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Discussion Paper No.194

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Shumei Hirai
Chuo University

Ferenc Szidarovszky
University of Pécs

October 2012

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INSTITUTE OF ECONOMIC RESEARCH
Chuo University
Tokyo, Japan

Existence and Uniqueness of Equilibrium in Asymmetric Contests with Endogenous Prizes*

Shumei Hirai† and Ferenc Szidarovszky‡

Abstract

This paper considers contests in which the efforts of the players determine the value of the prize. Players may have different valuations of the prize and different abilities to convert expenditures to productive efforts. In addition, players may face different financial constraints. This paper presents a proof for the existence and uniqueness of a pure Nash equilibrium in asymmetric contests with endogenous prizes.

JEL Classification: D72, C72, L13

Keywords: Contests; Endogenous prize; Existence and uniqueness

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*The earlier version of this paper was presented at the 2012 China International Conference on Game Theory and Application, Qingdao University, Qingdao, China, August 29-30, 2012. The authors wish to thank participants and Professor Akio Matsumoto of Chuo University for their suggestions. The usual disclaimer applies.

†Assistant Professor, Department of Economics, Chuo University, 742-1, Higashi-Nakano, Hachioji, Tokyo, 192-0393, Japan, hirai-shumei@jcom.home.ne.jp/hirai@tamacc.chuo-u.ac.jp

‡Professor, Department of Applied Mathematics, University of Pécs, Ifjusag u.6, H-7624, Pécs, Hungary, szidarka@gmail.com/szidar@sie.arizona.edu

1 Introduction

A contest is a strategic game in which players expend costly effort in order to increase their probability of winning a given prize. Since the pioneering work of Tullock (1980) and Dixit (1987), there is now a large and growing literature on the theory and application of contests.[1] One of the most important questions is the existence and uniqueness of pure Nash equilibrium. It has been extensively studied under the assumption of an exogenous prize; see e.g. Pérez-Castrillo and Verdier (1992), Szidarovszky and Okuguchi (1997), Cornes and Hartley (2005), and Yamazaki (2008).

However, many contests, such as R&D contest and war, involve a form of effort that changes the size of the total prize as well as its distribution. For instance, higher R&D effort increases the likelihood of making a more valuable discovery. Hence, firms' R&D efforts have a positive externality on the value of the prize. In a war, the contested land will be damaged by the struggle for it, thereby reducing the economic value of the prize available to the ultimate victor. Thus, military conflict have a negative externality of the contest efforts. Chung (1996) has first analyzed a rent-seeking contest with an endogenous prize (rent), which is increasing in aggregate efforts of the players. Okuguchi (2005) and Corchón (2007) showed that there exists a unique symmetric pure Nash equilibrium in Chung's endogenous contest with a general contest success function. Shaffer (2006) considers the case in which increased effort reduces the value of the prize. In these studies, players are assumed to be identical in terms of abilities and valuations of the prize. In addition, the authors implicitly assume that all players have budget large enough such that the budget constraints do not bind at all.

In practical situations, there can be three types of heterogeneity among players in a contest. First, each player may have a different valuation of the prize (e.g., Hillman and Riley, 1989). Secondly, each player may have a different ability to convert expenditures to productive efforts (e.g., Baik, 1994). Thirdly, players may

[1] See the excellent survey by Konrad (2007).

face different financial constraints (e.g., Che and Gale, 1997). Hence, in this paper, we prove that there exists a unique asymmetric pure Nash equilibrium in a contest with these three type of heterogeneity among players and an endogenous prize. The method used by Szidarovszky and Yakowitz (1977) and Cornes and Hartley (2005) will be used to show the existence and uniqueness of the pure Nash equilibrium.

The rest of the paper is organized as follows. Section 2 explains the basic model and the assumptions. In section 3, we prove that there exists a unique pure Nash equilibrium. Concluding remarks are presented in Section 4.

2 The Model

Let n be the number of players in a contest. Players are assumed to be risk-neutral. If x_i is player i's expenditure in contests, then the probability for winning the prize is given as

$$p_i = \frac{f_i(x_i)}{\sum_{j=1}^{n} f_j(x_j)} \tag{1}$$

where $f_i(\cdot)$ is an increasing function for all i.[2] Szidarovszky and Okuguchi (1997) called $f_i(\cdot)$ player i's production function for lotteries. We assume that each player has a finite wealth $\tilde{L}_i$, which is the budget constraint on what player i can spend on contests: $x_i \leq \tilde{L}_i$. Then, in line with most of the existence literature, we adopt the following assumption.

Assumption 1. *For all i the function f_i satisfies the following conditions:*
f_i is twice differentiable, $f_i(0) = 0$, and $f_i'(x_i) > 0$, $f_i''(x_i) < 0$ for all $x_i \in [0, \tilde{L}_i]$.

Notice that players' production functions and budgets do not necessarily have to be identical. A particularly well-studied form for f_i is $f_i(x_i) = a_i x_i^r$, where $r > 0$ and $a_i > 0$. This asymmetric form was given an axiomatic foundation by Clark and Riis (1998), following an earlier axiomatization by Skaperdas (1996) of the symmetric form.

[2]Another interpretation of p_i is that each player i receives a fraction $\frac{f_i(x_i)}{\sum_{j=1}^{n} f_j(x_j)}$ of the contested prize.

It will prove convenient to change variables by setting $y_i = f_i(x_i)$ for each i. Then the function $f_i(\cdot)$ may be thought of as transforming individual expenditure x_i into effective efforts y_i. We will henceforth refer to x_i as the *expenditure*, and y_i as the *effort*, of player i. Since f_i is monotonic, it has a well-defined inverse function, $g_i(y_i) = f_i^{-1}(y_i)$. Then, Assumption 1 (A.1 in what follows) implies that

$$g_i(0) = 0, \text{ and } g_i'(y_i) > 0, \; g_i''(y_i) > 0 \text{ for all } y_i \in [0, f_i(\bar{L}_i)]. \tag{2}$$

The function $g_i(y_i)$ describes the total cost to player i of generating the level y_i of effort.

Next, we introduce the following assumptions on the prize as a function of the aggregate effort by all players. Let $Y = \sum_{j=1}^{n} y_j$ be the aggregate effort by all players and $L_i = f_i(\bar{L}_i)$ be the player i's maximum effort due to his or her budget constraint.

Assumption 2. *For all i the value of the prize is endogenously determined by the aggregate effort: $R_i(Y)$. $R_i(Y)$ is twice differentiable and satisfies $R_i(Y) > 0$ for $Y \in [0, \sum_{i=1}^{n} L_i]$ and weakly concave in $Y \in [0, \sum_{i=1}^{n} L_i]$.*

Notice that the weak concavity property in A.2 allows for positive as well as for negative externalities of the aggregate effort. For example, a functional form of R_i is $R_i(Y) = \bar{R}_i + b_i Y$, where $\bar{R}_i > 0$. $\bar{R}_i$ is player i's intrinsic value of the prize and b_i is i's coefficient of enhancement (if $b_i > 0$) or destruction (if $b_i < 0$) of the prize by aggregate efforts. A.2, together with A.1 , ensures that a player's expected payoff is strictly concave function of her own effort. In addition, A.2 implies that the elasticity of the prize with respect to change in the aggregate effort is at most unity for positive Y. We will write $\epsilon_i = Y R_i'(Y)/R_i(Y)$ for the elasticity of the prize of player i.

Then, the expected payoff of player i is described by

$$\pi_i(y_i, Y_{-i}) = R_i(Y)p_i - x_i = R_i(y_i + Y_{-i})\frac{y_i}{y_i + Y_{-i}} - g_i(y_i), \tag{3}$$

where $Y_{-i} = \sum_{j \neq i}^{n} y_j$. Expression (3) holds if at least one player makes a positive effort. If $y_i = 0$ for all i we assume that no player wins the prize so that $\pi_i(0, 0) = 0$.

Player i is assumed to maximize (3) with respect to y_i subject to $y_i \in [0, L_i]$. Our analysis of contests is formulated as a simultaneous-move game and the solution concept we use throughout paper is that of a pure Nash equilibrium.

3 Existence Analysis

We can now calculate the best response of player i. Assume first that $Y_{-i} > 0$, so that the other players spend a positive amount of resources on contest activities, then

$$\frac{\partial \pi_i}{\partial y_i} = R_i'(y_i + Y_{-i})\frac{y_i}{y_i + Y_{-i}} + R_i(y_i + Y_{-i})\frac{Y_{-i}}{(y_i + Y_{-i})^2} - g_i'(y_i) \tag{4}$$

and under assumptions A.1 and A.2

$$\frac{\partial^2 \pi_i}{\partial y_i^2} = R_i''(y_i + Y_{-i})\frac{y_i}{y_i + Y_{-i}} - 2R_i(y_i + Y_{-i})\frac{Y_{-i}}{(y_i + Y_{-i})^3}(1 - \epsilon_i) - g_i''(y_i) < 0. \tag{5}$$

Hence π_i is strictly concave in y_i. The concavity of the payoff functions implies that the best response functions can be obtained in the form

$$\phi_i(Y_{-i}) = \begin{cases} 0 & \text{if } \frac{R_i(Y_{-i})}{Y_{-i}} - g_i'(0) \leq 0, \\ L_i & \text{if } R_i'(L_i + Y_{-i})\frac{L_i}{L_i + Y_{-i}} + R_i(L_i + Y_{-i})\frac{Y_{-i}}{(L_i + Y_{-i})^2} - g_i'(L_i) \geq 0, \\ y_i^* & \text{otherwise,} \end{cases} \tag{6}$$

where y_i^* is the unique solution of the strictly monotonic equation

$$R_i'(y_i + Y_{-i})\frac{y_i}{y_i + Y_{-i}} + R_i(y_i + Y_{-i})\frac{Y_{-i}}{(y_i + Y_{-i})^2} - g_i'(y_i) = 0 \tag{7}$$

in interval $(0, L_i)$. Observe that by our assumptions the left hand side of (7) strictly decreases and is continuous in y_i, positive at $y_i = 0$ and negative at $y_i = L_i$, therefore there is a unique solution, $\phi_i(Y_{-i})$. We notice that if $Y_{-i} = 0$, player i's payoff has a maximum at a finite and positive value of effort, which can be obtained from the solution of equation (7) with $Y_{-i} = 0$ due to assumptions A.1 and A.2. If it is below L_i, then it is the best response of player i at $Y_{-i} = 0$ and if it is above L_i, then L_i is the best response. It is well known that a vector $(\bar{y}_1, \cdots, \bar{y}_n)$ is an equilibrium if and only if for all i, $\bar{y}_i$ is the best response with fixed values of $\bar{Y}_{-i}$.

From (6), we can also rewrite the best responses as functions of the aggregate effort of all players:

$$\Phi_i(Y) = \begin{cases} 0 & \text{if } \frac{R_i(Y)}{Y} - g_i'(0) \leq 0, \\ L_i & \text{if } R_i'(Y)\frac{L_i}{Y} + R_i(Y)\frac{Y-L_i}{Y^2} - g_i'(L_i) \geq 0, \\ y_i^{**} & \text{otherwise,} \end{cases} \tag{8}$$

where y_i^{**} solves equation

$$R_i'(Y)\frac{y_i}{Y} + R_i(Y)\frac{Y - y_i}{Y^2} - g_i'(y_i) = 0 \tag{9}$$

in interval $(0, L_i)$. Notice that in the third case of (8), the left hand side is positive at $y_i = 0$, negative at $y_i = L_i$, and strictly decreasing, since it has a negative derivative given by

$$\frac{\partial}{\partial y_i}\{R_i'(Y)\frac{y_i}{Y} + R_i(Y)\frac{Y - y_i}{Y^2} - g_i'(y_i)\} = -\frac{R_i(Y)(1-\epsilon_i)}{Y^2} - g_i''(y_i) < 0,$$

where the sign comes from assumptions A.1 and A.2. Therefore there is a unique solution of equation (9), which is a continuously differentiable function of $Y > 0$ by the implicit function theorem. Following Wolfstetter (1999, p. 91), we call this function the *inclusive reaction function* of player i, which is proposed by Szidarovszky and Yakowitz (1977).

Then, consider the single-variable equation

$$\sum_{i=1}^{n} \Phi_i(Y) - Y = 0, \tag{10}$$

which must hold at an equilibrium. The left hand side, denoted by $H(Y)$, has the following properties. It is continuous, since all $\Phi_i(Y)$ are continuous, $H(0) \geq 0$, since $\Phi_i(Y) \geq 0$ for all i and $H(\sum_{i=1}^{n} L_i) \leq 0$, since $\Phi_i(Y) \leq L_i$. Therefore, there is at least one solution. In order to discuss the uniqueness of equilibrium we will need the derivative of the inclusive reaction function. Implicitly differentiating equation (9) with respect to Y and considering $y_i = \Phi_i(Y)$, we have

$$R_i''\frac{y_i}{Y} + R_i'\frac{\Phi_i'Y - y_i}{Y^2} + \frac{R_i'Y - R_i}{Y^2}(1 - \frac{y_i}{Y}) - \frac{R_i(\Phi_i'Y - y_i)}{Y^3} - g_i''\Phi_i' = 0$$

implying that

$$\Phi_i'(Y) = \frac{y_i Y R_i'' - R_i(1-\epsilon_i)(1-\frac{2y_i}{Y})}{R_i(1-\epsilon_i) + g_i'' Y^2}.$$

Here the denominator is always positive but the sign of the numerator is not determined by assumptions A.1 and A.2. Hence, $\Phi_i'(Y)$ is not necessarily monotonic. In view of equation (10), this fact creates slight additional difficulties in order to prove uniqueness of equilibrium.

So, consider function

$$h_i(Y, s_i) = R_i'(Y)s_i + \frac{R_i(Y)}{Y}(1-s_i) - g_i'(s_i Y) \tag{11}$$

with $s_i = y_i/Y$. The function h_i is the marginal payoff of player i expressed in terms of aggregate effort and share. Notice that under assumptions A.1 and A.2

$$\frac{\partial h_i}{\partial Y} = R_i''(Y)s_i - \frac{R_i(Y)}{Y^2}(1-\epsilon_i)(1-s_i) - g_i''(s_i Y)s_i < 0$$

as $s_i \leq 1$, furthermore

$$\frac{\partial h_i}{\partial s_i} = -\frac{R_i(Y)}{Y}(1-\epsilon_i) - g_i''(s_i Y)Y < 0.$$

Hence $h_i(Y, s_i)$ decreases in both variables. Define now the player i's *share function* $S_i(Y) = \Phi_i(Y)/Y$, which is proposed by Cornes and Hartley (2005). It follows from player i's inclusive reaction function (8) and equation (11) that we have

$$S_i(Y) = \begin{cases} 0 & \text{if } h_i(Y,0) \leq 0, \\ 1 & \text{if } Y \leq L_i \text{ and } h_i(Y,1) \geq 0, \\ L_i/Y & \text{if } Y > L_i \text{ and } h_i(Y, L_i/Y) \geq 0, \\ s_i^* & \text{otherwise,} \end{cases} \tag{12}$$

where s_i^* is the unique solution of equation

$$R_i'(Y)s_i + \frac{R_i(Y)}{Y}(1-s_i) - g_i'(s_i Y) = 0 \tag{13}$$

in interval $(0,1)$ if $Y \leq L_i$ or in interval $(0, L_i/Y)$ if $Y > L_i$. Notice that if the first case of (12) occurs, then by the monotonicity of $h_i(Y, s_i)$, neither cases 2, 3 and 4 of (12) can occur, and if one of cases 2, 3 and 4 holds, then the first case must not occur. Hence for a given value of $Y > 0$, exactly one case holds. The left hand side of equation (13) is positive at $s_i = 0$, negative at $s_i = 1$ (for $Y \leq L_i$) or at $s_i = L_i/Y$ (for $Y > L_i$), and strictly decreases in s_i. Therefore, there is a unique solution s_i^* which is differentiable by the implicit function theorem.

In our further analysis we will need the derivative of the share function. By differentiating equation (13) with respect to Y and considering $s_i = S_i(Y)$, we have

$$R_i'' s_i + R_i' S_i' + \frac{R_i' Y - R_i}{Y^2}(1 - s_i) - \frac{R_i}{Y} S_i' - g_i''(s_i Y)(S_i' Y + s_i) = 0$$

implying that

$$S_i'(Y) = \frac{(g_i'' - R_i'')s_i Y + \frac{R_i(1-s_i)}{Y}(1 - \epsilon_i)}{-R_i(1 - \epsilon_i) - g_i'' Y^2} < 0.$$

The inequality follows since the denominator is negative and the numerator is positive for $Y > 0$ in light of assumptions A.1 and A.2. So, $S_i(Y)$ is continuous with constant and strictly decreasing segments. Then, equation (10) can be also rewritten as

$$\sum_{i=1}^{n} S_i(Y) - 1 = 0, \tag{14}$$

where the left hand side is non-increasing. Assume that there are two different solutions $\bar{Y} < \bar{Y}'$. Then, $\bar{Y}' > 0$ and at least one $S_i(\bar{Y}') > 0$. In this case either $S_i(\bar{Y}) = S_i(\bar{Y}') = 1$ or $S_i(\bar{Y}) > S_i(\bar{Y}')$ and for all $j \neq i$, $S_j(\bar{Y}) \geq S_j(\bar{Y}')$. In the first case player i has two different maximizers with $Y_{-i} = 0$, which is impossible, since π_i is strictly concave. In the second case

$$\sum_{i=1}^{n} S_i(\bar{Y}) > \sum_{i=1}^{n} S_i(\bar{Y}')$$

which is also an obvious contradiction. Therefore, the equilibrium value of Y is unique. Given an equilibrium $\bar{Y}$, the corresponding unique strategy profile $(\bar{y}_1, \cdots, \bar{y}_n)$ is found by multiplying $\bar{Y}$ by each player's share evaluated at $\bar{Y}$: $\bar{y}_i = \bar{Y} S_i(\bar{Y})$. Hence we proved the following result:

Theorem 1. *Under assumptions A.1 and A.2, there exists a unique pure Nash equilibrium in asymmetric contests with endogenous prizes.*

Finally, notice that for each player i and any fixed value of Y_{-i}, the solution $y_i = 0$ always gives zero payoff value for this player. Therefore, at the best response, it must be non-negative. Hence, under assumptions A.1 and A.2, each player enjoys non-negative expected payoff at the equilibrium.

4 Conclusions

This paper provides a proof of the existence and uniqueness of the equilibrium in asymmetric contests with endogenous prizes where equations (12) and (14) suggest a practical method to compute it. The results can be applied to many areas, such as R&D contests, military conflicts, labor tournaments, and cooperative productions in which the size of prize is endogenously determined.

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