Delayed Nonlinear Cournot and Bertrand Dynamics with Product Differentiation

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Abstract

Dynamic duopolies will be examined with product differentiation and isoelastic price functions. We will first prove that under realistic conditions the equilibrium is always locally asymptotically stable. The stability can however be lost if the firms use delayed information in forming their best responses. Stability conditions are derived in special cases, and simulation results illustrate the complexity of the dynamism of the systems. Both price and quantity adjusting models are discussed.

1 Introduction

Since the pioneering work of Cournot (1838), many researchers have examined the different variants of oligopoly models. A comprehensive summary of the earlier work has been presented in Okuguchi (1976), and some extended models including multiproduct oligopolies are given in Okuguchi and Szidarovszky(1999). The existence and uniqueness of the equilibrium was first the main focus of the studies and then the interest has turned to the different dynamic extensions. The field of dynamic oligopolies is very rich. It includes models with discrete and continuous time scales, oligopolies with and without product differentiation, quantity and price adjusting schemes, multi-product models, rent-seeking and market-share games, labor managed oligopolies to mention only a few. The complexity of such models is very well illustrated in Puu (2003) and in Puu and Sushko (2002).

In this paper we will examine dynamic duopolies with product differentiation and isoelastic price functions. In a recent paper Matsumoto and Onozaki (2006) have analyzed such models with both linear and nonlinear demand functions. The profitability of quantity and price strategies were compared and the authors demonstrated circumstances under which complex dynamics occur. Yousefi and Szidarovszky (2006) have presented a simulation study with random model parameters in which the number of equilibria, stability conditions, equilibrium prices were compared in price and quantity adjusting models. Both discrete and continuous time scales were considered.

2 Differentiated Nonlinear Duopoly Model

There are two firms, firm 1 and firm 2, and two goods, x_1 and x_2, in a market. The goods are differentiated, so that each firm faces a different demand curve and sells its good at different price. Inverse demand functions are given by

$$\begin{aligned} P_1 &= \frac{1}{\alpha_1 x_1 + \beta_1 x_2 + \gamma_1} \\ P_2 &= \frac{1}{\beta_2 x_1 + \alpha_2 x_2 + \gamma_2} \end{aligned} \tag{1}$$

with $\alpha_i \in R^+$ and $\beta_i, \gamma_i \in R$. Here γ_i defines the maximum price $P_i^M = \frac{1}{\gamma_i}$ when zero-productions take place. Solving the above equations for x_i gives direct demand functions,

$$\begin{pmatrix} x_1 \\ \\ x_2 \end{pmatrix} = \frac{1}{\alpha_1\alpha_2 - \beta_1\beta_2} \begin{pmatrix} \alpha_2 & -\beta_1 \\ \\ -\beta_2 & \alpha_1 \end{pmatrix} \begin{pmatrix} \dfrac{1}{P_1} - \gamma_1 \\ \\ \dfrac{1}{P_2} - \gamma_2 \end{pmatrix}, \tag{2}$$

where $\frac{1}{P_i} - \gamma_i > 0$ or $P_i^M > P_i$ should hold due to the specifications of inverse demand functions. Substituting new variables q_i and θ_i defined by

$$q_i = \frac{P_i}{1 - \gamma_i P_i} \text{ and } \theta_i = \frac{\beta_i}{\alpha_i}$$

into (2) gives

$$\begin{pmatrix} x_1 \\ \\ x_2 \end{pmatrix} = \frac{1}{1 - \theta_1\theta_2} \begin{pmatrix} \dfrac{1}{\alpha_1} & -\dfrac{\theta_1}{\alpha_2} \\ \\ -\dfrac{\theta_2}{\alpha_1} & \dfrac{1}{\alpha_2} \end{pmatrix} \begin{pmatrix} \dfrac{1}{q_1} \\ \\ \dfrac{1}{q_2} \end{pmatrix}.$$

Further, introducing a new variable p_i defined as

$$p_i = \alpha_i q_i \text{ or } p_i = \frac{\alpha_i}{\gamma_i} \frac{P_i}{(P_i^M - P_i)},$$

turns the direct demand into a simplified form,

$$\begin{cases} x_1 = \dfrac{1}{1 - \theta_1\theta_2}\left(\dfrac{1}{p_1} - \dfrac{\theta_1}{p_2}\right), \\ \\ x_2 = \dfrac{1}{1 - \theta_1\theta_2}\left(\dfrac{1}{p_2} - \dfrac{\theta_2}{p_1}\right). \end{cases} \tag{3}$$

To keep the regular property that demand responds negatively to a change in its price, we make the following assumption:

Assumption 1. $0 < \theta_i < 1$.

Solving (3) for p_1 and p_2 yields the inverse demand function with new variables,

$$\begin{cases} p_1 = \dfrac{1}{x_1 + \theta_1 x_2}, \\ \\ p_2 = \dfrac{1}{\theta_2 x_1 + x_2}, \end{cases} \tag{4}$$

where θ_i indicates a degree of differentiation of good i to the other good: two are perfect substitute for $\theta_i = 1$, and one firm monopolizes a market for $\theta_i = 0$. Assumption 1 is reasonable because the case with differentiated goods can be considered to be intermediate between the two extreme cases, the perfect substitute case with $\theta_i = 1$ and the monopoly case with $\theta_i = 0$. In the following, we use the simplified versions of the inverse and direct demand functions, (4) and (3).

3 Cournot Competition

3.1 Cournot Equilibrium

Firm k produces differentiated good x_k with constant marginal cost c_k and sells it with price p_k. It determines output so as to maximize its profit,

$$\pi_k = \frac{x_k}{x_k + \theta_k x_{3-k}} - c_k x_k,$$

for $k = 1, 2$. Solving the first order conditions of interior optimum yields reaction functions of firms. For the sake of the latter analysis, the implicit forms are given here:

$$\theta_k x_{3-k} = c_k (x_k + \theta_k x_{3-k})^2. \tag{5}$$

These implicit expressions define reaction curves in the quantity space. An intersection of these curves determines Cournot outputs. Dividing (5) with $k = 1$ by the one with $k = 2$ leads to

$$\frac{\theta_1}{\theta_2}\frac{x_2}{x_1} = \frac{c_1}{c_2}\left(\frac{x_1 + \theta_1 x_2}{\theta_2 x_1 + x_2}\right)^2. \tag{6}$$

By introducing new variables

$$z = \frac{x_2}{x_1} \text{ and } c = \frac{c_2}{c_1},$$

we can re-write the ratio of the reaction functions, (6), in terms of these new variables,

$$c\frac{\theta_1}{\theta_2}z = \left(\frac{1 + \theta_1 z}{\theta_2 + z}\right)^2. \tag{7}$$

Since this is a cubic equation in z, it is possible to derive its explicit solutions. However, they are too complicated to use in the following analysis. Thus, instead of solving (7) explicitly, we view this equation as the intersection of the straight line with the quadratic polynomial and confirm an existence (i.e., intersection) of a ratio of Cournot outputs. Let us denote the left hand and right hand sides of (7), respectively, by $f_c(x)$ and $g(z)$, namely

$$f_c(z) = c\frac{\theta_1}{\theta_2}z \text{ and } g(z) = \left(\frac{1+\theta_1 z}{\theta_2 + z}\right)^2.$$

It can be checked that $g(z)$ is positive for all z with a positive intercept on the vertical axis, bounded from below, strictly decreasing, and strictly convex in z,[1]

$$g(0) = \left(\frac{1}{\theta_2}\right)^2 > 1, \ \lim_{z\to\infty} g(z) = \theta_1^2 < 1 \text{ and } g'(z) < 0 \text{ and } g''(z) > 0.$$

Since $f_c(z)$ is linear and strictly increasing with $f_c(0) = 0$, the two curves cross exactly once under Assumption 1. We denote the solution by α that is a ratio of Cournot outputs produced by the two firms. It is a function of parameters c, θ_1 and θ_2. That is,

$$c\frac{\theta_1}{\theta_2}\alpha = \left(\frac{1+\theta_1\alpha}{\theta_2+\alpha}\right)^2 \Rightarrow \alpha = \alpha(c,\theta_1,\theta_2) \text{ and } \alpha = \frac{x_2^C}{x_1^C}.$$

The value of α can be any positive number depending on the value of c and is strictly decreasing in c. In particular, it converges to infinity or zero as c goes to zero or infinity. Substituting $x_2^C = \alpha x_1^C$ into (6) and solving the resultant equation for x_1 provides explicit expressions of Cournot outputs in terms of exogenously determined parameters, c, θ_1 and θ_2:

$$\begin{aligned} x_1^C &= \frac{\alpha\theta_1}{c_1(1+\alpha\theta_1)^2} = \frac{\theta_2}{c_2(\theta_2+\alpha)^2}, \\ x_2^C &= \frac{\theta_1}{c_1(\theta_1+\alpha^{-1})^2} = \frac{\alpha\theta_2}{c_2(\theta_2+\alpha)^2}. \end{aligned} \tag{8}$$

We now consider separately continuous Cournot dynamical systems without and with time delays.

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[1] Differentiating $g(z)$ and $g'(z)$ yields

$$g'(z) = \frac{2(1+\theta_1 z)(\theta_1\theta_2 - 1)}{(\theta_2+z)^3} < 0,$$

and

$$g''(z) = \frac{2(3-\theta_1\theta_2+2\theta_1 z)(1-\theta_1\theta_2)}{(\theta_2+z)^4} > 0$$

where the directions of inequalities are due to Assumption 1.

3.2 Continuous Dynamics without Time Delays

Solving (5) for output gives the explicit form of reaction functions

$$R_1(x_2) = \sqrt{\frac{\theta_1 x_2}{c_1}} - \theta_1 x_2,$$

$$R_2(x_1) = \sqrt{\frac{\theta_2 x_1}{c_2}} - \theta_2 x_1.$$

The continuous dynamic system is

$$(C_1) : \begin{cases} \dot{x}_1(t) = k_1 \left(R_1(x_2(t)) - x_1(t) \right), \\ \\ \dot{x}_2(t) = k_2 \left(R_2(x_1(t)) - x_2(t) \right), \end{cases}$$

where the dot over a variable means a time derivative, and k_i $(i = 1, 2)$ is an adjustment coefficient and assumed to be positive. The Jacobian is

$$J_C = \begin{pmatrix} -k_1 & k_1\gamma_1 \\ \\ k_2\gamma_2 & -k_2 \end{pmatrix},$$

where γ_i is the derivative of firm i's reaction function evaluated at Cournot equilibrium,

$$\gamma_1 = \frac{\alpha^{-1} - \theta_1}{2} \text{ and } \gamma_2 = \frac{\alpha - \theta_2}{2}. \tag{9}$$

The characteristic equation is derived as

$$\lambda^2 + (k_1 + k_2)\lambda + k_1 k_2 (1 - \gamma_1\gamma_2) = 0.$$

The linear coefficient is positive. Next we will show that the constant term is also positive implying that the roots have negative real parts. Clearly,

$$\gamma_1\gamma_2 = \frac{1}{4}\left(1 + \theta_1\theta_2 - (\alpha\theta_1 + \frac{1}{\alpha}\theta_2)\right). \tag{10}$$

Since $\alpha + \frac{1}{\alpha} \geq 2$ for any $\alpha > 0$, we have $\alpha\theta_1 + \frac{1}{\alpha}\theta_2 \geq 2\min(\theta_1, \theta_2)$. Therefore,

$$\gamma_1\gamma_2 \leq \frac{1}{4}\left(1 + [\theta_1\theta_2 - 2\min(\theta_1, \theta_2]\right) < \frac{1}{4}$$

where the last inequality is due to $\theta_1\theta_2 - 2\theta_k = \theta_k(\theta_{3-k} - 2) < 0$. Notice in addition that the value of $\gamma_1\gamma_2$ can be any real value between $-\infty$ and $\frac{1}{4}$ by the appropriate choice of α. Thus, we have the following results:

Theorem 1 *Given Assumption 1, Cournot continuous model is always locally asymptotically stable.*

3.3 Continuous Dynamics with Time Delays

Assume that firm k has a time lag T_k in collecting and implementing information on the output of the competition as well as a time lag S_k in its own output. Similar situation occurs when the firms want to react to average information rather than to sudden changes. Then the dynamic system with fixed time lags is written as

$$(C_2):\begin{cases} \dot{x}_1(t) = k_1\left(R_1(x_2(t-T_1)) - x_1(t-S_1)\right), \\ \\ \dot{x}_2(t) = k_2\left(R_2(x_1(t-T_2)) - x_2(t-S_2)\right). \end{cases}$$

This is a system of delayed- (or difference-) differential equations. However, for the dynamical system with fixed delays, the characteristic polynomial becomes a mixed polynomial-exponential equation with infinitely many roots. So spectrum becomes infinite, and therefore stability analysis becomes complicated. Fixed time delays are not realistic in real economies, since the length of any delay is uncertain. Therefore continuously distributed time lags describe the situation more accurately. For the dynamical system with continuously distributed time lags, we have finite spectrum, and it is well known that the integro-differential equation is equivalent to a finite set of ordinary differential equations. Thus, if firm k's expectation of competitor's output is denoted by x^e_{3-k} and that of its own output by x^ε_k, then the dynamism can be written as the system of integro-differential equations

$$\dot{x}_1(t) = k_1\left(R_1(x^e_2(t)) - x^\varepsilon_1(t)\right),$$

$$\dot{x}_2(t) = k_2\left(R_2(x^e_1(t)) - x^\varepsilon_2(t)\right),$$

where for $k = 1, 2$,

$$x^e_k(t) = \textstyle\int_0^t w(t-s, T_k, m_k)x_k(s)ds,$$

$$x^\varepsilon_k(t) = \textstyle\int_0^t w(t-s, S_k, \ell_k)x_k(s)ds.$$

Here the weighting function $w(t-s, \Gamma, n)$ is defined as

$$w(t-s,\Gamma,n) = \begin{cases} \dfrac{1}{\Gamma}e^{-\frac{t-s}{\Gamma}} & \text{if } n = 0, \\ \\ \dfrac{1}{n!}\left(\dfrac{n}{\Gamma}\right)^{n+1}(t-s)^n e^{\frac{-n(t-s)}{\Gamma}} & \text{if } n \geq 1, \end{cases}$$

where n is a nonnegative integer and Γ is a positive real parameter. Since this system is equivalent to a system of ordinary differential equations (Chiarella and Szidarovszky (2002)), all tools known from the stability theory of differential equations can be applied in this case as well.

To examine local dynamics of the above system in a neighborhood of the equilibrium point, we consider the linearized system,

$$\begin{aligned}\dot{x}_{1,\delta}(t) &= k_1\left(\gamma_1\int_0^t w(t-s,T_1,m_1)x_{2,\delta}(s)ds-\int_0^t w(t-s,S_1,l_1)x_{1,\delta}(s)ds\right)\\ \dot{x}_{2,\delta}(t) &= k_2\left(\gamma_2\int_0^t w(t-s,T_2,m_2)x_{1,\delta}(s)ds-\int_0^t w(t-s,S_2,l_2)x_{2,\delta}(s)ds\right)\end{aligned}$$

where $x_{k,\delta}(t)$ is the deviation of $x_k(t)$ from its equilibrium level. Looking for the solution in the usual exponential form

$$x_{k,\delta}(t)=v_k e^{\lambda t},\ \ k=1,2,$$

we substitute this into the linearized system to obtain

$$(\lambda+k_1\textstyle\int_0^t w(t-s,S_1,l_1)e^{-\lambda(t-s)}ds)v_1-k_1\gamma_1\int_0^t w(t-s,T_1,m_1)e^{-\lambda(t-s)}dsv_2=0,$$

$$-k_2\gamma_2\textstyle\int_0^t w(t-s,T_2,m_2)e^{-\lambda(t-s)}dsv_1+(\lambda+k_2\int_0^t w(t-s,S_2,l_2)e^{-\lambda(t-s)}ds)v_2=0.$$

Notice next that allowing $t\to\infty$ yields

$$\int_0^\infty w(s,\Gamma,n)e^{-\lambda s}ds=\left(1+\frac{\lambda\Gamma}{q}\right)^{-(n+1)}$$

with

$$q=\begin{cases}1 & \text{if } n=0,\\ n & \text{if } n\geq 1.\end{cases}$$

So we have finally

$$\begin{pmatrix}A_1(\lambda) & B_1(\lambda)\\ B_2(\lambda) & A_2(\lambda)\end{pmatrix}\begin{pmatrix}v_1\\ v_2\end{pmatrix}=0,$$

where

$$\begin{aligned}A_i(\lambda) &= \left(\lambda\left(1+\frac{\lambda S_i}{q_i}\right)^{(l_i+1)}+k_i\right)\left(1+\frac{\lambda T_i}{r_i}\right)^{(m_i+1)},\\ B_i(\lambda) &= -k_i\gamma_i\left(1+\frac{\lambda S_i}{q_i},\right)^{(l_i+1)},\end{aligned}$$

with

$$q_i = \begin{cases} 1, & \text{if } l_i = 0 \\ \\ l_i, & \text{if } l_i \geq 1 \end{cases}$$

and

$$r_i = \begin{cases} 1, & \text{if } m_i = 0 \\ \\ m_i, & \text{if } m_i \geq 1. \end{cases}$$

Non-trivial solution exits if and only if

$$A_1(\lambda)A_2(\lambda) - B_1(\lambda)B_2(\lambda) = 0$$

,or

$$\prod_{i=1}^{2} \left(\lambda \left(1 + \tfrac{\lambda S_i}{q_i} \right)^{(l_i+1)} + k_i \right) \left(1 + \tfrac{\lambda T_i}{r_i} \right)^{(m_i+1)} - \prod_{i=1}^{2} k_i \gamma_i \left(1 + \tfrac{\lambda S_i}{q_i} \right)^{(l_i+1)} = 0. \tag{11}$$

If there are no time delays, $T_1 = T_2 = 0$ and $S_1 = S_2 = 0$, then (11) is reduced to

$$(\lambda + k_1)(\lambda + k_2) - k_1 k_2 \gamma_1 \gamma_2 = 0,$$

which is the same characteristic equation as the one that we already derived above. We will next show some simple special cases, where analytical results can be obtained. The more complicated cases can be examined by using computer methods.

Case 1. $T_1 > 0$ **and** $T_2 = 0$.

Let us begin with the simplest case. Assume that only firm 1 has the information lag about its rival's output, $T_1 > 0$ and $T_2 = 0$, furthermore neither firm has lag in its own output, $S_1 = S_2 = 0$. We also assume that $m_1 = 0$. The characteristic equation, (11), becomes

$$(\lambda + k_1)(\lambda + k_2)\left(1 + \lambda T_1\right) - k_1 k_2 \gamma_1 \gamma_2 = 0, \tag{12}$$

which is cubic in λ:

$$T_1 \lambda^3 + (1 + T_1(k_1 + k_2))\lambda^2 + (k_1 + k_2 + T_1 k_1 k_2)\lambda + k_1 k_2 (1 - \gamma_1 \gamma_2) = 0. \tag{13}$$

All coefficients are positive, so roots have negative real parts, according to Routh-Hurwitz condition,[2] if and only if

$$\left(1 + T_1(k_1 + k_2)\right)\left(k_1 + k_2 + T_1 k_1 k_2\right) > T_1 k_1 k_2 (1 - \gamma_1 \gamma_2).$$

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[2] A necessary and sufficient condition that all the roots of equation

$$a_0 \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n = 0$$

With fixed T_1, k_1, and k_2, the condition holds if

$$\gamma_1\gamma_2 > -\frac{(k_1+k_2)(1+T_1k_1)(1+T_1k_2)}{T_1k_1k_2}.$$

In Figure 1, in which $k_1 = k_2 = 0.8$, the shaded region is a set of $(T_1, \gamma_1\gamma_2)$ for which the above inequality is violated. As can be seen, the Cournot equilibrium becomes unstable for large negative $\gamma_1\gamma_2$ while it is stable if there is no or small time lag as Theorem 1 assures. Thus it can be said that a time lag on competitor's output might have a destabilizing effect, which we sum up as follows.

Insert Figure 1 Here.

Theorem 2 *An information lag on competitor's output might destabilize the otherwise stable Cournot continuous model.*

Let us go back to equation (11) to show the existence of a limit cycle. According to the Hopf bifurcation theorem, we can establish the existence if the Jacobian of the dynamical system evaluated at the equilibrium has a pair of pure imaginary roots and the real part of these roots vary with a bifurcation parameter.[3] We first select $1-\gamma_1\gamma_2 \equiv z$ as the bifurcation parameter and then calculate its value at the point for which loss of stability just occurs. It is obtained by substituting the stability condition with equality into the bifurcation parameter,

$$z^* = 1-\gamma_1\gamma_2 = \frac{(1+T_1(k_1+k_2))\,(k_1+k_2+T_1k_1k_2)}{T_1k_1k_2}.$$

In this case, the cubic equation, (13), can be written as

$$T_1\lambda^3 + (1+T_1(k_1+k_2))\lambda^2 + (k_1+k_2+T_1k_1k_2)\lambda + k_1k_2(1-\gamma_1\gamma_2)$$

$$= \left(\lambda + \frac{1+T_1(k_1+k_2)}{T_1}\right)(T_1\lambda^2 + (k_1+k_2+T_1k_1k_2)) = 0,$$

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with real positive coefficients have negative real parts is that the following conditions hold,

$$\begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} > 0, \quad \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} > 0, \cdots$$

[3] See, for example, Guckenheimer and Holmes (1983) for more details of the Hopf bifurcation theorem.

that can be explicitly solved for λ. One of the characteristic roots is negative real and the other two are pure imaginary:

$$\begin{aligned} \lambda_1 &= -\frac{1+T_1(k_1+k_2)}{T_1} < 0, \\ \lambda_{2,3} &= \pm i\sqrt{\frac{k_1+k_2+T_1k_1k_2}{T_1}} = \pm i\xi. \end{aligned}$$

To apply the Hopf bifurcation theorem, we need to check whether the real part of the complex roots is sensitive to a change in the bifurcation parameter. Suppose λ as a function of z, $\lambda(z)$, then by implicit differentiation of equation (13) we have

$$3T_1\lambda^2\frac{d\lambda}{dz} + 2\lambda(1+T_1(k_1+k_2))\frac{d\lambda}{dz} + (k_1+k_2+T_1k_1k_2)\frac{d\lambda}{dz} + k_1k_2 = 0$$

implying that

$$\frac{d\lambda}{dz} = -\frac{k_1k_2}{3T_1\lambda^2 + 2\lambda(1+T_1(k_1+k_2)) + (k_1+k_2+T_1k_1k_2)}.$$

Rationalizing the right hand side and noticing that the terms with λ are imaginary and the constant and quadratic terms are real yields the following form of the real part of the derivative of λ with respect to the bifurcation parameter:

$$Re\left(\frac{d\lambda}{dz}\right) = -\frac{k_1k_2(3T_1\lambda^2+k_1+k_2+T_1k_1k_2)}{(3T_1\lambda^2+k_1+k_2+T_1k_1k_2)^2 + (2\xi)^2(1+T_1(k_1+k_2))^2} \neq 0,$$

since at the critical value,

$$3T_1\lambda^2 + k_1 + k_2 + T_1k_1k_2 = -2(k_1+k_2+T_1k_1k_2) \neq 0 \text{ and } \xi \neq 0.$$

Therefore the Hopf bifurcation theorem applies, and thus a birth of limit cycle is assured around the equilibrium at the critical value.

In performing numerical simulation we first derived the 3-dimensional system of ordinary differential equations which is equivalent to our systems (as described in Chiarella and Szidarovszky (2002)) and then selected the values of parameters. Returning to Figure 1, we set $T_1 = T_m$ where $T_m = 1/\sqrt{k_1k_2}$. The corresponding value of $\gamma_1\gamma_2$ for $T_1 = T_m$ is

$$\gamma_m = -\frac{(k_1+k_2)(\sqrt{k_1}+\sqrt{k_2})^2}{k_1k_2},$$

which is the maximum value $\gamma_1\gamma_2$ under the current circumstance. Setting $k_1 = k_2 = 0.8$ yields $\gamma_m = -8$. We further set $c = 0.01$ and $\theta_2 = 0.5$. Since, at the Cournot equilibrium, γ_m satisfies

$$\gamma_m = \frac{(\alpha(\theta_1, c, \theta_2) - \theta_2)(\alpha(\theta_1, c, \theta_2)^{-1} - \theta_1)}{4},$$

solving the equation gives $\theta_1 = 0.803$. Taking account of these parameter values for which the system loses its stability, we specify the parameter values as follows:

$$k_1 = k_2 = 0.8, \ \theta_1 = 0.815, \ \theta_2 = 0.5, \ c_1 = 1, \ \text{and } c_2 = 0.00975.$$

Figure 2 displays a complete limit cycle surrounding the Cournot equilibrium point denoted by C.

Insert Figure 2 Here.

Case 2. $T_1 > 0$ **and** $T_2 > 0$.

In this case we allow both firms to have a lag about the competitor's output. We assume again that $m_1 = l_1 = 0$. Then (11) becomes

$$(\lambda + k_1)(\lambda + k_2)(1 + \lambda T_1)(1 + \lambda T_2) - k_1 k_2 \gamma_1 \gamma_2 = 0,$$

that can be written as a quartic equation in λ,

$$a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0,$$

where coefficients are defined as

$$\begin{aligned}
a_0 &= T_1T_2, \\
a_1 &= T_1 + T_2 + T_1T_2(k_1 + k_2), \\
a_2 &= 1 + T_1T_2k_1k_2 + (k_1 + k_2)(T_1 + T_2), \\
a_3 &= k_1 + k_2 + k_1k_2(T_1 + T_2), \\
a_4 &= k_1k_2(1 - \gamma_1\gamma_2).
\end{aligned}$$

Since all coefficients are positive, the Routh-Hurwitz theorem implies that roots have negative real parts if and only if

$$\begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} > 0 \text{ and } \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & a_4 & a_3 \end{vmatrix} > 0.$$

The first condition is satisfied as a simple calculation shows that the second order determinant is always positive. It depends on the value of $\gamma_1\gamma_2$ whether the second condition is satisfied. Solving the second inequality for $\gamma_1\gamma_2$ gives the stability condition

$$\gamma_1\gamma_2 > -\frac{(k_1+k_2)(1+k_1T_1)(1+k_2T_1)(T_1+T_2)(1+k_1T_1)(1+k_2T_2)}{k_1k_2\left(T_1+T_2+T_1T_2(k_1+k_2)\right)^2}.$$

which is clearly violated if $\gamma_1\gamma_2$ is negative with large absolute values.

The parameter space of positive T_1 and negative $\gamma_1\gamma_2$ is divided into three areas in Figure 3 in which we set $T_2 = 1$. The white area represents a set of parameters for which the equilibrium is stable. The shaded area represents a set of the same parameters for which the equilibrium is unstable. It consists of two subregions, the light-gray region and the dark-gray region. The form is the unstable region constructed under the assumption of asymmetric information lag, $T_1 > 0$ and $T_2 = 0$, which is identical with the shaded region in Figure 1. On the other hand, the latter is the extended unstable region due to the assumption of symmetric information lags, $T_1 > 0$ and $T_2 > 0$. It can be observed that introducing the additional time lag T_2 enlarges the unstable region.

Theorem 3 *If each firm has information lag on its competitor's output, then the destabilizing effect strengthens.*

Insert Figure 3 Here.

Case 3. $T_1 > 0$ **and** $S_1 > 0$.

Instead of the information lag on the competitor's production level, we introduce an information lag on the firm's own output, $S_1 > 0$ and examine how such an alternation affects Cournot dynamics. The characteristic equation (11) becomes

$$(\lambda(1+\lambda S_1)+k_1)(\lambda+k_2)(1+\lambda T_1) - k_1k_2\gamma_1\gamma_2(1+\lambda S_1) = 0,$$

which is also a quartic equation in λ,

$$a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0,$$

where the coefficients are defined as

$$\begin{aligned}
a_0 &= S_1T_1,\\
a_1 &= S_1 + T_1 + k_2S_1T_1,\\
a_2 &= 1 + k_2S_1 + (k_1+k_2)T_1,\\
a_3 &= k_1 + k_2 + k_1k_2(T_1 - \gamma_1\gamma_2S_1),\\
a_4 &= k_1k_2(1-\gamma_1\gamma_2).
\end{aligned}$$

It is natural to assume for a firm that the information lag on competitor's output is longer than the lag on it's own output.

Assumption 2. $S_i < T_i$ for $i = 1, 2$.

The coefficient a_3 is positive under Assumption 2 and $\gamma_1\gamma_2 < \frac{1}{4}$. Applying the Routh-Hurwitz conditions, which is the same as in Case 2, we derive the stability conditions as,

$$\gamma_1\gamma_2 > -\frac{T_1(1+(k_1+k_2)T_1)+S_1(1+k_2T_1)(1+k_2(S_1+T_1))}{k_1k_2S_1^2T_1},$$

and by solving the second condition [4]

$$\gamma_1\gamma_2 > -\frac{A+B\sqrt{C}}{2k_1k_2S_1^3T_1},$$

where

$$\begin{aligned} A = \;& -T_1^2 + S_1T_1(-1+(k_1-k_2)T_1) + k_2S_1^3(1+k_2T_1) - \\ & S_1^2T_1(k_1+k_2+k_1k_2T_1), \\ B = \;& (S_1+T_1+k_2S_1T_1), \\ C = \;& (T_1+(k_2S_1-k_1T_1)S_1)^2 + 4k_1S_1^2T_1(1+k_2T_1). \end{aligned}$$

The second condition is stronger than the first as the following inequality always holds:

$$-\frac{A-B\sqrt{C}}{2k_1k_2S_1^3T_1} > -\frac{T_1(1+(k_1+k_2)T_1)+S_1(1+k_2T_1)(1+k_2(S_1+T_1))}{k_1k_2S_1^2T_1}.$$

Thus the stability of the equilibrium is guaranteed if $\gamma_1\gamma_2$ is nonnegative or negative with small absolute value:

$$\gamma_1\gamma_2 > -\frac{A-B\sqrt{C}}{2k_1k_2T_1S_1^3}.$$

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[4] The stability condition also imposes the upper bound on $\gamma_1\gamma_2$,

$$-\frac{A-B\sqrt{C}}{2k_1k_2S_1^3T_1} > \gamma_1\gamma_2.$$

It can be shown however that the upper bound is decreasing in T_1 and S_1. Putting $S_1 = T_1$ and increasing T_1 to infinity yields the convergence of the upper bound to unity, which is the minimum value of the upper bound and is greater than $\frac{1}{4}$, the maximum value of $\gamma_1\gamma_2$. Thus this inequality is as ineffective constraint.

In Figure 4 in which we set $k_1 = k_2 = 0.8$ and $S_1 = 1$, the parameter region of T_1 and $\gamma_1\gamma_2$ is divided into three subregions. The white region implies stability of the equilibrium while the shaded region implies instability. As in Figure 3, instability in the light-gray subregion is due to the lag on competitor's output. As can be seen, the light-grey region is enlarged by introducing the lag on the firm's own output. Thus it can be said that the time lag S_1 also has a destabilizing effect.

Insert Figure 4 Here.

Case 4. $T_1 > 0$**,** $T_2 > 0$ **and** $S_1 > 0$, $S_2 > 0$.

This is the most general case. By selecting $m_1 = m_2 = l_1 = l_2 = 1$, equation(11) becomes a polynomial of degree six,

$$\begin{aligned}&(\lambda(1+\lambda S_1)+k_1)(\lambda(1+\lambda S_2)+k_2)(1+\lambda T_1)(1+\lambda T_2)\\&-k_1k_2\gamma_1\gamma_2(1+\lambda S_1)(1+\lambda S_2)=0,\end{aligned}$$

which can be written as

$$a_0\lambda^6 + a_1\lambda^5 + a_2\lambda^4 + a_3\lambda^3 + a_4\lambda^2 + a_5\lambda^1 + a_6 = 0,$$

where the coefficients are defined by

$$\begin{aligned}
a_0 &= S_1S_2T_1T_2,\\
a_1 &= S_1S_2(T_1+T_2)+T_1T_2(S_1+S_2),\\
a_2 &= S_1S_2+(S_1+S_2)(T_1+T_2)+(1+k_2S_1+k_1S_2)T_1T_2,\\
a_3 &= (S_1+S_2)+(1+k_2S_1+k_1S_2)(T_1+T_2)+(k_1+k_2)T_1T_2,\\
a_4 &= 1+k_2S_1+k_1S_2+(k_1+k_2)(T_1+T_2)+k_1k_2(T_1T_2-\gamma_1\gamma_2S_1S_2),\\
a_5 &= k_1+k_2+k_1k_2((T_1-\gamma_1\gamma_2S_1)+(T_2-\gamma_1\gamma_2S_2)),\\
a_6 &= k_1k_2(1-\gamma_1\gamma_2).
\end{aligned}$$

All coefficients are positive by the same reasons as in Case 3. The Routh-Hurwitz conditons in this case are

$$\begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} > 0, \quad \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix} > 0, \quad \begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ a_5 & a_4 & a_3 & a_2 \\ 0 & a_6 & a_5 & a_4 \end{vmatrix} > 0,$$

The first condition can be confirmed. It is possible to solve the second and third conditions for $\gamma_1\gamma_2$. However the expressions are so complicated and difficult to explain that we represent only the numerical result showing how the instability region is affected.[5]

In Figure 5, the equilibrium becomes unstable for any combination of T_1 and $\gamma_1\gamma_2$ in the shaded region, which consists of four areas distinguished by different levels of gray color. The area labelled $T_1 > 0$ is the unstable set in Case 1. The area increases by the area labelled $T_2 > 0$ if the information lag T_2 is introduced as discussed in Case 2. Replacing T_2 with S_1 increases the unstable region by the area labelled $S_1 > 0$ and decreases by the small area surrounded by two bold lines in the lower-left corner. It is thus undetermined which effects is stronger, the destabilizing effect caused by T_2 or the one by S_1. Finally the area labelled $S_2 > 0$ represents an increase of the unstable region if all of four lags are taken into account. Figure 5 exhibits that the unstable region enlarges as the number of lags increases. However, different specification of parameters gives rise qualitatively a different result. In Figure 6, two different cases can be observed: one is that T_2 has a stronger destabilizing effect than S_1 as the enlargement of the unstable region caused by T_2 is much larger than the one by S_2; the other shows that increasing the number of lags stabilizes the market as indicated by the contraction of the area labelled S_2.

Insert Figures 5 and 6 Here.

Given θ_1 and θ_2 equation (10) indicates that $\gamma_1\gamma_2$ can become larger negative for either smaller α or larger α. We have checked that α is decreasing in c. Thereby α is smaller or larger according to the fact that c is larger or smaller. Since c is the ratio of production costs, a larger or smaller c is due to production inefficiency between the two firms. We summarize this instability result as follows.

Theorem 4 *Strong production efficiency can be a source of Cournot instability if continuously distributed time lag is involved in obtaining and implementing information about rival's output.*

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[5] It is numerically confirmed that the stability condition derived from the third condition is stronger than the one from the second condition. So only the stronger condition is depicted in Figure 5.

4 Bertrand Competition

4.1 Bertrand Equilibrium

Using the direct demand functions, (3), firms set prices of the products to maximize profits:

$$\pi_1 = \frac{1}{1-\theta_1\theta_2}\left(\frac{1}{p_1} - \frac{\theta_1}{p_2}\right)(p_1 - c_1), \tag{14}$$

and

$$\pi_2 = \frac{1}{1-\theta_1\theta_2}\left(\frac{1}{p_2} - \frac{\theta_2}{p_1}\right)(p_2 - c_2). \tag{15}$$

Assuming interior optimum, the first-order conditions imply the following implicit forms of the reaction functions: for the first firm,

$$c_1 p_2 = \theta_1 p_1^2$$

and for the second firm

$$c_2 p_1 = \theta_2 p_2^2 \; .$$

Solving these equations together for the unknown prices provides the Bertrand equilibrium prices,

$$p_1^B = \sqrt[3]{\frac{c_1^2 c_2}{\theta_1^2 \theta_2}}$$

and

$$p_2^B = \sqrt[3]{\frac{c_1 c_2^2}{\theta_1 \theta_2^2}}.$$

We then substitute these prices into the direct demand function (3) to obtain the Bertrand equilibrium outputs:

$$\begin{cases} x_1^B = \dfrac{1}{1-\theta_1\theta_2}\sqrt[3]{\dfrac{\theta_1^2\theta_2}{c_1c_2}}\left\{\sqrt[3]{\dfrac{1}{c_1}} - \sqrt[3]{\dfrac{\theta_1^2\theta_2}{c_2}}\right\} \\[2ex] x_2^B = \dfrac{1}{1-\theta_1\theta_2}\sqrt[3]{\dfrac{\theta_1\theta_2^2}{c_1c_2}}\left\{\sqrt[3]{\dfrac{1}{c_2}} - \sqrt[3]{\dfrac{\theta_1\theta_2^2}{c_1}}\right\}. \end{cases} \tag{16}$$

In order to eliminate negative production levels, we assume

Assumption 3. $\theta_1^2\theta_2 < c < \dfrac{1}{\theta_1\theta_2^2}$.

4.2 Continuous Dynamics without Time Delays

Solving the implicit forms of Bertrand reaction functions for price gives the explicit form of reaction functions

$$\begin{cases} R_1^B(p_2) = \sqrt{\dfrac{c_1 p_2}{\theta_1}}, \\ \\ R_2^B(p_1) = \sqrt{\dfrac{c_2 p_1}{\theta_2}}. \end{cases}$$

The continuous dynamic system is

$$(B_1) : \begin{cases} \dot{p}_1(t) = \kappa_1 \left(R_1^B(p_2(t)) - p_1(t) \right), \\ \\ \dot{p}_2(t) = \kappa_2 \left(R_2^B(p_1(t)) - p_2(t) \right), \end{cases}$$

where the dot over a variable means a time derivative, κ_i $(i = 1, 2)$ is an adjustment coefficient and assumed to be positive. The Jacobian is

$$J^B = \begin{pmatrix} -\kappa_1 & \kappa_1 \gamma_1^p \\ \\ \kappa_2 \gamma_2^p & -\kappa_2 \end{pmatrix},$$

where derivatives of firm k's reaction functions are

$$\gamma_1^B = \frac{1}{2} \sqrt[3]{\frac{c_1 \theta_2}{c_2 \theta_1}} \text{ and } \gamma_2^B = \frac{1}{2} \sqrt[3]{\frac{c_2 \theta_1}{c_1 \theta_2}}. \tag{17}$$

So $\gamma_1^B \gamma_1^B = \frac{1}{4}$. The characteristic equation is

$$\lambda^2 + (\kappa_1 + \kappa_2)\lambda + \kappa_1 \kappa_2 (1 - \gamma_1^B \gamma_2^B) = 0.$$

Since the coefficients are positive, the real part of characteristic roots are always negative. We summarize this results as follows:

Theorem 5 *Bertrand continuous model is always locally asymptotically stable.*

4.3 Continuous Dynamics with Time Delays

Assume now that firm k has continuously distributed time lags in the output of its competitor as well as in its own output. When the time delays are taken into account, the Bertrand integro-differential equation system becomes

$$(B_2) : \begin{cases} \dot{p}_1(t) = \kappa_1 \left(R_1^B(p_2^e(t)) - p_1^\varepsilon(t) \right), \\ \\ \dot{p}_2(t) = \kappa_2 \left(R_2^B(p_1^e(t)) - p_2^\varepsilon(t) \right), \end{cases}$$

where the expected price is

$$p_k^e(t) = \int_0^t w(t-s, T_k, m_k) p_k(s) ds \quad \text{for } k = 1, 2,$$

and

$$p_k^\varepsilon(t) = \int_0^t w(t-s, S_k, l_k) p_k(s) ds.$$

By almost the same procedure as the one we presented above, we have

$$\prod_{i=1}^{2} \left(\lambda \left(1 + \tfrac{\lambda S_i}{q_i}\right)^{(l_i+1)} + \kappa_i \right) \left(1 + \tfrac{\lambda T_i}{r_i}\right)^{(m_i+1)} - \prod_{i=1}^{2} \kappa_i \gamma_i^B \left(1 + \tfrac{\lambda S_i}{q_i}\right)^{(l_i+1)} = 0,$$

where q_i and r_i are the same as in equation (11).

If there is no time delay, then $T_1 = T_2 = S_1 = S_2 = 0$, so this equation reduces to

$$(\lambda + \kappa_1)(\lambda + \kappa_2) - \kappa_1 \kappa_2 \gamma_1^B \gamma_2^B = 0,$$

which is the same equation that was derived before.

Case 1. $T_1 > 0$ and $T_2 > 0$.

Assume next that $S_1 = S_2 = 0$, that is, the firms have no delays in their own outputs. In this case the characteristic equation becomes

$$(\lambda + \kappa_1)(\lambda + \kappa_2) \left(1 + \frac{\lambda T_1}{q_1}\right)^{(m_1+1)} \left(1 + \frac{\lambda T_2}{q_2}\right)^{(m_2+1)} - \kappa_1 \kappa_2 \gamma_1^B \gamma_2^B = 0.$$

We will easily prove that the system is always locally asymptotically stable. To check whether the system can be locally unstable, we assume that $Re(\lambda) \geq 0$. Then we have

$$\mid \lambda + \kappa_1 \mid \geq \kappa_1, \quad \mid \lambda + \kappa_2 \mid \geq \kappa_2, \quad \mid 1 + \frac{\lambda T_1}{q_1} \mid \geq 1, \quad \mid 1 + \frac{\lambda T_2}{q_2} \mid \geq 1.$$

Thus

$$(\lambda + \kappa_1)(\lambda + \kappa_2) \left(1 + \frac{\lambda T_1}{q_1}\right)^{(m_1+1)} \left(1 + \frac{\lambda T_2}{q_2}\right)^{(m_2+1)} \geq \kappa_1 \kappa_2.$$

On the other hand we have

$$\mid \kappa_1 \kappa_2 \gamma_1^B \gamma_2^B \mid = \frac{\kappa_1 \kappa_2}{4} < \kappa_1 \kappa_2.$$

Therefore, λ such that $Re(\lambda) \geq 0$ can't solve the equation. Thus, the equilibrium is locally asymptotically stable. We summarize this result in the following way:

Theorem 6 *Bertrand equilibrium is locally asymptotically stable even if time delays are introduced in the outputs of the competitors.*

Case 2. $T_1 > 0$**,** $S_1 > 0$ **and** $T_2 = S_2 = 0$.

We assume that $m_1 = l_1 = 0$ as in the cases of Cournot dynamics. The characteristic equation is

$$(\lambda(1 + \lambda S_1) + \kappa_1)(\lambda + \kappa_2)(1 + \lambda T_1) - \kappa_1\kappa_2\gamma_1^B\gamma_2^B(1 + \lambda S_1) = 0,$$

which is a quartic equation in λ and can be rewritten as

$$a_0\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_0 = 0.$$

This is the same as the one in Case 3 of Cournot dynamics except the simplifying equation $\gamma_1^B\gamma_2^B = \frac{1}{4}$. It can be confirmed by lengthy calculation that all coefficients are positive and the Routh-Hurwitz stability conditions are fulfilled. That is,

$$a_1a_2 - a_0a_3 = S_1(1 + \kappa_2 T_1) + T_1(1 + \kappa_1 T_1 + \kappa_2 T_2) + \kappa_2 S_1^2(1 + \frac{1}{4}\kappa_1 T_1 + \kappa_2 T_2) > 0$$

and

$$(a_1a_2 - a_0a_3)a_3 - a_0^2a_4 > 0$$

as $T_1 \geq S_1$. Notice that this condition is realistic. The first inequality is obvious, and the second can be proved based on the facts, that the value and derivative with respects to T_1 of the left hand side at $T_1 = S_1$ are both positive. Furthermore its second derivative with respect to T_1 is also positive. Hence we have the following result:

Theorem 7 *The equilibrium is locally asymptotically stable even if only one firm faces time delays.*

Case 3. $T_1 > 0$**,** $T_2 > 0$ **and** $S_1 > 0, S_2 > 0$.

With $m_1 = m_2 = l_1 = l_2 = 0$, the characteristic equation becomes

$$(\lambda(1+\lambda S_1)+\kappa_1)(\lambda(1+\lambda S_2)+\kappa_2)(1+\lambda T_1)(1+\lambda T_2) - \frac{\kappa_1\kappa_2}{4}(1+\lambda S_1)(1+\lambda S_2) = 0,$$

which is a polynomial equation of degree 6.

The application of the Routh-Hurwitz criterion to check stability is too complicated in this case. Instead looking for analytical results we performed a computer study. In a large number (several thousands) of cases we could always observe local asymptotic stability. So we presume that Bertrand dynamics are always asymptotically stable, but we could not prove it in general.

5 Conclusion

The local asymptotical stability of Cournot and Bertrand dynamics were examined under the assumption that there is a time delay for the firms in collecting and implementing information about the outputs of the rivals and also about their own outputs. We have proved that both dynamics are locally asymptotically stable without time lag. This stability can be however lost in Cournot dynamics if time delays are introduced. Stability conditions were derived and in the case when instability occurs, bifurcation was observed. For Bertrand dynamics we could prove that local asymptotic stability is preserved when only one firm faces time lags. We could not prove similar result in the general case, but simulation study indicates that stability is maintained even in the general case. For the sake of mathematical simplicity, we considered only exponential kernel functions ($m = l = 0$). The analysis of the asymptotical behavior of the equilibrium with positive m and l values will be the subject of a future paper.

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