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Models with Random Exchangeable Structures, and Coexistence of Several Types Agents in the Long-Run: New Implementations of Schumpeter's Dynamics

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Abstract

We propose a procedure for building macroeconomic models of a large number of interacting agents by dividing the collection of agents by "types" into clusters or subsets. These clusters form random exchangeable partitions. We argue that this approach is useful in implementng some of the ideas originally proposed by Schumpter by adding more information on dynamic aspects of his proposals.

Introduction

Economists often face problems of examining collective behavior of a large number of interacting agents, where agents are generally heterogeneous in their characteristics such as their risk attidutes, and their circumstances, that is, constraints they face, and so forth.

We use the word "type" to denote agents of different categories or classes. This word should be interpreted broadly. Agents may be differentiated by the behavioral rules they use, by their risk attitudes, the products they produce, services they render, and so forth. We assume that the number of types are at most countable.

One of the important components in macroeconomic modeling is the specification of entry behavior of agents. We do not assume in advance the number of types is fixed. Agents of new or unanticipated types may enter the market. Economists do not have a well-established procedure for building macroeconomic models with this contingency, that is, they do not deal with situations of unanticipate knowledge in the terminology by Zabell (1992).

We observe here a close parallel with the ideas of random exchangeable partitions by Kingman (1978 a, b). Zabell (1992) discusses the distinction between the problems in which the types of agents is fixed in advance, and those with new, so far unsuspected types of agents may enter in the future. This latter is the situation of unanticipated knowledge according to Zabell. The entry by agents or goods in economics is exactly this situation since new goods, optimizaation procedures, manufacturing methods and so on are known to arise in the future, even though we do not know exactly when they enter the economy. The conditional expectations that an agent next enters the market is of a new type is related to the laws of sucession of Johnson, as exposited by Zabell (1982, 1992).

This paper proposes a new approach to macroeconomic model building which explicitly deals with clusters of agents, including new ones. We use stationary distributions of order statistics of cluster sizes to draw inferences on model behaviors in markets in stochastic equilibria. We adapt tools that are originally developed by population genetists such as Ewens, Watterson, and by mathematicians and statisticians such as Kingman, Zabell, Hoppe, among others.

Exchangeable Random Partitions

This section explains briefly that the notion of exchangeable random partitions, which is due to Kingman (1978a, 1978b). This is used to discuss a large number of agents forming clusters of various sizes. We follow Zabell (1992) to describe the notion. Appendix has further details in order to make this paper more self-contained.

This notion extends that of exchangeable sequences of de Finetti, which is appropriate when the number of categories is finite and known in advance. Two exchangeable sequences have the same probability if one is a rearrangement of the other. Thus, the sequences are invariant with respect to permutations of time indices of the sequences of samples or agents who enter the market. In exhcnageable partitons, the sequences are not only invariant with respect to time indicies but also with respect to permutations of category or cluster indices.

Given that there are n agents, we denote by $[n]$ the set, $\{1, 2, \dots, n\}$. The set $[n]$ is partitioned into K_n subsets or clusters. A random partion Π_n is a random object whose values are partitions of a set $[n]$. This set is partitioned into $\{A_1, A_2, \dots A_K\}$ where A_i has n_i elements. This is denoted by $|A_i| = n_i$. Partition vector $\mathbf{a}$ is a vector made up of a_i, which is the number of clusters or blocks with i elements, $i = 1, 2, \dots, n$.[1] We have the accounting identity relations: $\sum_{i=1}^n a_i = K$, is the total number of types in the model, and $\sum_{i=1}^n i a_i = n$ is the total number of agents in the model. To emphasize that the number of types varies with the sample, we sometimes write K_n as the number of clusters of $[n]$.

Random partitions are called exchangeable if two partitions with the

[1] This definition is due to Zabell. Kingman has the same concept, but his name is specific to genetic application. So we follow Zabell.

same partition vectors have the same probabilities. Note that the notion of exchangebility of random sequences, which has to do with invariance of probability with respect to permutation of time indices of samples or agents, are extended in random partitions to invariance of probability with repsect to time indicies and to category or type indicies. See Appendix for further details.

It may help to recall the occupancy problem in elementary probability textbooks in which identical-looking fresh tennis balls are to be placed in unmarked undistinguishable boxes. Then, the only way to describe the configurations of how balls are distributed among boxes is in terms of partition vectors of balls. There are so many boxes containing one ball, two balls, and so on. More precisely, partition vector $\mathbf{a} = (a_1, a_2, \dots, a_n)$ are such that a_i counts the number of boxes with exactly i balls, that is, the number of categories or types with exactly i agents.

Let $X_1, X_2, \dots$ be an infinite sequence of random variables taking on K distinct values, which are possible categories or types of agents, and where $\mathbf{n}$ is the frequency vector, that is, empirical distribution. n_i is the number of Xs observed to be of type i. By exchangeability, each of the seqeunces with the same frequendcy vector is equi-probable. Therefore, In the case of exchangeable random sequences we have

$$\Pr(X_1, X_2, \dots, X_n | \mathbf{n}) = \frac{n_1! n_2! \cdots n_K!}{n!}.$$

In the case of exchangeable random partitions with partition vector $\mathbf{a}$, its probability is given by

$$\Pr(\mathbf{a}|n) = \frac{n!}{\theta^{[n]}} \prod_{j=1}^{n} \frac{\theta^{a_j}}{j^{a_j} a_j!},$$

where $\theta^{[n]} = \theta(\theta+1)(\theta+2)\cdots(\theta+n-1)$ denotes an ascending factorial. It has one parameter θ which controls the correlatedness of agents. Higher the value of θ a randomly selected two agents are less correlated.

There are a number of ways to derive this. One way is to relate the partition vector to the cyclic product representation of permutation by interpreting a_i as the number of cycles of length i. Then we ask how many permutations are there with a given partition vector. Permutations are not distinct either because all cycles containing the same elements in the same cyclic order are the same, or because relative positions of cycles is immaterial. In a cycle with j elements, there are j possible first elements of the cycles. Hence ther are j^{a_j} duplicates. Since the relative positions of a_j cycles in the permutation do not matter. This gives rise to $a_j!$ fators of duplications. Therefore, among $n!$ permutations the number of distinct permutations with the same partition vector $\mathbf{a}$ is given by

$$C_n(\mathbf{a}) = \frac{n!}{1^{a_1} 2^{a_2} \cdots n^{a_n} a_1! a_2! \cdots a_n!}.$$

The probability is therefore given by the inverse of this expression when all permutations are equally likely. This is the special case with $\theta = 1$. When permutations with k cycles are weighted proportional to θ^k with positive θ, we derive the formula shown above. See also Aoki (2002, A.9) for further details.

We use partition vectors as state vectors in our model.

Instead of the Polya urn, the generalized Polya urn with one ball of special color (black in Hoppe and called mutator by Zabell) generates the correct conditional expectation when the Polya urn is modified by a device that when the black ball is drawn, the black ball and a ball of new type is returned into the urn. See Hoppe (1987).

Ewens distribution is the stationary distribution function of the partition vector and is given by the last equation above. This distribution is also known as the Ewens sampling formula or MED (multivarite Ewens distribution). See Ewens (1972) or Johnson, Kotz, and Balakrishnan (1997).

There are also several ways of deriving the above as the stationary distributions of back-ward Chapman-Kolmogorov equation (also called master equations in the physics literature and in Aoki (1996, 2002) by associating continous-time Markov chains to state vectors. Kingman also used this equation in some of his writings without identifying it as such when he introduces the notion of coalesence, Kingman (1978a, b, 1982).

Market Share Distributions

We use shares of a market by n agents as a motivating example to arrive at the Poisson-Dirichlet distribution which plays an important role in what follows. We follow Kingman (1993) and suppose that $x_1, x_2, \ldots x_n$ are the market shares of n firms, that is $x_i \geq 0$, and $\sum_i^n x_i = 1$. Suppose that these share have the Dirichlet distribution with parameter α, $\mathcal{D}(\alpha, n)$. Its density is given by

$$f(x_1, x_2, \ldots, x_n) = \frac{\Gamma(n\alpha)}{\Gamma(\alpha)^n}(x_1 x_2 \cdots x_n)^{\alpha-1}.$$

This is a probability density on the simplex Δ_n. This density is symmetric, and the shares are exchangeable.

Next, consider their order statistics, that is, rearrange xs and let

$$x_{(1)} \geq x_{(2)} \cdots \geq x_{(n)}.$$

We let n goes to infinity, and let $n\alpha$ approach θ. The distribution of the infinite random sequence $\xi = (\xi_1, \xi_2, \ldots)$, satisfying $\xi_1 \geq \xi_2 \geq \cdots$, $\sum_i \xi_i = 1$ depends only on θ and is called Poisson-Dirichlet distribution, $PD(\theta)$.

The important of this distribution in genetics and ecology has been noted by Kingman and many others. We show that this distribution is also important in economic applications as well. For small values of θ a few shares dominate. For example, see Aoki (2000). In this paper, it is shown that the largest two shares may determine market clearing prices under certain conditions.

Unfortunately, this distribution is rather difficult to deal with, even though Watterson and Guess have used elementary methods to obtain joint probabability distributions for the largest r shares. See also Arratia, Barbour, and Tavaré (2003), and Pitman (2003) for additional details.

There is a simpler distribution than PD distribution, called size-biased version, from which PD distribution can be derived. See Kingman (1993, Chap. 9), Hoppe (1987), or Carlton (1999).

We say that $\{Q_n\}$ has a GEM distribution with parameter θ, when it is generated from $\{W_n\}$, which are i.i.d. Beta $(1, \theta)$ distribution,

$$Q_1 = W_1, Q_n = (1 - W_1) \cdots (1 - W_{n-1}) W_n, n = 2, 3, \ldots.$$

When these Q's are rearranged in non-increasing order as $Q_{(1)} \geq Q_{(2)} \geq \ldots$, then the sequence $\{Q_{(n)}\}$ is known to have a Poisson-Dirichlet distribution, denoted by $PD(\theta)$. See Carlton (1999, p. 7).

There is one more important notion, called invariance under size-biased permutation, Kingman (1993, p.98) Given a sequence $\{P_n\}$, permute this sequence to generate a sized-bised sequence $\{\tilde{P}_n\}$ by

$$\Pr(\tilde{P}_1 = P_n | P_1, P_2, \ldots) = P_n,$$

and for $j = 2, 3, \ldots$

$$\Pr(\tilde{P}_{j+1} = P_n | \tilde{P}_1, \ldots \tilde{P}_j, P_1, P_2, \ldots) = P_n / [1 - \tilde{P}_1 - \tilde{P}_2 \ldots - \tilde{P}_j],$$

provided that P_n is not equal to any of the $\tilde{P}_i, i = 1, 2, \ldots, j$.

When the distribution of the resampled sequence is the same as the original one, the sequence is called invariant under the size-biased permutation. Pitman (1996) has shown that the probability sequence $\{P_n\}$ is invariant under size-biased permutation if and only if the sequence is a GEM (θ) for some positive θ. The ranked sequence $\{P_n\}$ of GEM (θ) is a Poisson-Dirichlet distribution $PD(\theta)$.

Law of Sucession: Entries by New Type

Johnson did not consider entries by unknown types, but his sufficientness postulate addresses the question of what is the type of next entrant. Let $X_1, X_2, \ldots, X_N$ be an exchangeable sequence of K-valued random variables such that for every $n \leq N$ the joint probability

$$\Pr(X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n) > 0,$$

for all $(i_1, \ldots i_n)$, and

$$\Pr(X_{n+1} = i | X_1, X_2, \ldots, X_n) = f_i(n_i),$$

where $f_i(n_i) = a_i + b^{(n)} n_i$, with a_i non-negative, for K greater than 2, Zabell (1982, Th2.1).

When some unknown type, that is, a type not foreseen or anticipated enter, the law of sucession becomes[2]

$$\Pr(X_{n+1} = \text{new} | X_1, X_2, \ldots, X_n) = \frac{\theta}{n + \theta}.$$

[2]Later, Pitman generalizes the one-parameter PD(θ) to a two-parameter version.

Distribution of Order Statistics

Given that we have observed a number of types of agents so far, summarized by the empirical distribution, and given the Poisson-Dirichlet prior, we obtain

$$\Pr(X_{N+1} = c_k | \mathbf{n}) = \frac{n_k}{N + \theta}.$$

Dynamics of Clustering Processes

We use continous-time Markov chains to model economic models in general. To express transition rates in partition vectors, define $\mathbf{u}_1 = \mathbf{e_1}$, and for $i \geq 2$

$$\mathbf{u}_i = \mathbf{e}_i - \mathbf{e}_{i-1},$$

where the vector $\mathbf{e}_i$ has the only non-zero entry at the ith component which is one. When an agent leaves a group of j agents, the number of clusters of size j is reduced by one and that of size $j-1$ increases by one. That is, a_j is reduced by one and a_{j-1} is increased by one. This transition rate is therefore expressed by $w(\mathbf{a}, \mathbf{a} + \mathbf{u}_j)$. For example we may define a process with

$$w(\mathbf{a}, \mathbf{a} + \mathbf{u}_j) = \lambda \frac{j a_j}{n}.$$

Kelly (1979, p. 146) has discussed a birth-death process which can be expressed in this matter. He has derived the equilibrium distribution of the process

$$\Pr(\mathbf{a}) = const \times \prod_{j=1}^{\infty} \frac{\beta^{a_j}}{a_j!},$$

where $\beta_j = x^j/j$, with $x = \lambda/\mu$.

Economic Applications

There are many examples with entries by unknown or unanticipated agents.[3] Here, we mention two that appeared elsewhere. Aoki (2000) gives one application of the Ewens sampling formula in finance, and mentions power-laws as the tail-distribution for the returns of certain financial assets. In this application, stocks of a holding company is traded by a large number of agents. With $\theta = 0.3$, two largest groups of agents are shown to capture nearly 80 percent of the market shares and hence dominate the market excess demands for the shares, which in turn determines the stationary distributions of returns. This example shows that for small values of θ, that is, when agents are correlated to some significant degree, two types of agents may not be a bad approximation.

Aoki (2002, Sec.7.3 and 7.4) have several models of industries wtih innovators and imitators. Aoki (1996) has an example of technical diffusion. In this model the new and old technology co-exist in long-run equilibrium.

[3] The notion of coalescence of Kingman (1982) may be applied to trace genealogy of products or how firms grow. We will treat this subject in another paper.

There are two equilibria. In the good equilibrium a majority of the firms adopt new technology, while in the bad equilibrium only a minority of firms ends up adopting the new technolgy. This is an example in the spirit of Iwai (2001) since statistical equilibria of two technologies co-exist in the long-run. The economy of this model will not approach a classical or neoclassical equilibria of uniform technology. Iwai states that this phenomena contradicts the long-held tradition in economics about the determination of the normal rates of profit.

In another application Aoki and Yoshikawa (2003) models of entries of new production factors into various sectors of economy using the laws of succession. The notion of holding times of a continous-time Markov chain is used to probabilistically determine which sector jump first. Different sectors grow at different rates. This example will be outlined later.

Example of innovating and imitating sectors

Consider a sector of economy composed of two types of firms. Firms of type 1 are technically advanced and benefit from occaisional innovations, while firms of type 2 are not so blessed with innovations. Let $n_1(t)$ and $n_2(t)$ be the number of firms in the two sectors (measured in some convenient and fixed units.) We use a continous-time Markov chain to describe the time evolution of $\mathbf{n(t)}$ with two components. Let $\mathbf{e}_i$, $i = 1, 2$ be two-dimensional vectors with the only non-zero element 1 in the first and second component, respectively. Then, the transition rates of the process is determined uniquely by the transition rates

$$q(\mathbf{n}, \mathbf{n} + \mathbf{e}_1) = c_1 n_1 + f_1,$$

$$q(\mathbf{n}, \mathbf{n} + \mathbf{e}_2) = c_2 n_2,$$

$$q(\mathbf{n}, \mathbf{n} - \mathbf{e}_j) = d_j n_j, \ j = 1, 2,$$

$$q(\mathbf{n}, \mathbf{n} + \mathbf{e}_1 - \mathbf{e}_2) = \mu g_1 n_2 (n_1 + h_1),$$

and

$$q(\mathbf{n}, \mathbf{n} + \mathbf{e}_2 - \mathbf{e}_1) = \mu g_2 n_2.$$

In the above $g_i = c_i/d_i$, $i = 1, 2$.

Let $P(\mathbf{n}(t))$ be the probability distribution of the vector $\mathbf{n}(t)$. Its time evolution is governed by the backward Chapman-Kolmogorov equation (also called as master equation, the name we use from now on),

$$\frac{\partial P(\mathbf{n}(t))}{\partial t} = I(\mathbf{n}(t)) - O(\mathbf{n}(t)),$$

where the first term represents the inflow and the second outflow of probability fluxes. For example, the outflow is given by

$$O(\mathbf{n}(t)) = P(\mathbf{n}(t))[\sum_{j=1}^{2} \{q(\mathbf{n}, \mathbf{n} + \mathbf{e}_j) + q(\mathbf{n}, \mathbf{n} - \mathbf{e}_j)\}$$

$$+ q(\mathbf{n}, \mathbf{n} + \mathbf{e}_1 - \mathbf{e}_2) + q(\mathbf{n}, \mathbf{n} + \mathbf{e}_2 - \mathbf{e}_1)]$$

.

The expressions for the inflows are similar, and consist of six terms, the first of which is $q(\mathbf{n}+\mathbf{e}_1, \mathbf{n})P(\mathbf{n}(\mathbf{t})+\mathbf{e}_1$.

The time evolution of the probability distribution must be obtained by solving this master equation. It can be equally be done by solving the partial differential equation for the probability generating function

$$G(z_1, z_2, t) = \sum_{n_1} \sum_{n_2} z_1^{n_1} z_2^{n_2} P(\mathbf{n}(t)).$$

Since this is also complicated, we solve for the moments by using the cumulant generating function

$$K(\theta_1, \theta_2, t) = \ln G(e^{-\theta_1}, e^{-\theta_2}, t).$$

See Aoki (2002, Sec 7.1.3) on converting probability generating functions into cumulant generating function.

The equations for the first moments, that is the means of $n_1(t)$ and $n_2(t)$ are, dropping the time argument for shorter notation

$$\frac{d\kappa_1}{dt} = f_1 - (d_1 - c_1)\kappa_1 + \lambda d_2 f_1 \kappa_2 + \alpha\lambda(\kappa_{12} + \kappa_1\kappa_2),$$

and

$$\frac{d\kappa_2}{dt} = -(d_2 - c_2 + \lambda d_2 f_1)\kappa_2 - \alpha\lambda(\kappa_{12} + \kappa_1\kappa_2),$$

where $\alpha = d_1 d_2(1 - g_2)$ and $\lambda = \mu/(d_1 d_2)$.

In this example the ordinary differential equation for the cumulant $\kappa_i(t)$ the means of $n_i(t)$, $i = 1, 2$ involves the cross covariance term $\kappa_{12}(t)$. The ordinary differential equations for the components of the covariance matrix, $\kappa_{11}(t)$, $\kappa_1 2(t)$, and $\kappa_2 2(t)$, are closed, that is, do not involve any higher order cumulants, and can be solved.

There are two important cases; one with parameters $1 > g_2 > g_1$; and the other $1 < g_2 < 1 + \lambda f_1$, and $g_2 < g_1$.

What this example shows is again the co-existence in the long-run of two types of firms of disparate technical capabilities. This illustrate the central conclusion of Iwai in a more straightforward manner. See Aoki, Nakano, and Yoshida for the actual expressions for the coupled differential equations of the cumulants and other technical details.

Example: Model of Fluctuations and Growth

Next, we consider a simple stochastic model of economy composed of several sectors, in which fluctuation and growth of output happen, together with creations or entries of new sectors with new goods. This model is used in Aoki and Yoshikawa (2003) to demonstrate a possibility that demand patterns affect the aggregate output. Here this example illustrates the applications of laws of sucession. In the literature, economic fluctuations are usually explained as a direct outcome of the individual agent behavior. The focus is on individual agents. Often, elaborate microeconomic models of optimization or rational expectations are the starting points. The more stronly one wishes to interpret aggregate fluctuations as something 'rational' or 'optimal', the more likely one is led to this essentialy microeconomic approach.

This model proposes a different approach to explain economic growth and fluctuations. The focus is not on individual agents, nor on elaborate microeconomic optimization modeling. Rather, the focus is on the probabilistic manner by which a large number of agents enter and interact.

Resources are stochastically allocated to existing sectors in response to aggregate demands for the goods of sectors, and a new sector appear stochastically, as outlined in our discussion of law of succession.

Because we assume zero adjustment cost for the sizes of sectors, our model is a model of economy with underutilized factors of production, such as hours of work of emplyees.[4] For empirical studies of such economies see Davis, Haltiwanger and Schuh (1996).

We assume that sector i has productiviy coefficient, c_i, which is exogenously given and fixed. Assume, for convenience, that sectors are arranged in the decreasing order in productivity. Sector i employs N_i units of factor of production. It is a non-negative integer-valued random variable. We call its value as "size" of the sector. When $N_i(t) = n_i$, $i = 1, 2, \ldots, K$, the output of sector i is $c_i n_i$, and the total output (GDP) of this economy is

$$Y(t) := \sum_{i=1}^{K} c_i n_i(t). \tag{1}$$

Demand for the output of sector i is denoted by $s_i Y(t)$, where $s_i > 0$ is the share of sector i, and $\sum_i s_i = 1$. The shares are also assumed to be exogeously given and fixed.

We denote the excess demand for goods of sector i by

$$f_i(t) := s_i Y(t) - c_i n_i(t), \tag{2}$$

$i = 1, 2, \ldots, K$. Denote the set of sectors with positive excess demands by

$$I_+ = \{i; f_i > 0\},$$

and similarly for the set of sectors with negative demands by[5]

$$I_- = \{j; f_j \leq 0\}.$$

To shorten notation, summations over these subsets are denoted as $\sum_+$ and $\sum_-$. Denote by n_+ the number of ns in the set I_+, that is, we write

$$n_+ := \sum_+ n_i,$$

[4]Clower (1965) and Leijonhufvud (1968) pointed out that quantity adjustment might be actually more important than price adjustment in economic fluctuations. Although this insight spawned a vast literature of the so-called 'non-Walrasian' or 'disequilibrium analysis, this approach suffers either from basically static, or deterministic nature of analysis.

[5]To be definite we include sectors with zero excess demands as well.

where the subscript $+$ is a short-hand for the set I_+, and similarly

$$n_- := \sum_- n_j,$$

for the sum over the sectors with negative excess demands. Let $n = n_+ + n_-$.

Sectors with non-zero excess demands attempt to reduce the sizes of excess demands by adjusting their sizes, up or down by unit amounts, depending on the signs of the excess demands. No adjustment cost is included in the model. The size n_is may be interpreted as some measure of capacity utlization factor in situations where capacity constraint is not binding.[6]

We set this model as a continous-time Markov chain to avoid the phenomenon of a lock-step adjustment by several different sectors in the economy.

Transition Rate Specifications

The transition probabilities are such that

$$\Pr(N_i(t+h) = n_i + 1 | N_i(t) = n_i) = \gamma_i h + o(h),$$

for $i \in I_+$, and

$$\Pr(N_i(t+h) = n_i - 1 | N_i(t) = n_i) = \rho_i h + o(h),$$

for $i \in I_-$. We assume that γs and ρs depend on the total number of sizes and the current size of the sector that adjusts,

$$\gamma_i = \gamma_i(n_i, n),$$

and

$$\rho_j = \rho(n_j, n).$$

This is an example of applying W. E. Johnson's sufficientness postulate we have earlier discussed. We have discussed specifications of entry and exit probabilities in Aoki (2000). See also Costantini and Garibaldi (1979, 1989), who give clear discussions on reasons for these specifications. We follow Zabell in specifying γ_i to depend only on n_i and n, and similarly for ρ_i.

We specify the entry rate, that is, the rate of size increase by

$$\gamma_a(n_a, n) = \frac{\alpha + n_a}{\theta_+ + n_+},$$

and that of the exit rate, namely, the rate of size decrease by

$$\rho_a(n_a, n) = \frac{n_a}{n_-},$$

where subscripts $+$ and $-$ refer to the signs of the excess demands.

[6] With fixed numbers of employees in each sector, hours worked per period may be an example of units of production factor entering and leaving production processes without cost of hiring or firing.

If α is much smaller than θ_+, then

$$\gamma_i(n_i, n) \approx \frac{n_+}{\theta + n_+} \frac{n_i}{n_+}.$$

So long as θ is kept constant, the above expression implies that the choice of K and α do not matter, provided α is much smaller than K. It is also clear that γ_i is nearly the same as the fraction n_i/n, which is the probability for exit. Then, time histories of n_i are nearly those of a near fair coin tosses. We have K such coin tosses available at each jump. The sector that jumps determines which coin toss is selected from these K coins.

We set $\alpha = 0$ to discuss economies with fixed numbers of sectors, and set it to a positive number to allow for new sectors to emerge. In the latter case, a new sector emerges with probability $\theta/(\theta+n)$, while a size of sector i increases by one when the sector has positive excess demand with probability $(\alpha + n_i)/(\theta + n)$. See Ewens (1972).

Holding Times

The question of which sector acts first is resolved by means of the holding time of this continuous-time Markov chain. We assume that the time it takes for sector i to adjust its size by one unit, up or down, T_i, is exponentially distributed,

$$\Pr(T_i > t) = \exp(-b_i t),$$

where b_i is either γ_i or ρ_i depending on the sign of the excess demand. This time is called sojourn time or holding time in the probability literature. We assume that the random variables Ts of the sectors with non-zero excess demand are independent.

The sector that adjusts first is determined by the sector with the shortest holding time. Let T^* be the minimum of all the holding times of the sectors with non-zero excess demands. Lawler calculates that for $a \in I_+$

$$\Pr(T_a = T^*) = \frac{\gamma_a}{\gamma_+ + \rho_-},$$

where $\gamma_+ = \sum_+ \gamma_i$, and $\rho_- = \sum_- \rho_j$, and if $a \in I_-$, then the probability of the jump in sector a is given by

$$\rho_a/(\gamma_+ + \rho_-),$$

and similarly for γs. See Lawler (1995, 56) or Aoki (1996, Sec.4.2).

Aggregate Outputs and Demands

After a change in the size of a sector, the total output of the economy changes to

$$Y(t+h) = Y(t) + sgn\{f_a(t)\}c_a,$$

where a is the sector that jumped first by the time $t + h$.[7]

[7] For the sake of simplicity we may think of the skeleton Markov chain, in which the directions of jump are chosen appropriately but the holding times themselves are replaced by a fixed unit time interval. Limiting behavior of the original and the skeletal version are known to be the same under certain technical conditions, which hold for this example. See Cinlar (1975), or Norris (1997, 87).

After the jump, this sector's excess demand changes to

$$f_a(t+h) = f_a(t) - c_a(1-s_a)sgn\{f_a(t)\}. \tag{3}$$

Other, non-jumping sectors have the excess demands changed to

$$f_i(t+h) = f_i(t) + sgn\{f_a(t)\}s_i c_a, \tag{4}$$

for $i \neq a$.

These two equations show the effects of an increase of size in one sector. An increase by c_a of output increases GDP by the same amount. However, sector a experiences an increase of its demand by only a fraction s_a of it, while all other sectors experience increase of their demands by $s_i c_a$, $i \neq a$. Eq. (4) shows a source of externality for this model that affects the model behavior significantly. The index sets I_+ and I_- also change in general.

Defining $\Delta Y(t) := Y(t+h) - Y(t)$, and $\Delta f_i(t) = f_i(t+h) - f_i(t)$, rewrite (1), (3) and (4) as

$$\Delta Y(t) = sgn\{f_a(t)\}c_a,$$

and

$$\Delta f_a(t) = -(1-s_a)\Delta Y(t),$$

and

$$\Delta f_i(t) = s_i \Delta Y(t),$$

for $i \neq a$.

See Aoki (2002, Sec.8.6) for some results of simulations and some examples of patterns of growth and fluctuations, as the values of θ and demand patterns are varied. .

Simulation Analysis

A number of simulations have been run on this model. One clear conclusion is that the GDP responds to the demand patterns, i.e., the specification of $\{s_i\}$. The more demands on productive sectore, higher is GDP. Another is the larger the value of θ, the more new sectors are created. See Aoki (2002, Sec.8.6) for details on the simulation results

Concluding Remarks

This paper uses the framework of continous-time Markov chains and the master equations to derive some examples which show that several types of agents can co-exist stochastically in the long-run. This supports the claims by Iwai (2001) that technologies with a wide range of efficiency will indeed coexist even in the long run, and shed new lights and add new elements to the sorts of discussions in Aghion and Howitt (1992), for example. In this paper we add a set of new concepts and tools drawn from random combinatorial analysis and claim that exchangeable random partitions and their stationary distributions are the right tools to deal with entries by new types of agents or goods.

We have not discussed the notion of coalescence by Kingman (1982), and by Watterson (1984). These have important implications on the lines of developments of new products, and the familty trees of how firms grow. We plan to explore their implications in connection with Schumperterian dynamics, and to discuss questions of distributions of sizes and ages of firms.

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Appendices

We collect here some relevant material from Zabell (1992).

Exchangeable Random Sequences

Let $X_1, X_2, \dots$ be an infinite sequence of random variables taking on k discrete values, $c_1, c_2, \dots, c_k$, say. These are possible categories or types of decisions or choices by agents,or cells in the literature on occupancy numbers in probability into which the outcomes or realizations of the sequence are classified.

The sequence is said to be exchangeable, if for every n, the 'cylinder set' probabilities

$$\Pr(X_1 = e_1, X_2 = e_2, \dots, X_N = e_N) = \Pr(e_1, e_2, \dots, e_N)$$

are invariant under all possible permutations of the subscripts of Xs.[8] In other words, two sequences have the same probability if one is the rearrangement of the other.

Let n_i denote the number of times the j-th type occurs in the sequence. The vector $\mathbf{n} = (n_1, n_2, \dots, n_k)$ is called the frequency vector. The vector $\mathbf{n}/N$ is known as the empirical distribution in statistics. Note that given any two sequences, one can be obtained from the other if and only if the two sequences have the **same** frequency vector or empirical distribution.

The observed frequency counts $n_j = n_j(X_1, X_2, \dots, X_N)$ are sufficient statistics in the language of statistics for the sequence $\{X_1, X_2, X_N\}$ because probabilities conditional on the frequency counts depend only on $\mathbf{n}$, and are independent of the choice of the exchangeable probability Pr.

[8]These subscripts may be thought of time index or the order by which samples are taken.

By exchangeablity each of the sequences having the same frequency vector is equally probable or equally likely. There are $N!/n_1!n_2!\cdots n_k!$ such sequences, and consequently

$$\Pr(X_1, X_2, \ldots, X_N|\mathbf{n}) = \frac{n_1!n_2!\cdots n_k!}{N!}.$$

We have the representation theorem for exchangeable sequences by de Finetti (1937). He established that if an infinite sequence of k-valued random variables $X_1, X_2, \ldots$ is excahngeable, then the infinite limiting frequency

$$Z := \lim_{N\to\infty}(\frac{n_1}{N}, \frac{n_2}{N}, \cdots, \frac{n_k}{N})$$

exists almost surely; and if

$$\mu(A) = \Pr(Z \in A)$$

denotes the distribution of this limiting frequency, then

$$\Pr(X_1 = e_1, X_2 = e_2, \ldots, X_N = e_N) = \int_{\Delta_K} p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} d\mu(p_1, p_2, \ldots, p_{k-1}).$$

To apply de Finetti theorem, we must choose a specific 'prior' or mixing measure $d\mu$. One way is to follow Johnson and postulate that all ordered k-partitions of N are equally likely. This Johnson fpostulate uniquely determines $d\mu$ to be

$$d\mu(p_1, p_2, \ldots p_k) = dp_1 dp_2 \cdots dp_{k-1},$$

a flat prior.

Less arbitrary is the Johnson,s sufficientness postulate[9]

$$\Pr(X_{N+1} = c_i|X_1, X_2, \ldots, X_N) = \Pr(X_{N+1} = c_i|\mathbf{n}) = f(n_i N),$$

if $\Pr(X_1 = c_1, \ldots, X_N = c_N) > 0$ for all cs. This formula states that in predicting that the next outcome is c_i, n_i is the only relevant information contained in the sample.

Zabell shows that Johnson's sufficientness postulate implies that

$$\Pr(X_{N+1} = c_i|\mathbf{n}) = \frac{n_i + \alpha}{N + k\alpha},$$

where α is the parameter of the symmetrical Dirichlet prior which is the mixing measure $d\mu$. Note that the Polya urn model [Feller (1968, 119–21)] can produce the same conditional probability.

[9]This term is adopted to avoid confusion with the usual meaning of sufficiency in statistics. (see Good (1965, p. 26).

Partition Exchangeability

In economic applications in which groups of agents of different types interact, such as multiple-agent models of stock markets, we face exactly the same problem which confronted statisticians in dealing with the so-called sampling of species problem. Sometimes this problem is referred to as the ecological problem in the emerging literature on multi-agent or agent-based modelling in economics: Suppose we take a snap shot of all the agents in the market at a point in time, and observe all different trading strategies in use. Some or most trategies (types of agents) have been seen in such snapshots taken earlier. There may be new ones not so far observed, however. As Zabell clearly explains this is not the problem of observing the event to which we assign 0 probability, that is the event whose probability we judge to be 0. Rather, the problem is when we observe an event whose existence we did not even previously suspect. A new strategy is invented, or new type of agents are born, and so on. Zabell calls it the problem of unanticipated knowledge. To deal with this problem we need Kingman's construction of exchangeable partition.
A probability function P is partition exchangeable if the cylinder set probabilities $\Pr(X_1 = e_1, X_2 = e_2, \dots, X_N = e_N)$ are invariant with respect to permutations of the time index **and** the category index.

Define the frequencies of the frequencies (called abundances in the population genetics or sampling of species literature) by what Zabell names partition vector $\mathbf{a} = (a_1, a_2, \dots a_N)$ where a_r is the number of n_j which is exactly equal to r.[10] In the above example, the original sample has the frequency $n_1 = 1, n_2 = 1, n_3 = 0, n_4 = 3, n_5 = 2, n_6 = 2,$

$$\mathbf{n} = (1,1,0,3,2,2) = 0^1 1^2 2^2 3^1,$$

where the last expression is the notation of Andrew (1971) used to indicate cyclic products of permutations with a_i cycles of size i. In this example $a_1 = 2, a_2 = 2, a_3 = 1 a_4 = a_5 = \cdots = a_{10} = 0$. Note also that $a_0 = 1$.

The partition vector plays the same role relative to partition exchangeable sequences that the frequency vector plays for exchangeable sequences. Two sequences are equivalent, in the sense that one can be obtained from the other by a permutation of the time set and a permutation of the category set, if and only the two sequences have the *same* partition vector.

We formally define that a random partition is exchangeable if any two partitions π_1 and π_2 having the same partition vector have the same probability

$$\mathbf{a}(\pi_1) = \mathbf{a}(\pi_2) \rightarrow P(\pi_1) = P(\pi_2).$$

Since partition exchangeable sequences are exchageable, they can be represented by $d\mu$ on the K-simplex $\triangle_K$. To prepare our way for letting K becomes infinite, we use the order statistics. Denote by $\triangle_K^*$ the simplex of

[10] Kingman named it differently because he was working in population genetics. We use Zabell's more neutral name. In Sachkov (1997, 82) it is called state vector of second specification.

the ordered probabilities

$$\triangle_K^* := \{(p_1^*, p_2^*, \ldots, p_K^*); p_1^* \geq p_2^* \geq \cdots \geq p_K^* \geq 0, \sum_{i=1}^{K} p_i^* = 1\}.$$

In the case of the partition exchangeable sequences, the conditional probability becomes

$$P(X_{N+1} = c_i | X_1, X_2, \ldots, X_N) = f(n_i; \mathbf{a}).$$

In taking the snapshots of a market situation and counting the numbers of agents by types of strategies they are using at that point in time, the relevant information is an exchan geable random partition of the set $\mathbf{N} := \{1, 2, \ldots, N\}$, where N is the total numbers of agents in the market at that time. We observe the first type, then possibly later the second type and so on. We need not identify what the first type is, for example. We have merely a partition of $\mathbf{N}$

$$\mathbf{N} = A_1 \cup A_2 \cup \cdots,$$

where $A_i \cap A_j = \phi$, $i \neq j$, and where

$$A_1 := \{t_1^1, t_1^2, \ldots; 1 = t_1^1 < t_1^2 < \cdots\}.$$

This means that the type of the first agent observed (sampled) is called type 1. Agent of the same type may be observed (sampled) at later times t_1^2, t_1^3 and so on. Agent of a different type, called type 2, is first observed (sampled) at t_2^1. We construct a set

$$A_2 := \{t_2^1, t_2^2, \ldots; t_2^1 < t_2^2 < \cdots\},$$

where t_2^1 is the first positive integer not in the set A_1.
Example Given a sample of size 10, taken in the order 6, 3, 4,2,3,1,6,2,2,3 we have a partition

$$\{1, 2, \ldots, 10\} = \{1, 7\} \cup \{2, 5, 10\} \cup \{3\} \cup \{4, 8, 9\}.$$

Its partition vector is $\mathbf{a}$=(2,1,2,0,0,...0). This indicates that there are two singletons, 1 subgroup with two numbers, and two clusters with three elements each, and others are emply. The sum $\sum_i a_i = 5$ gives the total number of clusters, that is 5 different groupings have been observed in this sample.