{pmatrix} \\ &= 2. \end{aligned}$$

Therefore the system is controllable for either parameter, a or b.

If we consider a as a control parameter,[6] we can construct the state feedback,

$$\Delta a = \begin{pmatrix} c_{a1} & c_{a2} \end{pmatrix} \begin{pmatrix} \Delta x_t \\ \Delta y_t \end{pmatrix}. \tag{19}$$

The characteristic equation of controlled system is

$$\begin{aligned} \mid A + B_a C - \mu I \mid &= \begin{vmatrix} -\dfrac{c_{a1}}{2a(a+b)} - \mu & \dfrac{(b^2-a^2)-c_{a2}}{2a(a+b)} \\ \dfrac{a-b}{2b} & -\mu \end{vmatrix} \\ &= \mu^2 - trJ_P\mu + det\mathbf{J}_P = 0, \end{aligned}$$

where

$$trJ_P = -\frac{c_{a1}}{2a(a+b)} \text{ and } det\mathbf{J}_P = \frac{(a-b)(b^2-a^2-c_{a2})}{4ab(a+b)}.$$

[5] See Theorem 1 of Ueta and Kawakami [1995].

[6] Choosing b as the control parameter generates qualitatively the same result.

If trJ_P and $det\mathbf{J}_P$ are in the shaded triangular region shown in Figure A of the Appendix, the module of the eigenvalues is less than unity and thus the controlled system becomes stable. The control c_{ai} can be solved by the characteristic equation. In the special case of $trJ_a = det\mathbf{J}_a = 0$ (i.e., the dean beat control), control parameters are given by

$$c_{a1} = 0 \text{ and } c_{a2} = b^2 - a^2. \tag{20}$$

Thus the controlled system is

$$\begin{cases} x_{t+1} = \sqrt{\dfrac{y_t}{a + v(y_t)}} - y_t, \\ \\ y_{t+1} = \sqrt{\dfrac{x_t}{b}} - x_t, \end{cases} \tag{21}$$

where, given arbitrary small value of ϵ,

$$v(y_t) = \begin{cases} (b^2 - a^2)(y_t - y^c) & \text{if } \mid y_t - y^c \mid \leq \varepsilon \\ \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

The small value of ϵ means that the application of linear control theory succeeds only in a sufficiently small neighborhood around the Cournot point. According to (22), the controlled system turns on only if values of (x_t, y_t) are inside the ϵ-neighborhood of the Cournot point and the original system turns on otherwise.

We also present two numerical simulations in which the unstable Cournot point is stabilized through the feedback control. Setting $\epsilon = 0.01$, we choose different initial points on the reaction curve of firm Y; $x_0 = x^c - 0.095$ and $y_0 = g(x_0)$ in the first simulation and $x_0 = x^c - 0.011$ and $y_0 = g(x_0)$ in the second simulation. As in the previous cases, the red trajectory is the uncontrolled one and the blue is the controlled one. Figure 8 illustrates the return maps and time series of the first simulation. There we superimpose the blue trajectory on the red trajectory. In the right part of the figure, we can see only the blue trajectory in the first 6 periods, which means that two trajectories start from the same initial point and take exactly the same values in the first 6 periods. In the left part it can be seen that the blue trajectory is inside the ϵ-neighborhood (that is, in the ellipse circle) in the 6th period. The dynamic system is now switched to the controlled one (21) from the uncontrolled one (3). The effect of control is clearly seen in the right part of Figure 8. The controlled blue trajectory jumps to the Cournot point in the next period and stays there since then while the uncontrolled red trajectory moves oscillating away from the Cournot point. Figure 9 presents uncontrolled and controlled trajectories of the second simulation. Since the blue trajectory is seen to converge to the Cournot point, the control also works effectively in the second simulation. However, on the contrary to the first simulation, it takes longer periods of time until convergence is accomplished. In fact, it takes more than 70 periods for the controlled system to be turned on. These simulations show that although almost all initial conditions lead to controllable trajectories, the time until control is achieved has initial-point dependency.

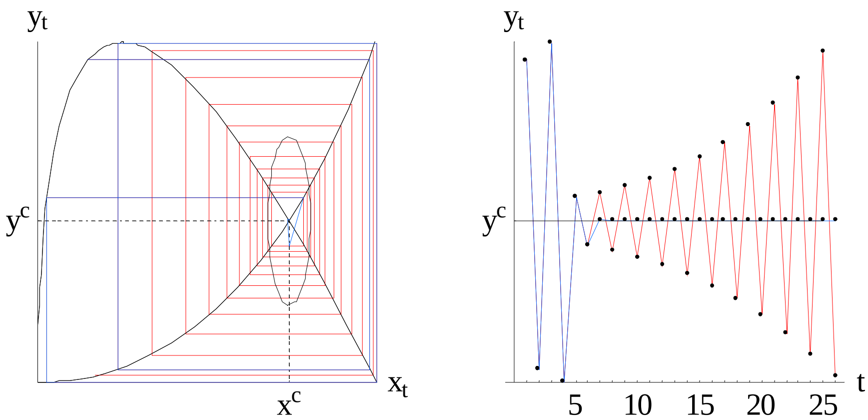


Figure 8. Rapid Control.

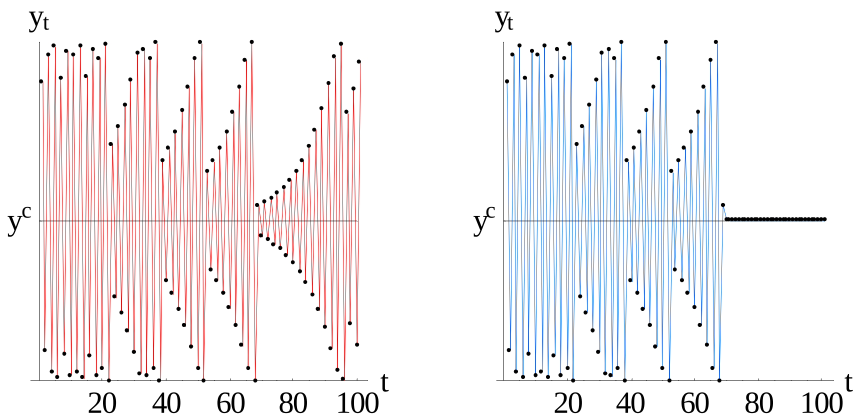


Figure 9. Slow Control.

The feedback control may be thought as a "public" method in the form of government interventions to stabilize a chaotic market. As defined in (21), the control directly affects the marginal cost of firm X. The form of intervention is, however, critically dependent on the behavior of firm Y. Since $b > a$ due to Assumption, the sign of $v(y_t)$ can be either positive or negative according to y_t is larger or smaller than y^c. The government, therefore, introduces a production tax aiming at decreases in firm X's production if firm Y over-produces its output (i.e., $v(y) > 0$) and gives a production subsidy aiming at increases firm X's production if firm Y under-produces its output (i.e., $v(y) < 0$). Admittedly, the feedback control in terms of economic policy is restrictive in the sense that the government is required to possess enormous amount of information on not only firms' production level but also on the evolution of the economy to make an accurate decision. Further, we naturally raise the policy question of whether government assistance should be targeted toward a weaker firm or a stronger firm. In the numerical example where $b > a$ is assumed, firm X is thought to be stronger because it dominates the market share and earns the larger profits. It may seem paradoxical that the government is more favour for the stronger firm and stabilizes the chaotic market at the expense of profits of the weaker firm. However, the example does help us to focus on an issue which has not been adequately explored: if the chaotic market is controllable, which firms should be more favoured. Finally we mention that the controllability of the production level in N-firm oligopolies linear price function has been early examined in Okuguchi and Szidarovsky [1999]

5 Summary and Concluding Remarks

Our purpose of this study is to consider economic implications of generating as well as controlling chaotic fluctuations in economic models. To this end, we use the nonlinear duopoly model developed by Puu [2000] and carry out the following results.

(I) A firm with a lower marginal cost of production (i.e., efficient firm) dominates the market and yields more profits than a firm with a higher marginal cost (i.e., inefficient firm) at the Cournot point.

(II) In case of $b > a$, economically feasible periodic as well as aperiodic fluctuations involving chaos are evident when the ratio of marginal costs are higher than $3 + 2\sqrt{2}$ but less than or equal to $\frac{25}{4}$.

(III) From the long-run point of view, the efficient firm is harmful and the inefficient firm is beneficial in the chaotic market because the long-run average profit of the former is less than its Cournot profit, and the one of the latter is larger.

(IV) The naive expectation formation is inconsistent in the sense that the autocorrelations between outputs are non-zero so that firms have an incentive to change their expectation formations.

(V) When firms adaptively forecast their rival's behavior, the chaotic market can be stabilized unless their adjustment parameters are close to unity.

(VI) Economic policy in the form of government interventions (subsidy or tax on production) can fully stabilize the chaotic market.

(I) and (III) imply that the efficient firm prefers the stable market while the inefficient firm prefers the chaotic market. (V) and (VI) imply that the chaotic market can be stabilized by either individual effort or public effort. Chaos in Cournot competition is in a double bind. In other words, generating chaos is favoured for the inefficient firm and not for the inefficient firm; controlling chaos is favoured for the efficient firm and not for the inefficient firm. In short, a loser now in one market will be later to win in the other market.

Appendix A

In this appendix, we derive stability results on a two-dimensional discrete map,

$$\begin{cases} x_{t+1} = \phi(x_t, y_t), \\ \\ y_{t+1} = \varphi(x_t, y_t), \end{cases} \tag{A-1}$$

where x_t and y_t represent the components of the iteration process at time t. We assume that the map has an equilibrium point, x^* and y^*. To investigate its stability, we make a linear approximation in the neighborhood of the equilibrium point,

$$\begin{pmatrix} \Delta x_{t+1} \\ \Delta y_{t+1} \end{pmatrix} = J \begin{pmatrix} \Delta x_t \\ \Delta y_t \end{pmatrix} \tag{A-2}$$

where $\Delta x_t = x_t - x^*$ and $\Delta y_t = y_t - y^*$ denote differences of variables at time t from the equilibrium point, and J is the Jacobi matrix evaluated at the equilibrium point,

$$J = \begin{pmatrix} \dfrac{\partial\phi(x^*, y^*)}{\partial x} & \dfrac{\partial\phi(x^*, y^*)}{\partial y} \\ \dfrac{\partial\varphi(x^*, y^*)}{\partial x} & \dfrac{\partial\varphi(x^*, y^*)}{\partial y} \end{pmatrix}. \tag{A-3}$$

The eigenvalues of the Jacobi matrix are the solutions of the characteristic equation,

$$\lambda^2 - trJ\lambda + \det J = 0 \tag{A-4}$$

where

$$trJ = \frac{\partial\phi(x^*, y^*)}{\partial x} + \frac{\partial\varphi(x^*, y^*)}{\partial y},$$

$$\det J = \frac{\partial\phi(x^*, y^*)}{\partial x}\frac{\partial\varphi(x^*, y^*)}{\partial y} - \frac{\partial\varphi(x^*, y^*)}{\partial x}\frac{\partial\phi(x^*, y^*)}{\partial y}.$$

It has been demonstrated that the equilibrium (x^*, y^*) is asymptotically stable if both eigenvalues have modulus smaller than one,

$$|\lambda_1| < 1 \text{ and } |\lambda_2| < 1. \tag{A-5}$$

Due to the outcome of the general Routh-Hurwitz stability condition, we have the following result which ensures these inequality conditions. Since these results are straightforward, we still omit a proof just although we often use these in the analysis.

Stability Result: *The polynomial* $\lambda^2 - trJ\lambda + \det J$ *has roots less than unity in absolute value if and only if*

$$\begin{array}{ll} (S1) & \det J < 1, \\ (S2) & \det J > trJ - 1 \\ (S3) & \det J > -trJ - 1. \end{array}$$

Stability Result is graphically summarized in Figure A in which the shaded triangular region shows the stable region satisfying the conditions $(S1)$, $(S2)$ and $(S3)$. If trJ and $detJ$ of (A-4) are placed in the dark-gray region, the eigenvalues are real and thus the equilibrium is monotonically stable. If they are in the light-gray region, the eigenvalues are complex and thus the equilibrium is oscillatory stable. Two regions are separated by $detJ = \frac{(trJ)^2}{2}$, the locus of trJ and $detJ$ for which the discriminant is zero. Boundaries of the stable region are constructed from the following three lines: the *flutter boundary* with the equation $detJ = 1$, the *divergence boundary* with the equation $detJ = trJ - 1$, the *flip boundary* with the equation $detJ = -trJ - 1$.

On the divergence boundary at least one of the eigenvalues is equal to 1. Crossing this boundary allows trajectories to approach infinity. On the flip boundary at least one of the eigenvalues is equal to -1. Each point on the flip boundary corresponds to a two-periodic cycle, and movement outside the domain of stability generates the Feigenbaum type period doubling sequence, leading to chaos. On the flutter boundary, $|\lambda_1| = 1$ and $|\lambda_2| = 1$. It is possible to describe the types of bifurcation at all points of the flutter boundary. The equality condition $|\lambda_1| = |\lambda_2| = 1$ means that $\lambda_{1,2} = e^{\pm i 2\pi\theta}$ with some $0 \le \theta \le 1$. Therefore,

$$trJ = \lambda_1 + \lambda_2 = 2\cos 2\pi\theta.$$

The dynamic system can generate a periodic trajectory on the flutter boundary according to the value of trJ. In particular, it generates a period-3 cycle when $trJ = -1$, a period-4 cycle when $trJ = 0$, a period-5 cycle when $trJ = \frac{\sqrt{5}-1}{2}$, and so on.

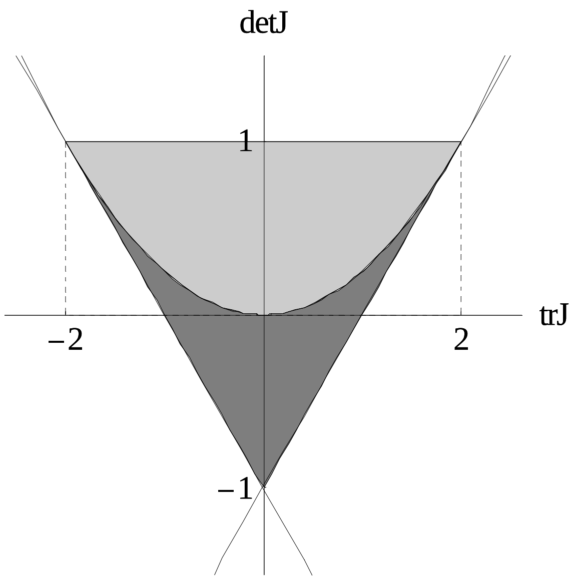


Figure A Stable Region.

Appendix B

In this appendix, we show the process with adaptive adjustment is equivalent to the process with adaptive expectations.

The dynamic process with adaptive expectations is four-dimensional and given by

$$\left\{\begin{array}{l} x_{t+1} = f(\lambda_y y_t + (1-\lambda_y) y_t^e) \\ y_{t+1} = g(\lambda_x x_t + (1-\lambda_x) x_t^e) \\ x_{t+1}^e = \lambda_x x_t + (1-\lambda_x) x_t^e \\ y_{t+1}^e = \lambda_y y_t + (1-\lambda_y) y_t^e \end{array}\right. \tag{B-1}$$

Its Jacobian is

$$J = \begin{pmatrix} 0 & f'\lambda_y & 0 & f'(1-\lambda_y) \\ g'\lambda_x & 0 & g'(1-\lambda_x) & 0 \\ \lambda_x & 0 & 1-\lambda_x & 0 \\ 0 & \lambda_y & 0 & 1-\lambda_y \end{pmatrix}. \tag{B-2}$$

Let the eigenvector be denoted by (x, y, u, v). Then the eigenvector equations are

$$\begin{array}{l} f'\lambda_y y + f'(1-\lambda_y) v = \mu x, \\ g'\lambda_x x + g'(1-\lambda_x) u = \mu y, \\ \lambda_x x + (1-\lambda_x) u = \mu u, \\ \lambda_x y + (1-\lambda_y) v = \mu v. \end{array} \tag{B-3}$$

Subtracting the third equation multiplied by g' from the second equation and also subtracting the fourth equation multiplied by f' from the first equation yield

$$\mu y = \mu u g' \text{ and } \mu x = \mu v f'.$$

Dividing both equations by μ where $\mu \neq 0$ is assumed gives

$$y = ug' \text{ and } x = vf'$$

which are substituted into the third and fourth equation of (B-3) to obtain

$$\begin{array}{l} \lambda_x v f' + (1-\lambda_x) u = \mu u, \\ \\ \lambda_y u g' + (1-\lambda_y) v = \mu v. \end{array} \tag{B-4}$$

This is equivalent to eigenvalue problem of the following matrix

$$\begin{pmatrix} 1-\lambda_x & vf' \\ ug' & 1-\lambda_y \end{pmatrix}. \tag{B-5}$$

(B-5) is the same as the Jacobian of (11). Therefore the dynamic process with adaptive adjustment is equivalent to the dynamic process with adaptive expectations.

References

[1] Bala, V., M. Majumdar, and T. Mitra, "A Note on Controlling a Chaotic Tatonnement," *Journal of Economic, Behavior and Organization*, 33, 411-420, 1998.

[2] Boldrin, M., and M. Woodford, "Equilibrium Models Displaying Endogenous Fluctuations and Chaos: a Survey," *Journal of Monetary Economics*, 25, 189-222, 1990.

[3] Day, R., "Irregular Growth Cycles," *American Economic Review*, 72, 406-414, 1982.

[4] Day, R., *Complex Economic Dynamics*, MIT Press, 1994.

[5] Hommes, C., "On the Consistency of Backward-looking Expectations: the Case of the Cobweb," *Journal of Economic Behavior and Organization*, 33, 333-362, 1998.

[6] Kass, L., "Stabilizing Chaos in a Dynamic Macroeconomic Model," *Journal of Economics, Behavior and Organization*, 33, 313-332, 1998.

[7] Huang, W-H., "Theory of Adaptive Adjustment," *Discrete Dynamics in Nature and Society*, 5, 247-263, 2001.

[8] Kopel, M., "Simplex and Complex Adjustment Dynamics in Cournot Duopoly Models," *Chaos, Solitions and Fractals*, 7, 2031-2048, 1996.

[9] Kopel, M., "Improving the Performance of an Economic System: Controlling Chaos," *Journal of Evolutionary Economics*, 7, 269-289, 1997.

[10] Okuguchi, K., and F. Szidarovszky, *The Theory of Oligopoly with Multi-Product Firms*, Springer, 1999 (2nd edition)

[11] Matsumoto, A., and Y. Nonaka, "Statistical Dynamics in Chaotic Cournot Model with Complementary Goods," mimeo, 2004.

[12] Majumdar, T, T. Mitra, and K. Nishimua, *Optimization Chaos*, Springer, 2000.

[13] Mendes, V., and D. Mendes, "Control of Hyperchaos in an OLG Economic Model," mimeo, 2000.

[14] Romeiras, F. J., C. Grebogi, E. Ott, and W.P. Dayawana, "Controlling Chaotic Dynamical Systems," *Physica D,* 58, 165-192, 1992.

[15] Puu, T., *Attractors, Bifurcations, and Chaos: Nonlinear Phenomena in Economics*, Springer, 2000.

[16] Puu, T. and I. Sushko, *Oligopoly Dynamics: Models and Tools*, Springer, 2002.

[17] Rand, D., "Exotic Phenomena in Games and Duopoly Models," *Journal of Mathematical Economics*, 5, 173-184, 1978.

[18] Sakai, H., and H. Tokumaru, "Autocorrelations of a Certain Chaos," *IEEE Transactions on Acoustics, Speech and Signal Processing*, ASSP-28, 588-590, 1980.

[19] Ueta, T., and Kawakami, H., "Composite Dynamic System for Controlling Chaos," *IEICE Trans. Fundamentals*, E78-A(6), 708-714, 1995.