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Abstract.

In this paper, we investigate a nonlinear macrodynamic model of business cycles which describes the dynamic interaction of two regions which are connected through inter-regional trade and inter-regional capital movement with fixed exchange rates. Our model is formulated as a five-dimensional system of nonlinear differential equations, which is a two-regional extension of the Kaldorian business cycle model. We study the local stability / instability and the condition for the existence of the cyclical fluctuation analytically, and we also present some numerical examples which support our analytical results.

Keywords : Two-regional model, business cycles, fixed exchange rates, Kaldorian approach

1. Introduction

The purpose of this paper is to investigate a nonlinear macrodynamic model of business cycles which describes the dynamic interaction of two regions which are connected through inter-regional trade and inter-regional capital movement with fixed exchange rates. The model which is developed in this paper can be applicable to the analysis of the dynamic interaction of two countries, for example, Japan and the United States, in the period of fixed exchange rates, 1950s and 1960s. However, this model can also be applicable to the analysis of the dynamic interaction of two countries in the European Union, for example, Germany and France, after the currency integration in 2001, because the currency integration is the extreme form of the system of fixed exchange rates.

Our model have two theoretical origins. One of them is the recent development of the dynamic analysis in regional economics and international macroeconomics. Typical examples of such works are Nijkamp and Reggiani(1992), Puu(1997), Rosser(1991), and Asada, Chiarella, Flaschel and Franke(2003). Second origin is the nonlinear business cycle theory originated by Kaldor(1940) and developed by Lorenz(1993), Gandolfo(1996) and others, which is called the 'Kaldorian' business cycle theory. This paper is a sequel to Asada(1995) and Asada, Inaba and Misawa(2001). Asada(1995) developed a Kaldorian business cycle model in a small open economy, and investigated analytically and numerically both of the case of fixed exchange rates and the case of flexible exchange rates developing the nonlinear three-dimensional systems of differential equations. Asada, Inaba and Misawa(2001) studied a Kaldorian two-regional model of business cycles with fixed exchange rates, which consists of the nonlinear five-dimensional system of difference equations (discrete time system). The analytical treatment of such a high-dimensional nonlinear system with discrete time is quite difficult, so that Asada, Inaba, and Misawa(2001) analyzed the model mainly numerically. However, recently some economists developed the analytical approach to high-dimensional nonlinear dynamical system with continuous time. We can find such works in Chiarella and Flaschel(2000), Chiarella, Flaschel, Groh and Semmler(2000), Chiarella, Flaschel, Franke and Semmler(2002), Asada and Flaschel(2002), Asada and Yoshida(2003), Asada, Chiarella, Flaschel and Franke(2003) etc. The method of analysis in this paper is based on the method which was developed in these recent works.

The model in this paper is in fact the continuous time version of the model which was developed by Asada, Inaba and Misawa(2001). This means that our model consists of the five-dimensional nonlinear differential equations. Analytical treatment of such high-dimensional system is not easy even if we consider the system with continuous time, but it is possible to investigate some qualitative natures of such a system analytically. We study the local stability / instability and the conditions for the existence of the cyclical fluctuation analytically, and we also present some numerical examples which support our analytical result.

2. Formulation of the model

Our model consists of the following system of equations.[1]

$$\dot{Y}_i = \alpha_i[C_i + I_i + G_i + J_i - Y_i] \quad ; \quad \alpha_i > 0 \tag{1}$$

$$\dot{K}_i = I_i \tag{2}$$

$$C_i = c_i(Y_i - T_i) + C_{0i} \quad ; \quad 0 < c_i < 1, \quad C_{0i} \geqq 0 \tag{3}$$

$$I_i = I_i(Y_i, K_i, r_i) \quad ; \quad \partial I_i / \partial Y_i > 0, \quad \partial I_i / \partial K_i < 0, \quad \partial I_i / \partial r_i < 0 \tag{4}$$

$$T_i = \tau_i Y_i - T_{0i} \quad ; \quad 0 < \tau_i < 1, \quad T_{0i} \geqq 0 \tag{5}$$

$$M_i / p_i = L_i(Y_i, r_i) \quad ; \quad \partial L_i / \partial Y_i > 0, \quad \partial L_i / \partial r_i < 0 \tag{6}$$

$$J_1 = J_1(Y_1, Y_2, E) \quad ; \quad \partial J_1 / \partial Y_1 < 0, \quad \partial J_1 / \partial Y_2 > 0, \quad \partial J_1 / \partial E > 0 \tag{7}$$

$$Q_1 = \beta\{r_1 - r_2 - (E^e - E)/E\} \quad ; \quad \beta > 0 \tag{8}$$

$$A_1 = J_1 + Q_1 \tag{9}$$

$$p_1 J_1 + E p_2 J_2 = 0 \tag{10}$$

$$p_1 Q_1 + E p_2 Q_2 = 0 \tag{11}$$

$$p_1 A_1 + E p_2 A_2 = 0 \tag{12}$$

$$\dot{M}_1 = p_1 A_1 \tag{13}$$

$$M_1 + E M_2 = \overline{M} \tag{14}$$

where the subscript i $(i = 1,2)$ is the index number of a region, and the meanings of other symbols are as follows. $Y_i =$ real regional income, $K_i =$ real physical capital stock, $M_i =$ nominal money stock, $I_i =$ net real private investment expenditure on physical capital, $G_i =$ real government expenditure (fixed), $p_i =$ price level (fixed), $r_i =$ nominal rate of interest, $E =$ exchange rate, $E^e =$ expected exchange rate of near future, $J_i =$ balance of current account (net export) in real terms, $Q_i =$ capital account in real terms, $A_i = J_i + Q_i =$ total balance of payments in real terms, $\alpha_i =$ adjustment speed in goods market, $\beta =$ degree of capital mobility.

Eq. (1) is the Kaldorian quantity adjustment process in the goods market. This equation implies that the real output fluctuates according as the excess demand in the goods market is positive or negative. Eq. (2) means that the net investment contributes to the changes of the real capital stock. Equations (3), (4), and (5) are standard Keynesian consumption function, Kaldorian-Keynesian investment function, and income tax function respectively. Eq. (6) is the equilibrium condition in the money market, which is nothing but the standard Keynesian 'LM equation'. Equations (7) and (8) are the current account function and the capital account function of region 1 respectively. Eq. (9) is the definitional equation of the total balance of payments of region 1. Equations (10), (11), and (12) imply that the current account surplus, the capital account surplus, and the surplus of total balance of payments of a region must be accompanied by the same amounts of the current account deficit, the capital account deficit, and the deficit of total balance of payments of another region respectively in this two-regional model. Eq. (13) means the money supply of region 1 increases or decreases according as the total balance of payments of region 1 is in surplus or in deficit. Eq. (14) means that the total nominal money supply of two regions are fixed. In other words, money moves between two regions through interregional trade and interregional capital movement in our model.

In this paper, we abstract from the changes of prices and we assume the fixed price economy. Therefore, we can normalize the price levels of two regions as follows without loss of generality.

$$p_1 = p_2 = 1 \tag{15}$$

The above system of equations (1) – (15) can be applicable to both of the system of fixed exchange rates and that of flexible exchange rates. In this paper, however, we shall concentrate on the analysis of the system of fixed exchange rates like two countries in the European Union. Therefore, we can suppose the following relationship.[2]

$$E = E^e = \overline{E} = \text{ constant.} \tag{16}$$

In this case, we can transform the above system into the following five-dimensional system of nonlinear differential equations, which is a fundamental dynamical system in our model.

$$
\begin{aligned}
&(\text{i}) \quad \dot{Y}_1 = \alpha_1[c_1(1-\tau_1)Y_1 + c_1 T_{01} + C_{01} + G_1 + I_1(Y_1, K_1, r_1(Y_1, M_1)) \\
&\qquad\qquad + J_1(Y_1, Y_2, \overline{E}) - Y_1] \equiv F_1(Y_1, K_1, Y_2, M_1; \alpha_1) \\
&(\text{ii}) \quad \dot{K}_1 = I_1(Y_1, K_1, r_1(Y_1, M_1)) \equiv F_2(Y_1, K_1, M_1) \\
&(\text{iii}) \quad \dot{Y}_2 = \alpha_2[c_2(1-\tau_2)Y_2 + c_2 T_{02} + C_{02} + G_2 + I_2(Y_2, K_2, r_2(Y_2, \{\overline{M} - M_1\}/\overline{E})) \\
&\qquad\qquad - (1/\overline{E})J_1(Y_1, Y_2, \overline{E}) - Y_2] \equiv F_3(Y_1, Y_2, K_2, M_1; \alpha_2) \\
&(\text{iv}) \quad \dot{K}_2 = I_2(Y_2, K_2, r_2(Y_2, \{\overline{M} - M_1\}/\overline{E})) \equiv F_4(Y_2, K_2, M_1) \\
&(\text{v}) \quad \dot{M}_1 = J_1(Y_1, Y_2, \overline{E}) + \beta[r_1(Y_1, M_1) - r_2(Y_2, \{\overline{M} - M_1\}/\overline{E})] \\
&\qquad\qquad \equiv F_5(Y_1, Y_2, M_1; \beta)
\end{aligned} \tag{17}
$$

3. Nature of the equilibrium solution

The equilibrium solution $(Y_1^*, K_1^*, Y_2^*, K_2^*, M_1)$ of the system (17) is determined by the following system of equations.

$$(\text{i}) \quad H_1(Y_1, Y_2) \equiv c_1(1-\tau_1)Y_1 + c_1 T_{01} + C_{01} + G_1 + J_1(Y_1, Y_2, \overline{E}) - Y_1 = 0$$

$$(\text{ii}) \quad H_2(Y_1, K_1, M_1) \equiv I_1(Y_1, K_1, r_1(Y_1, M_1)) = 0$$

$$(\text{iii}) \quad H_3(Y_1, Y_2) \equiv c_2(1-\tau_2)Y_2 + c_2 T_{02} + C_{02} + G_2 - (1/\overline{E})J_1(Y_1, Y_2, \overline{E}) - Y_2 = 0$$

$$(\text{iv}) \quad H_4(Y_2, K_2, M_1) \equiv I_2(Y_2, K_2, r_2(Y_2, \{\overline{M} - M_1\}/\overline{E})) = 0$$

(v) $H_5(Y_1, Y_2, M_1; \beta) \equiv J_1(Y_1, Y_2, \overline{E}) + \beta[r_1(Y_1, M_1) - r_2(Y_2, \{\overline{M} - M_1\} / \overline{E})] = 0$ (18)

Equilibrium regional incomes Y_1* and Y_2* are determined by two equations (18) (i) and (iii). Totally differentiating these equations, we have the following relationships.

$$(dY_2 / dY_1)\big|_{H_1=0} = -H_{11} / H_{12} = \{1 - c_1(1-\tau_1) + m_1\} / m_2 > 0 \quad (19)$$

$$(dY_2 / dY_1)\big|_{H_2=0} = -H_{21} / H_{22} = \{(1/\overline{E})m_1\} / \{1 - c_2(1-\tau_2) + (1/\overline{E})m_2\} > 0 \quad (20)$$

where $m_1 = -\partial J_1 / \partial Y_1 > 0,$ $m_2 = \partial J_1 / \partial Y_2 > 0,$ $H_{11} = -\{1 - c_1(1-\tau_1) + m_1\} < 0,$ $H_{12} = m_2 > 0,$ $H_{21} = (1/\overline{E})m_1 > 0,$ and $H_{22} = -\{1 - c_2(1-\tau_2) + (1/\overline{E})m_2\} < 0.$ These relationships mean that both of the slopes of the graphs of equations (18)(i) and (iii) are positive. We *assume* that a subsystem which consists of (18)(i), (iii) has the unique solution $(Y_1*, Y_2*) > (0,0).$ Substituting this solution into Eq. (18)(iv), we have

$$H_5(Y_1*, Y_2*, M_1; \beta) \equiv J_1(Y_1*, Y_2*, \overline{E}) + \beta[r_1(Y_1*, M_1) - r_2(Y_2*, \{\overline{M} - M_1\} / \overline{E})] = 0 \quad (21)$$

; which is an equation with single unknown, M_1. Differentiation of Eq. (21) gives us

$$\partial H_5 / \partial M_1 = \beta[r^1_{M1} + (1/\overline{E})r^2_{M2}] < 0 \quad (22)$$

where $r^i_{Mi} = \partial r_i / \partial M_i = 1/(\partial L_i / \partial r_i) < 0 \quad (i = 1,2).$ The inequality (22) implies that we can exclude the possibility of the multiple solutions of M_1* as long as (Y_1*, Y_2*) is determined uniquely. Substituting $(Y_1, Y_2, M_1) = (Y_1*, Y_2*, M_1*)$ into the equations (18) (ii) and (iv), we obtain the following relationships.

$$H_2(Y_1*, K_1, M_1*) \equiv I_1(Y_1*, K_1, r_1(Y_1*, M_1*)) = 0 \quad (23)$$

$$H_4(Y_2^*, K_2, M_1^*) \equiv I_2(Y_2^*, K_2, r_2(Y_2^*, \{\overline{M} - M_1^*\} / \overline{E})) = 0 \qquad (24)$$

Eq. (23) (Eq. (24)) is a single equation which has only unknown, K_1 (K_2). Since $\partial H_2 / \partial K_1 = \partial I_1 / \partial K_1 < 0$ and $\partial H_4 / \partial K_2 = \partial I_2 / \partial K_2 < 0,$ we can also exclude the multiple solutions of (K_1^*, K_2^*) as long as (Y_1^*, Y_2^*) is determined uniquely. It is worth to note that the equilibrium values $(Y_1^*, K_1^*, Y_2^*, K_2^*, M_1^*)$ are independent of the parameter values α_i $(i = 1,2),$ and the values of (Y_1^*, Y_2^*) are independent of the parameter value $\beta,$ although the values of $(K_1^*, K_2^* M_1^*)$ depend on the parameter value $\beta.$

In the next section, we shall study the local stability / instability of the system around the equilibrium point by means of the linear approximation method by *assuming* that there exists the unique equilibrium point $(Y_1^*, K_1^*, Y_2^*, K_2^*, M_1^*) > (0,0,0,0,0)$ of Eq. (17).

4. Local stability analysis

We can write the Jacobian matrix of the system (17) which is evaluated *at the equilibrium point* as follows.[3]

$$J = \begin{bmatrix} F_{11} & F_{12} & F_{13} & 0 & F_{15} \\ F_{21} & F_{22} & 0 & 0 & F_{25} \\ F_{31} & 0 & F_{33} & F_{34} & F_{35} \\ 0 & 0 & F_{43} & F_{44} & F_{45} \\ F_{51} & 0 & F_{53} & 0 & F_{55} \end{bmatrix} = \begin{bmatrix} \alpha_1 G_{11} & \alpha_1 G_{12} & \alpha_1 G_{13} & 0 & \alpha_1 G_{15} \\ F_{21} & G_{12} & 0 & 0 & G_{15} \\ \alpha_2 G_{31} & 0 & \alpha_2 G_{33} & \alpha_2 G_{34} & \alpha_2 G_{35} \\ 0 & 0 & F_{43} & G_{34} & G_{35} \\ F_{51}(\beta) & 0 & F_{53}(\beta) & 0 & F_{55}(\beta) \end{bmatrix} \qquad (25)$$

where $G_{11} = \underset{(+)}{I_{Y1}^1} + \underset{(-)}{I_{r1}^1} \underset{(+)}{r_{Y1}^1} - \{1 - c_1(1-\tau_1) + m_1\},$ $\quad G_{12} = I_{K1}^1 < 0,$ $\quad G_{13} = m_2 > 0,$

$G_{15} = \underset{(-)}{I_{r1}^1} \underset{(-)}{r_{M1}^1} > 0,$ $\quad F_{21} = \underset{(+)}{I_{Y1}^1} + \underset{(-)}{I_{r1}^1} \underset{(+)}{r_{Y1}^1},$ $\quad G_{31} = (1/\overline{E}) m_1 > 0,$

$G_{33} = \underset{(+)}{I_{Y2}^2} + \underset{(-)}{I_{r2}^2} \underset{(+)}{r_{Y2}^2} - \{1 - c_2(1-\tau_2) + (1/\overline{E}) m_2\},$ $\quad G_{34} = I_{K2}^2 < 0,$

$G_{35} = -(1/\overline{E}) \underset{(-)}{I_{r2}^2} \underset{(-)}{r_{M2}^2} < 0,$ $\quad F_{43} = \underset{(+)}{I_{Y2}^2} + \underset{(-)}{I_{r2}^2} \underset{(+)}{r_{Y2}^2},$ $\quad F_{51}(\beta) = -\underset{(+)}{m_1} + \beta \underset{(+)}{r_{Y1}^1},$

$F_{53}(\beta) = \underset{(+)}{m_2} - \beta \underset{(+)}{r_{Y2}^2},$ and $F_{55}(\beta) = \beta[\underset{(-)}{r_{M1}^1} + (1/\overline{E}) \underset{(-)}{r_{M2}^2}] < 0.$

Now, let us assume as follows.

Assumption 1.

I_{Y1}^{1} and I_{Y2}^{2} are so large that we have $G_{11}>0$ and $G_{33}>0$ at the equilibrium point.

Remark 1.

Assumption 1 automatically implies that $F_{21}>0$ and $F_{43}>0$ at the equilibrium point.

Assumption 1 means that the sensitivities of investment with respect to the changes of national incomes of both regions are sufficiently large at the equilibrium point, which is nothing but the standard hypothesis of Kaldorian business cycle model (cf. Kaldor(1940), Asada(1995), Asada, Inaba and Misawa(2001)).

We can express the characteristic equation of this system as

$$f(\lambda) \equiv \left|\lambda I - J\right| = \lambda^5 + a_1\lambda^4 + a_2\lambda^3 + a_3\lambda^2 + a_4\lambda + a_5 = 0 \tag{26}$$

where we have the following relationships.

$$a_1 = -traceJ = -\alpha_1 G_{11} - G_{12} - \alpha_2 G_{33} - G_{34} - F_{55}(\beta) \equiv a_1(\alpha_1, \alpha_2, \beta) \tag{27}$$

$a_2 =$ sum of all principal second-order minors of J

$$= \alpha_1 \begin{vmatrix} G_{11} & G_{12} \\ F_{21} & G_{12} \end{vmatrix} + \alpha_1\alpha_2 \begin{vmatrix} G_{11} & G_{13} \\ G_{31} & G_{33} \end{vmatrix} + \alpha_1 \begin{vmatrix} G_{11} & 0 \\ 0 & G_{34} \end{vmatrix} + \alpha_1 \begin{vmatrix} G_{11} & G_{15} \\ F_{51}(\beta) & F_{55}(\beta) \end{vmatrix}$$

$$+ \alpha_2 \begin{vmatrix} G_{12} & 0 \\ 0 & G_{33} \end{vmatrix} + \begin{vmatrix} G_{12} & 0 \\ 0 & G_{34} \end{vmatrix} + \begin{vmatrix} G_{12} & G_{15} \\ 0 & F_{55}(\beta) \end{vmatrix} + \alpha_2 \begin{vmatrix} G_{33} & G_{34} \\ F_{43} & G_{34} \end{vmatrix}$$

$$+ \alpha_2 \begin{vmatrix} G_{33} & G_{35} \\ F_{53}(\beta) & F_{55}(\beta) \end{vmatrix} + \begin{vmatrix} G_{34} & G_{35} \\ 0 & F_{55}(\beta) \end{vmatrix} \equiv a_2(\alpha_1, \alpha_2, \beta) \tag{28}$$

$a_3 = -$(sum of all principal third-order minors of J)

$$-\alpha_1\alpha_2\begin{vmatrix} G_{11} & G_{12} & G_{13} \\ F_{21} & G_{12} & 0 \\ G_{31} & 0 & G_{33} \end{vmatrix} - \alpha_1\begin{vmatrix} G_{11} & G_{12} & 0 \\ F_{21} & G_{12} & 0 \\ 0 & 0 & G_{34} \end{vmatrix} - \alpha_1\begin{vmatrix} G_{11} & G_{12} & G_{15} \\ F_{21} & G_{12} & G_{15} \\ F_{51}(\beta) & 0 & F_{55}(\beta) \end{vmatrix}$$

$$-\alpha_1\alpha_2\begin{vmatrix} G_{11} & G_{13} & 0 \\ G_{31} & G_{33} & G_{34} \\ 0 & F_{43} & G_{34} \end{vmatrix} - \alpha_1\alpha_2\begin{vmatrix} G_{11} & G_{13} & G_{15} \\ G_{31} & G_{33} & G_{35} \\ F_{51}(\beta) & F_{53}(\beta) & F_{55}(\beta) \end{vmatrix}$$

$$-\alpha_1\begin{vmatrix} G_{11} & 0 & G_{15} \\ 0 & G_{34} & G_{35} \\ F_{51}(\beta) & 0 & F_{55}(\beta) \end{vmatrix} - \alpha_2\begin{vmatrix} G_{12} & 0 & 0 \\ 0 & G_{33} & G_{34} \\ 0 & F_{43} & G_{34} \end{vmatrix} - \alpha_2\begin{vmatrix} G_{12} & 0 & G_{15} \\ 0 & G_{33} & G_{55} \\ 0 & F_{33}(\beta) & F_{55}(\beta) \end{vmatrix}$$

$$-\begin{vmatrix} G_{12} & 0 & G_{15} \\ 0 & G_{34} & G_{35} \\ 0 & 0 & F_{55}(\beta) \end{vmatrix} - \alpha_2\begin{vmatrix} G_{33} & G_{34} & G_{35} \\ F_{43} & G_{34} & G_{35} \\ F_{53}(\beta) & 0 & F_{55}(\beta) \end{vmatrix} \equiv a_3(\alpha_1, \alpha_2, \beta) \qquad (29)$$

a_4 = sum of all principal fourth-order minors of J

$$=\alpha_1\alpha_2\begin{vmatrix} G_{11} & G_{12} & G_{13} & 0 \\ F_{21} & G_{12} & 0 & 0 \\ G_{31} & 0 & G_{33} & G_{34} \\ 0 & 0 & F_{43} & G_{44} \end{vmatrix} + \alpha_1\alpha_2\begin{vmatrix} G_{11} & G_{12} & G_{13} & G_{15} \\ F_{21} & G_{12} & 0 & G_{15} \\ G_{31} & 0 & G_{33} & G_{35} \\ F_{51}(\beta) & 0 & F_{53}(\beta) & F_{55}(\beta) \end{vmatrix}$$

$$+\alpha_1\begin{vmatrix} G_{11} & G_{12} & 0 & G_{15} \\ F_{21} & G_{12} & 0 & G_{15} \\ 0 & 0 & G_{34} & G_{35} \\ F_{51}(\beta) & 0 & 0 & F_{55}(\beta) \end{vmatrix} + \alpha_1\alpha_2\begin{vmatrix} G_{11} & G_{13} & 0 & G_{15} \\ G_{31} & G_{33} & G_{34} & G_{35} \\ 0 & F_{43} & G_{34} & G_{35} \\ F_{51}(\beta) & F_{53}(\beta) & 0 & F_{55}(\beta) \end{vmatrix}$$

$$+\alpha_2\begin{vmatrix} G_{12} & 0 & 0 & G_{15} \\ 0 & G_{33} & G_{34} & G_{35} \\ 0 & F_{43} & G_{34} & G_{35} \\ 0 & F_{53}(\beta) & 0 & F_{55}(\beta) \end{vmatrix} \equiv a_4(\alpha_1, \alpha_2, \beta) \qquad (30)$$

$$a_5 = -\det J = -\alpha_1\alpha_2 \begin{vmatrix} G_{11} & G_{12} & G_{13} & 0 & G_{15} \\ F_{21} & G_{12} & 0 & 0 & G_{15} \\ G_{31} & 0 & G_{33} & G_{34} & G_{35} \\ 0 & 0 & F_{43} & G_{34} & G_{35} \\ F_{51}(\beta) & 0 & F_{53}(\beta) & 0 & F_{55}(\beta) \end{vmatrix}$$

$$= -\alpha_1\alpha_2 \begin{vmatrix} G_{11}-F_{21} & 0 & G_{13} & 0 & 0 \\ F_{21} & G_{12} & 0 & 0 & G_{15} \\ G_{31} & 0 & G_{33}-F_{43} & 0 & 0 \\ 0 & 0 & F_{43} & G_{34} & G_{35} \\ F_{31}(\beta) & 0 & F_{53}(\beta) & 0 & F_{55}(\beta) \end{vmatrix}$$

$$= -\alpha_1\alpha_2 G_{12} \begin{vmatrix} G_{11}-F_{21} & G_{13} & 0 & 0 \\ G_{31} & G_{33}-F_{43} & 0 & 0 \\ 0 & F_{43} & G_{34} & G_{35} \\ F_{51}(\beta) & F_{53}(\beta) & 0 & F_{55}(\beta) \end{vmatrix}$$

$$= -\alpha_1\alpha_2 G_{12} G_{34} \begin{vmatrix} G_{11}-F_{21} & G_{13} & 0 \\ G_{31} & G_{33}-F_{43} & 0 \\ F_{51}(\beta) & F_{53}(\beta) & F_{55}(\beta) \end{vmatrix}$$

$$= -\alpha_1\alpha_2 G_{12} G_{34} F_{55}(\beta)[(G_{11}-F_{21})(G_{33}-F_{43}) - G_{13}G_{31}]$$

$$= -\alpha_1\alpha_2 \underset{(-)}{G_{12}}\ \underset{(-)}{G_{34}}\ \underset{(-)}{F_{55}}(\beta)[\{1-c_1(1-\tau_1)\}\{1-c_2(1-\tau_2)\} + \{1-c_1(1-\tau_1)\}(1/\overline{E})m_2$$

$$+\{1-c_2(1-\tau_2)\}m_1] \equiv a_5(\alpha_1,\alpha_2,\beta) > 0 \quad \text{for all} \quad (\alpha_1,\alpha_2,\beta) > (0,0,0) \qquad (31)$$

The Routh-Hurwitz terms Δ_i $(i = 1,2,\cdots,5)$ are defined as follows in this five-dimensional case.

(i) $\Delta_1 = a_1$

(ii) $\Delta_2 = \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix} = a_1 a_2 - a_3$

(iii) $$\Delta_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} = a_3\Delta_2 + a_1(a_5 - a_1 a_4) = a_1 a_2 a_3 - a_3{}^2 - a_1{}^2 a_4 + a_1 a_5$$

(iv) $$\Delta_4 = \begin{vmatrix} a_1 & a_3 & a_5 & 0 \\ 1 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & a_5 \\ 0 & 1 & a_2 & a_4 \end{vmatrix} = a_4\Delta_3 - a_5 \begin{vmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \\ 0 & 1 & a_2 \end{vmatrix}$$

$$= a_4\Delta_3 + a_5(-a_1 a_2{}^2 - a_5 + a_2 a_3 + a_1 a_4)$$

$$= a_4\Delta_3 + a_5(a_1 a_4 - a_5 - a_2\Delta_2)$$

(v) $$\Delta_5 = \begin{vmatrix} a_1 & a_3 & a_5 & 0 & 0 \\ 1 & a_2 & a_4 & 0 & 0 \\ 0 & a_1 & a_3 & a_5 & 0 \\ 0 & 1 & a_2 & a_4 & 0 \\ 0 & 0 & a_1 & a_3 & a_5 \end{vmatrix} = a_5\Delta_4 \tag{32}$$

It is well known that all of the roots of the characteristic equation (26) have negative real parts *if and only if* the following set of conditions is satisfied (cf. **Theorem A1** in **Appendix A**).

$$\Delta_i > 0 \quad \text{for all} \quad i \in \{1,2,\cdots,5\} \tag{33}$$

Now, we can prove the following properties under **Assumption 1**.

Proposition 1.

(i) Suppose that the parameter $\beta > 0$ is fixed at any level. Then, the equilibrium point of the system (17) is locally asymptotically stable for all sufficiently small values of $\alpha_1 > 0$ and $\alpha_2 > 0$..

(ii) Suppose that $\beta > 0$ is fixed at any level, and $\alpha_1 > 0$ or $\alpha_2 > 0$ is set to be sufficiently large. Then, the equilibrium point of the system (17) is locally unstable.

(iii) Suppose that $\alpha_1 > 0$ and $\alpha_2 > 0$ are fixed at such levels that the inequality

$$\alpha_1 \underset{(+)}{G_{11}} + \underset{(-)}{G_{12}} + \alpha_2 \underset{(+)}{G_{33}} + \underset{(-)}{G_{34}} > 0 \tag{34}$$

is satisfied. Then, the equilibrium point of the system (17) is locally unstable for all sufficiently large $\beta > 0$.

(Proof.) See **Appendix B**. □

Proposition 1 (i) and (ii) imply that the *high* speed of adjustment in the goods market disequilibrium in each region is a *destabilizing* factor, and the *slow* speed of adjustment is a *stabilizing* factor. **Proposition 1** (iii) implies that the *high* speed of capital mobility between regions, which is accompanied by relatively high speed of the adjustment in the goods markets, is a *destabilizing* factor in a two-regional model with fixed exchange rate system. We can interpret the above result in economic terms as follows. Suppose that Y_i, income of region i, is decreased to the level which is less than equilibrium level because of some exogenous shock. In this case, we can consider the following opposite two effects.

(E1) $Y_i \downarrow \;\Rightarrow\; J_i \uparrow \;\Rightarrow\; Y_i \uparrow$

$$\Downarrow$$

$$A_i \uparrow \;\Rightarrow\; M_i \uparrow \;\Rightarrow\; r_i \downarrow \;\Rightarrow\; I_i \uparrow \;\Rightarrow\; Y_i \uparrow$$

(E2) $Y_i \downarrow \;\Rightarrow\; r_i \downarrow \;\Rightarrow\; I_i \uparrow \;\Rightarrow\; Y_i \uparrow$

$$\Downarrow$$

$$Q_i \downarrow \;\Rightarrow\; A_i \downarrow \;\Rightarrow\; M_i \downarrow \;\Rightarrow\; r_i \uparrow \;\Rightarrow\; I_i \downarrow \;\Rightarrow\; Y_i \downarrow$$

(E1) illustrates the stabilizing 'current account effect'. The decrease of the regional income will increase the current account through the decrease of the regional import, which will contribute to the increase of regional income. This direct stabilization effect is illustrated in the upper half of the figure (E1). Another indirect stabilizing effect through the increase of the current account is illustrated in the lower half of the figure (E1). The increase of the regional current account will contribute to the increase of the regional money supply, which will stimulate the regional investment expenditure through the decrease of the regional rate of interest, which is without doubt stabilizing.

The lower half of the figure (E2) illustrates the destabilizing 'capital account effect'. The decrease of the regional income will induce the decrease of the regional rate of interest through the 'LM equation', which will stimulate the regional effective demand through the increase of the regional investment expenditure, which is a direct stabilizing effect. This effect is illustrated in the upper half of the figure (E2). On the other hand, the decrease of the regional rate of interest will induce the decrease of the regional capital account through the capital movement between regions, which will contribute to decrease the regional money supply, and as a result the regional effective demand will decrease through the decrease of the regional investment expenditure. This is the destabilizing 'capital account effect'. If the speed of capital mobility between regions (β) is high, the destabilizing 'capital account effect' will dominate the stabilizing 'current account effect'. If the speeds of the adjustment in the goods market (α_1 and α_2) are high, this destabilizing effect will be reinforced.

5. Existence of the cyclical fluctuation

The economic interpretation of the stabilizing / destabilizing mechanisms which is provided in the previous section already suggests the possibility of the cyclical fluctuation of the regional incomes, regional rates of interest, regional investment expenditures etc. through the complex interaction of the stabilizing and the destabilizing factors. In fact, we can establish the existence of the cyclical fluctuation analytically at the intermediate levels of the adjustment speeds in the goods markets of both regions. To interpret this result, let us fix $\beta>0$ at any level and fix $\alpha_2>0$ at the sufficiently small level, and let us select $\alpha_1>0$ as a bifurcation parameter.[4]

Proposition 1 (i) and (ii) mean that the equilibrium point of the system (17) is locally asymptotically stable for all sufficiently small values of $\alpha_1 > 0$, and it is locally unstable for all sufficiently large values of $\alpha_1 > 0$. Therefore, there exists, by continuity, at lease one 'bifurcation point' at which the local stability of the equilibrium point is lost as the parameter value $\alpha_1 > 0$ increases.[5] We can characterize the nature of this bifurcation point as follows.

Proposition 2.

(i) At the bifurcation point, the characteristic equation (26) has a pair of pure imaginary roots.

(ii) At the bifurcation point, the characteristic equation (26) does not have a root such that $\lambda = 0$.

(Proof.)

Suppose that the 'bifurcation', the existence of which is already ensured, occurs at $\alpha_1 = \alpha_1^0 > 0$. By the very nature of the bifurcation point, the characteristic equation (26) must have at least one root with zero real part at $\alpha_1 = \alpha_1^0$. However, we can exclude the root such as $\lambda = 0$ because of the fact that $f(0) = a_5 > 0$. Therefore, Eq. (26) must have a pair of pure imaginary roots at $\alpha_1 = \alpha_1^0$. □

Only the following two cases can occur in our model.

Case 1. At the bifurcation point, Eq. (26) has a pair of pure imaginary roots and three roots with negative real parts.

Case 2. At the bifurcation point, Eq. (26) has two pairs of pure imaginary roots and one negative real roots.

Case 1 corresponds to the so called 'simple' Hopf Bifurcation, and in this case we can establish the existence of the closed orbit at some parameter values α_1 which are sufficiently close to the bifurcation value.[6] The bifurcation point in **Case 2** is not the Hopf Bifurcation point, and in this case we cannot establish the existence of the closed orbits analytically.

We can consider that **Case 1** is the normal case, and **Case 2** will occur only by accident. However, even in **Case 2**, we can establish the existence of (two pairs of) complex roots at some range of the parameter values which are sufficiently close to the bifurcation value. This is enough to establish the existence of cyclical fluctuations, rather than the existence of the closed orbits, at some range of the parameter values.

6. A numerical illustration

In this section, we shall present some numerical examples which support the theoretical reasoning in the previous sections.[7] Let us assume the following parameter values and the functional forms of the investment function and the LM equation (the equilibrium condition for the money market), which are assumed to be common to both regions $(i = 1,2)$ for simplicity.

$$c_i = 0.8, \quad \tau_i = 0.2, \quad T_{0i} = 10 \tag{35}$$

$$I_i = 25\sqrt{Y_i} - 0.3K_i - r_i + 160 \tag{36}$$

$$r_i = 10\sqrt{Y_i} - M_i + 160 \tag{37}$$

Substituting the LM equation (37) into the investment function (35), we obtain the following reduced form of the investment function.

$$I_i = 15\sqrt{Y_i} - 0.3K_i + M_i \tag{38}$$

We specify other parameter values and the functional form of the current account function as follows.

$$C_{01} = 20, \quad G_1 = 30, \quad C_{02} = 40, \quad G_2 = 60, \quad \overline{M} = 600 \tag{39}$$

$$p_1 = p_2 = \overline{E} = 1 \tag{40}$$

$$J_1(Y_1, Y_2, \overline{E}) = -0.3Y_1 + 0.3Y_2 \tag{41}$$

In this case, the five-dimensional dynamical system (17) becomes as follows.

(i) $\dot{Y}_1 = \alpha_1[-0.66Y_1 + 15\sqrt{Y_1} - 0.3K_1 + M_1 + 0.3Y_2 + 58]$

(ii) $\dot{K}_1 = 15\sqrt{Y_1} - 0.3K_1 + M_1$

(iii) $\dot{Y}_2 = \alpha_2[-0.66Y_2 + 15\sqrt{Y_2} - 0.3K_2 - M_1 + 0.3Y_1 + 708]$

(iv) $\dot{K}_2 = 15\sqrt{Y_2} - 0.3K_2 - M_1 + 600$

(v) $\dot{M}_1 = -0.3Y_1 + 0.3Y_2 + \beta(10\sqrt{Y_1} - 10\sqrt{Y_2} - 2M_1 + 600)$ (17a)

The equilibrium values of the variables in this system become as follows.

$$Y_1{*} \cong 203.93, \quad Y_2{*} \cong 255.86, \quad M_1{*} \cong 291.43 + (7.7895/\beta),$$
$$K_1{*} \cong 1{,}685.45 + (25.965/\beta), \quad K_2{*} \cong 1{,}823.35 - (25.965/\beta) \tag{42}$$

The case of perfect capital mobility corresponds to $\beta \to +\infty$. In our model with imperfect capital mobility, the degree of capital mobility β is finite. However, the effects of the changes of β on the equilibrium values $M_1{*}$, $K_1{*}$, and $K_2{*}$ are almost negligible unless the value of β is unrealistically small.

Now, we shall compute the trajectories which are produced by the system (17a) by selecting several parameter values and the following common initial conditions of the variables, which are not extremely far from the equilibrium values.

$$Y_1(0) = 200, \quad Y_2(0) = 280, \quad M_1(0) = 300, \quad K_1(0) = 1680, \quad K_2(0) = 1830 \tag{43}$$

Figures 1 – 6 summarize the main results of our numerical simulations.[8]

Insert Fig. 1 – Fig. 6 here.

Figures 1, 2, and 3 are made under the common parameter values $\alpha_2 = 2$ and $\beta = 15$. In these figures, the degree of capital mobility between regions and the economic structure of region 2 are the same, and only the adjustment speed of the goods market in region 1 (α_1) is different. These figures suggest that the trajectories of the variables converge to the equilibrium levels monotonically when the adjustment speeds of the goods markets in both regions are relatively small, and the cyclical fluctuation emerges as the adjustment speed of the goods market in a region increases. These figures also suggest that the increase of the adjustment speed increases the amplitude of fluctuation and shorten the period of the cycle, and too large adjustment speed makes the equilibrium point unstable. The period of the cycle is about 13 years in Fig. 2, and it is about 10 years in Fig. 3. They correspond to the periods of the so called Juglar cycle or the major cycle. Fig. 4 is the phase portrait of the trajectory in the $Y_1 - Y_2$ plane. These figures show that the changes of the economic structure of a region affects the economic performance of another region considerably in the interdependent two-regional system, even if the economic structure of another region is unchanged. By the way, in case of Fig. 2 the economic structures of two regions are the same except the absolute levels of expenditures. In this case, the business cycles of two regions almost synchronize.

Figures 5 and 6 show the case in which $\beta = 80$ but α_1 and α_2 are the same as those of Figures 3 and 4. Comparison of the figures 3 and 5 (or alternatively, 4 and 6) reveals that relatively large difference of the degree of capital mobility ($\beta = 15$ and $\beta = 80$) makes very little difference of the qualitative and quantitative natures of the business cycles in both regions. We have very similar cyclical patterns for the range of the degree of capital mobility $15 \leqq \beta \leqq 99$. These figures also seem to suggest that the increase of the degree of capital mobility has a moderate stabilizing effect. However, it is not correct to say that the large degree of capital mobility always has a stabilizing effect. In fact, the solution becomes quite unstable and it explodes when $\beta \geqq 100$ under the parameter values $\alpha_1 = 5$ and $\alpha_2 = 2$. In this case, the solution becomes economically meaningless unless the other sources of the nonlineality of the system which are not considered in this paper, for example, the existence of the full employment ceilings and the nonnegativity of the *gross* (but not *net*) level of the investment expenditures of both regions, prevent the system from the infinite divergency. This means that the very large degree of capital mobility which is accompanied by the relatively large adjustment speeds of the goods market tends to destabilize the system, which is nothing but the conclusion which we obtained analytically in section 4 of this paper.

7. Notes on the further extension of the model

In this paper, we studied a two-regional Kaldorian business cycle model with fixed exchange rates analytically and numerically. As we already noted in the introductory part of the paper, this model can be applicable to the dynamic analysis of the economic interaction between two countries under the currency integration such as two countries in the European Union. Although we need some reformulation of the model if we intend to investigate the two-regional model with flexible exchange rates, we can use many building blocks which were presented in this paper even for the dynamic analysis of the system with flexible exchange rates. If we assume that the exchange rate E is not fixed but it is endogenously determined to ensure the equilibrium condition of the total balance of payments $A_1 = 0$, we have the equation such that

$$A_1 = J_1(Y_1, Y_2, E) + \beta[r_1(Y_1, M_1) - r_2(Y_2, M_2) - (E^e - E)/E] = 0. \tag{44}$$

Solving this equation with respect to E, we obtain

$$E = E(Y_1, Y_2, E^e; \beta). \tag{45}$$

In this case, we can assume that M_1 and M_2 are fixed, because money stock does not move between regions when the condition $A_1 = 0$ is satisfied. Substituting Eq. (45) into equations (17) (i) – (iv), we obtain the following system.

(i) $\dot{Y}_1 = F_1(Y_1, K_1, Y_2, E^e; \alpha_1, \beta)$

(ii) $\dot{K}_1 = F_2(Y_1, K_1)$

(iii) $\dot{Y}_2 = F_3(Y_1, Y_2, K_2, E^e; \alpha_2, \beta)$

(iv) $\dot{K}_2 = F_4(Y_2, K_2)$ (46)

This system consists of four dynamical equations with five endogenous variables (Y_1, K_1, Y_2, K_2, E^e), so that this system is not yet complete. If we add the dynamic equation which describes the mechanism of the expectation formation of the expected exchange rates, we can close the system. We shall write the mechanism of the expectation formation in a rather abstract form, that is,[9]

$$\dot{E}^e = \Phi(\bullet). \tag{47}$$

Obviously, the dynamic nature of this system crucially depends on the specification of the expectation formation mechanism, but this system is still five-dimensional system of nonlinear differential equations. Therefore, it is possible to study the dynamic nature of this two-regional Kaldorian model with flexible exchange rates by using the same method which was developed in this paper.

We can also consider other various possibilities of the extension of the model to enrich the analysis. The typical examples are as follows.

(1) Introduction of the price dynamics and the labor market dynamics.

(2) Introduction of the inventory dynamics.

(3) Introduction of the growth factor including the changes of the population and the technical change.

If we introduce the above factors, dimension of the dynamical system will further increase. In some cases, dimension of the system may exceeds ten. Chiarella and Flaschel (2000) and Chiarella, Flaschel, Groh and Semmler(2000) studied the high-dimensional dynamic models in a closed economy which contain all of the above factors (1) – (3), and they called such models KMG (Keynes-Metzler-Goodwin) models. As for the complicated two-regional version of the KMG model, see Asada, Chiarella, Flaschel and Franke(2003).

Appendix A. : Two useful theorems

In this appendix, we summarize two mathematical theorems which are useful for the derivation of the propositions in the text.

Let us consider the following characteristic equation.

$$f(\lambda) \equiv \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_{n-1}\lambda + a_n = 0 \tag{A1}$$

In this case, the Routh-Hurwitz terms Δ_i ($i = 1, \cdots, n$) are defined as follows.

$$\Delta_1 = a_1, \quad \Delta_2 = \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}, \quad \cdots\cdots,$$

$$\Delta_n = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 & \cdots & 0 \\ 1 & a_2 & a_4 & a_6 & \cdots & 0 \\ 0 & a_1 & a_3 & a_5 & \cdots & 0 \\ 0 & 1 & a_2 & a_4 & \cdots & 0 \\ 0 & 0 & a_1 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & a_n \end{vmatrix} = a_n\Delta_{n-1} \tag{A2}$$

Then, we have the following two theorems.

Theorem A1. (Routh-Hurwitz conditions for stable roots) (cf. Gandolfo(1996) pp. 221 – 222)

Consider the characteristic equation (A1) with $n \geqq 1$. Then, all of the roots of this equation have negative real parts *if and only if* the following set of conditions is satisfied.

$$\Delta_i > 0 \quad \text{for all} \quad i \in \{1, \cdots, n\} \tag{A3}$$

Theorem A2. (Liu's theorem) (cf. Liu(1994))

Consider the characteristic equation (A1) with $n \geqq 3$. Then, this equation has a pair of pure imaginary roots and $(n-2)$ roots with negative real parts *if and only if* the following set of conditions is satisfied.

$$\Delta_i > 0 \quad \text{for all} \quad i \in \{1, \cdots, n-2\}, \quad \Delta_{n-1} = 0, \quad a_n > 0 \tag{A4}$$

Theorem A1 is well known among the economists, while **Theorem A2** may be less known. **Theorem A2** provides us a useful criterion for the occurrence of the so called 'simple' Hopf Bifurcation in the general n-dimensional system of differential equations. The 'simple' Hopf Bifurcation is a type of the Hopf Bifurcation in which all the characteristic roots *except* a pair of purely imaginary ones have negative real parts. By the way, Asada and Yoshida(2003) provides a complete mathematical characterization of the Hopf Bifurcation including 'non-simple' one as far as the four-dimensional system is concerned.

Appendix B : Proof of Proposition 1

(i) From equations (27) – (32) we have the following relationships for all $\beta > 0$.

$$\Delta_1(0,0,\beta) \equiv a_1(0,0,\beta) = -\underset{(-)}{G_{12}} - \underset{(-)}{G_{34}} - \underset{(-)}{F_{55}}(\beta) > 0 \tag{B1}$$

$$a_2(0,0,\beta) = \underset{(-)}{G_{12}}\,\underset{(-)}{G_{34}} + (\underset{(-)}{G_{12}} + \underset{(-)}{G_{34}})\,\underset{(-)}{F_{55}}(\beta) > 0 \tag{B2}$$

$$a_3(0,0,\beta) = -\underset{(-)}{G_{12}}\,\underset{(-)}{G_{34}}\,\underset{(-)}{F_{55}}(\beta) > 0 \tag{B3}$$

$$\Delta_2(0,0,\beta) \equiv a_1(0,0,\beta)a_2(0,0,\beta) - a_3(0,0,\beta)$$

$$= -(\underset{(-)}{G_{12}} + \underset{(-)}{G_{34}})\{\underset{(+)}{a_2(0,0,\beta)} + F_{55}(\beta)^2\} > 0 \tag{B4}$$

$$a_4(0,0,\beta) = 0, \quad a_5(0,0,\beta) = 0 \tag{B5}$$

$$\Delta_3(0,0,\beta) \equiv a_3(0,0,\beta)\Delta_2(0,0,\beta) > 0 \tag{B6}$$

$$a_4(\alpha_1,0,\beta) = \alpha_1 G_{34} \begin{vmatrix} G_{11} & G_{12} & G_{15} \\ F_{21} & G_{12} & G_{15} \\ F_{51}(\beta) & 0 & F_{55}(\beta) \end{vmatrix} = \alpha_1 G_{34} \begin{vmatrix} G_{11} - F_{21} & 0 & 0 \\ F_{21} & G_{12} & G_{15} \\ F_{51}(\beta) & 0 & F_{55}(\beta) \end{vmatrix}$$

$$= \alpha_1 \underset{(-)}{G_{34}}\,(\underset{(-)}{G_{11} - F_{21}})\,\underset{(-)}{G_{12}}\,\underset{(-)}{F_{55}}(\beta) > 0 \quad \text{for all} \quad (\alpha_1, \beta) > (0,0) \tag{B7}$$

$$a_5(\alpha_1,0,\beta) = 0 \quad \text{for all} \quad (\alpha_1, \beta) > (0,0) \tag{B8}$$

$$\Delta_4(\alpha_1,0,\beta) \equiv a_4(\alpha_1,0,\beta)\Delta_3(\alpha_1,0,\beta) \tag{B9}$$

$$a_5(\alpha_1,\alpha_2,\beta)>0 \quad \text{for all} \quad (\alpha_1,\alpha_2,\beta)>(0,0,0) \tag{B10}$$

$$\Delta_5(\alpha_1,\alpha_2,\beta) \equiv a_5(\alpha_1,\alpha_2,\beta)\Delta_4(\alpha_1,\alpha_2,\beta) \tag{B11}$$

The inequalities (B1), (B4), and (B6) imply that we have

$$\Delta_1>0,\ \Delta_2>0,\ \Delta_3>0 \quad \text{for all sufficiently small} \quad (\alpha_1,\alpha_2)>(0,0) \tag{B12}$$

by continuity. It also means that we have $\Delta_3(\alpha_1,0,\beta)>0$ for all sufficiently small $\alpha_1>0$. It follows from this fact, (B7) and (B9) that we have $\Delta_4(\alpha_1,0,\beta)>0$ for all sufficiently small $\alpha_1>0$, which also means by continuity that

$$\Delta_4>0 \quad \text{for all sufficiently small} \quad (\alpha_1,\alpha_2)>(0,0). \tag{B13}$$

Finally, it follows from (B10), (B11) and (B13) that

$$\Delta_5>0 \quad \text{for all sufficiently small} \quad (\alpha_1,\alpha_2)>(0,0). \tag{B14}$$

(B12), (B13) and (B14) imply that all of the Routh-Hurwitz conditions for stable roots (a set of inequalities (33) in the text) are in fact satisfied for all sufficiently small $(\alpha_1,\alpha_2)>(0,0)$ when $\beta>0$ is fixed at any level.

(ii) Suppose that $\alpha_1>0$ or $\alpha_2>0$ is set to be sufficiently large for any given level of $\beta>0$. Then, we have $\Delta_1 = a_1<0$ because of **Assumption 1**, which violates one of the Routh-Hurwitz conditions for stable roots.

(iii) It is easy to see that $\Delta_2 = a_1 a_2 - a_3$ becomes to be a quadratic function of β such that

$$\Delta_2 = A\beta^2 + B\beta + C, \tag{B15}$$

where A, B, and C are independent of β, and

$$A = -\{\underset{(-)}{r^1_{M1}} + (1/\bar{E})\underset{(-)}{r^2_{M2}}\}[\{\underset{(-)}{r^1_{M1}} + (1/\bar{E})\underset{(-)}{r^2_{M2}}\}(\alpha_1 \underset{(+)}{G_{11}} + \underset{(-)}{G_{12}} + \alpha_2 \underset{(+)}{G_{33}} + \underset{(-)}{G_{34}})$$

$$+\underset{(+)}{\alpha_2\, r_{Y2}^2}\, \underset{(-)}{G_{35}}]. \tag{B16}$$

If the inequality (34) is satisfied, we have

$$\lim_{\beta\to+\infty}(\Delta_2/\beta^2) = A<0. \tag{B17}$$

Inequality (B17) means that we have $\Delta_2<0$ for all sufficiently large $\beta>0$, which violates one of the Routh-Hurwitz conditions for stable roots.

Notes

(1) This is in fact the continuous time version of the model in Asada, Inaba and Misawa(2001), which is formulated as a system of difference equations.

(2) In the case of two countries in the European Union after the year 2002, we can set $E = E^e = 1$.

(3) $I_{Yi}^i = \partial I_i/\partial Y_i$, $r_{Yi}^i = \partial r_i/\partial Y_i$, $m_1 = -\partial J_1/\partial Y_1$, $m_2 = \partial J_1/\partial Y_2$ etc. and all partial derivatives are evaluated at the equilibrium point.

(4) Obviously, we can select α_2 as a bifurcation parameter instead of α_1.

(5) Note that the equilibrium values of the variables are *independent* of the parameters α_1 and α_2.

(6) As for the Hopf Bifurcation theorem, see Gandolfo(1996) and Lorenz(1993). As for the concept of the 'simple' Hopf Bifurcation, see **Appendix A** of this paper. **Theorem A2** (Liu's theorem) in this appendix implies that we have a set of conditions $\Delta_1>0$, $\Delta_2>0$, $\Delta_3>0$, $\Delta_4 = 0$, and $a_5>0$ at such a bifurcation point.

(7) The numerical examples which are presented in this section is the adapted two-regional versions of Asada(1995)'s numerical example in a small open economy. The purpose of the numerical simulation in this section is rather limited. It is not purported to be quantitatively realistic, but it is simply purported to illustrate the qualitative

insight which was obtained theoretically.

(8) In these figures, t denotes time (years). We adopted the Euler's algorithm to approximate the system of differential equations $\dot{x}(t) = F(x(t))$ by the system of difference equations $x(t+\Delta t) = x(t) + (\Delta t)F(x(t))$ for the computer simulations, where x is the vector of the variables. We consider the unit time interval as a year, and we adopted the time interval $\Delta t = 0.01$ (years).

(9) In case of the adaptive expectations, the functional form $\Phi(\bullet)$ will become as $\Phi(\bullet) = \gamma\{E(Y_1, Y_2, E^e; \beta) - E^e\}$, where γ is a positive constant.

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