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Isoelastic Oligopolies under Uncertainty*

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Abstract

Single-product oligopolies are examined with uncertain isoelastic price functions and linear cost functions. Each firm wants to maximize its expected profit and also wants to minimize its uncertainty by minimizing the variance. This multiobjective optimization problem is solved by the weighting method, where the utility function of each firm is a linear combination of the expectation and variance of its profit. The existence and uniqueness of the equilibrium of the resulting n-person game is proved and an efficient algorithm is suggested to compute the equilibrium. The asymptotic behavior of the equilibrium is also investigated. Complete stability and bifurcation analysis is presented. The theoretical results are verified by computer simulation.

Keywords: multiobjective optimization, isoelastic demand, n-person Cournot game, demand uncertainty, nonlinear stability analysis

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1 Introduction

In an earlier paper [2] single-product oligopolies without product differentiation under uncertain price functions were investigated. It was assumed that each firm wants to maximize its expected profit and minimize the variance of the profit. We showed the existence and uniqueness of the equilibrium arising from the use of the weighting method under the usual differentiability and concavity conditions (see for example, [3] or [4]), when the best response functions of the firms are monotonic.

Isoelastic price functions were introduced into oligopoly theory in [5], however the special model here does not satisfy the usual assumptions on the concavity of the profit functions and the best responses are no longer monotonic. In this paper we revisit our earlier work [2] and investigate the existence and uniqueness of the equilibrium as well as suggest a simple procedure for the computation of the equilibrium in the isoelastic case. We also examine the dynamic extensions of the model, and perform stability analysis.

2 The Mathematical Model

Consider n firms that produce identical goods or offer the same service to a homogeneous market. This simple economic situation is known as a single-product oligopoly without product differentiation. Let q_j be the output of firm j, $Q_j = \sum_{i \neq j} q_i$ and $Q = \sum_{i=1}^{n} q_i$ denote the output of the rest of the industry and the total industry output, respectively. Assume that the inverse demand (or price) function is hyperbolic, $f(Q) = A/Q$, and the cost functions are linear: $C_j(q_j) = c_j q_j + d_j$. Each firm j believes that the price function is $f(Q) + \eta_j$, where η_j is a white noise with zero mean and given variance σ_j^2. We also assume that each firm wants to maximize its expected profit and also to minimize the uncertainty in its profit by minimizing variance. The profit of firm j has the form

$$\Pi_j = q_j \left[\frac{A}{Q_j + q_j} + \eta_j \right] - (c_j q_j + d_j) \tag{1}$$

with expectation

$$E(\Pi_j) = q_j \frac{A}{Q_j + q_j} - (c_j q_j + d_j) \tag{2}$$

and variance

$$Var(\Pi_j) = q_j^2 \sigma_j^2 \,. \tag{3}$$

It is assumed that firm j uses the weighting method to find its best response by maximizing a linear combination of the two objectives:

$$E(\Pi_j) - \frac{\alpha_j}{2} Var(\Pi_j) \tag{4}$$

where α_j shows the importance of reducing risk compared to maximizing expected profit. Expression (4) is also known as the certainty equivalent of the random outcome (1).

3 Best Responses

The first order conditions in maximizing the composite objective function (4) are as follows:

$$\frac{AQ_j}{(Q_j + q_j)^2} - (c_j + \alpha_j q_j \sigma_j^2) \begin{cases} \leqq 0 & \text{if } q_j = 0 \\ = 0 & \text{if } q_j > 0 \, . \end{cases} \tag{5}$$

The second order conditions

$$-\frac{2AQ_j}{(Q_j + q_j)^3} - \alpha_j \sigma_j^2 < 0$$

are clearly satisfied. The left hand side of (5) is strictly decreasing in q_j with fixed values of Q_j, its value is $A/Q_j - c_j$ at $q_j = 0$ and it converges to $-\infty$ as $q_j \to +\infty$ by assuming that both α_j and σ_j^2 are positive. Therefore if $Q_j \geqq A/c_j$, then the best response of firm j is zero, otherwise it is the unique positive solution of the equation

$$AQ_j - (c_j + \alpha_j q_j \sigma_j^2)(Q_j + q_j)^2 = 0 \, . \tag{6}$$

For a given value of Q_j, this is a cubic equation for q_j. Therefore in order to obtain a closed-form representation of the best response we consider Q_j as a function of q_j, which requires only the use of the quadratic formula, so that

$$Q_j = \frac{A \pm \sqrt{A^2 - 4Aq_j(c_j + \alpha_j q_j \sigma_j^2)}}{2(c_j + \alpha_j q_j \sigma_j^2)} - q_j \, . \tag{7}$$

Introduce the simplifying notation

$$\gamma_j = \frac{\alpha_j \sigma_j^2}{A} \text{ and } \delta_j = \frac{c_j}{\alpha_j \sigma_j^2}$$

to have

$$Q_j = \frac{1 \pm \sqrt{1 - 4q_j \gamma_j (q_j + \delta_j)}}{2\gamma_j (q_j + \delta_j)} - q_j \, . \tag{8}$$

Let next $x_j = q_j + \delta_j$, then $q_j = x_j - \delta_j$, so

$$Q_j = \frac{1 \pm \sqrt{1 - 4\gamma_j (x_j - \delta_j) x_j}}{2\gamma_j x_j} - x_j + \delta_j \, . \tag{9}$$

In order to have a real solution, we must have

$$4\gamma_j x_j^2 - 4\delta_j \gamma_j x_j - 1 \leqq 0 \, . \tag{10}$$

The left hand side has two real roots for x_j, one is positive and the other is negative. Let x_j^* denote the positive root of (10). The vertex of the left hand side of (10) is at $x_j = \frac{\delta_j}{2}$. In order to have nonnegative solution for q_j we need $x_j \geqq \delta_j$. Notice that at $x_j = \delta_j$, the left hand side of (10) is negative, so (10) holds with nonnegative q_j if and only if $x_j \in [\delta_j, x_j^*]$. If $x_j = x_j^*$, then we have a unique solution for Q_j, and if $\delta_j \leq x_j < x_j^*$, then there are two solutions.

Consider first the larger root:

$$Q_j^{(2)} = \frac{1 + \sqrt{1 - 4\gamma_j(x_j - \delta_j)x_j}}{2\gamma_j x_j} - x_j + \delta_j \,. \tag{11}$$

Notice first that at $x_j = \delta_j$, $Q_j^{(2)} = \frac{1}{\gamma_j \delta_j}$, and at $x_j = x_j^*$,

$$Q_j^{(2)} = \frac{1}{2\gamma_j x_j^*} - x_j^* + \delta_j < \frac{1}{2\gamma_j x_j^*} < \frac{1}{2\gamma_j \delta_j}$$

and

$$Q_j^{(2)} = \frac{1 - 2\gamma_j x_j^*(x_j^* - \delta_j)}{2\gamma_j x_j^*} > \frac{1 - 4\gamma_j x_j^*(x_j^* - \delta_j)}{2\gamma_j x_j^*} = 0 \,.$$

Furthermore with increasing value of x_j, the numerator of (11) decreases and the denominator increases, so $Q_j^{(2)}$ strictly decreases in x_j.

Consider next the smaller root

$$Q_j^{(1)} = \frac{1 - \sqrt{1 - 4\gamma_j(x_j - \delta_j)x_j}}{2\gamma_j x_j} - x_j + \delta_j \,. \tag{12}$$

Notice that at $x_j = \delta_j$, $Q_j^{(1)} = 0$, and at $x_j = x_j^*$,

$$Q_j^{(1)} = Q_j^{(2)} = \frac{1}{2\gamma_j x_j^*} - x_j^* + \delta_j = x_j^* - \delta_j \,.$$

A simple calculation shows that

$$Q_j^{(1)} = \frac{x_j(1 - \sqrt{1 - 4\gamma_j(x_j - \delta_j)x_j}) - 2\delta_j}{1 + \sqrt{1 - 4\gamma_j(x_j - \delta_j)x_j}} + \delta_j \tag{13}$$

which implies that for $x_j \in (\delta_j, x_j^*)$, $Q_j^{(1)}$ strictly increases in x_j. The relation between q_j and Q_j can be obtained by simple transformation $q_j = x_j - \delta_j$. In addition notice that

$$\frac{1}{\delta_j \gamma_j} = \frac{1}{\frac{c_j}{\alpha_j \sigma_j^2} \cdot \frac{\alpha_j \sigma_j^2}{A}} = \frac{A}{c_j}. \tag{14}$$

The resulting relation between q_j and Q_j is shown in Figure 1 where we see that the best reply with respect to Q_j has the form

$$R_j(Q_j) = \begin{cases} Q_j^{(1)} & \text{if } 0 \le Q_j < Q_j^m, \\ Q_j^{(2)} & \text{if } Q_j^m \le Q_j < \dfrac{A}{c_j}, \\ 0 & \text{if } \dfrac{A}{c_j} \le Q_j \end{cases}$$

where $Q_j^m := x_j^* - \delta_j$ and $q_j^m := R_j(Q_j^m) = Q_j^m$. Notice that q_j is strictly increasing if $Q_j < Q_j^m$, strictly decreasing in Q_j if $Q_j^m < Q_j < A/c_j$, and thus the 45 degree black line passes through the vertex (Q_j^m, q_j^m) of the blue curve.

We can also consider q_j as the function of the total production level $Q = Q_j + q_j$. If $Q \geq \frac{A}{c_j}$, then $q_j(Q) = 0$, otherwise q_j is the unique solution of equation (6) which now reads

$$-Aq_j + AQ - (c_j + \alpha_j q_j \sigma_j^2)Q^2 = 0$$

implying that

$$q_j(Q) = \frac{AQ - c_j Q^2}{A + \alpha_j \sigma_j^2 Q^2} \tag{15}$$

which is clearly positive if $0 < Q < A/c_j$.

Notice that $q_j(Q)$ satisfies the relations

$$q_j(0) = 0$$

and

$$q_j'(Q) = -\frac{A(\alpha_j \sigma_j^2 Q^2 + 2c_j Q - A)}{(A + \alpha_j \sigma_j^2 Q^2)^2}. \tag{16}$$

This derivative is positive if

$$\alpha_j \sigma_j^2 Q^2 + 2c_j Q - A < 0\,. \tag{17}$$

The left hand side of (17) has two roots, one is positive and the other is negative. The positive root equals

$$Q^{(P)} = \frac{-c_j + \sqrt{c_j^2 + A\alpha_j \sigma_j^2}}{\alpha_j \sigma_j^2} = \frac{(c_j^2 + A\alpha_j \sigma_j^2) - c_j^2}{\alpha_j \sigma_j^2 [c_j + \sqrt{c_j^2 + A\alpha_j \sigma_j^2}]} < \frac{A}{2c_j},$$

and for $0 < Q < Q^{(P)}$ (17) holds, so q_j is increasing in Q. Clearly, derivative (16) is negative if $Q^{(P)} < Q < A/c_j$. Notice also that $q_j'(0) = 1$ and $q_j(Q^{(P)}) = Q^{(P)}/2$. Since $q_j'(Q)$ strictly decreasing in Q, the function q is strictly concave in the interval $(0, A/c_j)$. The graph of function $q_j(Q)$ is shown in Figure 1 where the best reply with respect to Q has the form

$$\tilde{R}_j(Q) = \begin{cases} q_j(Q) & \text{if } 0 \leq Q < \dfrac{A}{c_j}, \\ 0 & \text{if } \dfrac{A}{c_j} \leq Q, \end{cases}$$

where $Q^{(P)}$ is a maximizer and $Q^{(P)}/2$ is the maximum. Notice that $q_j(0) = 0$, $q_j'(0) = 1$, $q_j'(Q) < 1$ for $0 < Q < A/c_j$, the function q_j increases until $Q = Q^{(P)}$, decreases afterwards, becomes zero at $Q = A/c_j$ and remains zero for all $Q > A/c_j$. Notice also that $Q_j^m = Q^{(P)}/2$.

4 Existence and Uniqueness of Equilibrium

The equilibrium industry output is the solution of the equation

$$H(Q) = \sum_{j=1}^{n} q_j(Q) - Q = 0. \tag{18}$$

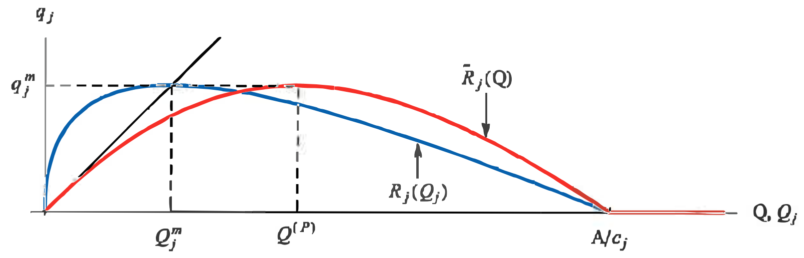


Figure 1: Graphs of q_j as a function of Q_j (blue curve) and as a function of Q (red curve)

Notice that $Q = 0$ is clearly a solution. Since this trivial equilibrium has no economic meaning, we assume that $Q > 0$ and have the following result:

Proposition 1 *There exists a unique industry equilibrium output, and a unique Nash equilibrium.*

Proof. (1) *Existence of the equilibrium*: Since $q_j'(0) = 1$ for all j, for sufficiently small values of Q the left hand side of (18) is positive, and if $Q \geq \max_j \left\{\frac{A}{c_j}\right\}$, then $q_j(Q) = 0$ for all j, so the left hand side becomes negative. Since it is continuous, there is at least one solution.

(2) *Uniqueness of the equilibrium*: Suppose that equation (18) can have multiple positive solutions, that is, there is a $\bar{Q} > 0$ such that $H(\bar{Q}) \leq 0$ and $H'(\bar{Q}+) \geq 0$, since for sufficiently small positive values of $Q, H(Q) > 0$. Let $\mathbb{K}$ denote the set of firms such that $q_j(\bar{Q}) > 0$. Then

$$H(\bar{Q}) = \sum_{j \in \mathbb{K}} \frac{A\bar{Q} - c_j\bar{Q}^2}{A + \alpha_j\sigma_j^2\bar{Q}^2} - \bar{Q} \leq 0$$

implying that

$$\sum_{j \in \mathbb{K}} \frac{A - c_j\bar{Q}}{A + \alpha_j\sigma_j^2\bar{Q}^2} \leq 1.$$

Furthermore

$$\begin{aligned} H'(\bar{Q}+) &= \sum_{j \in \mathbb{K}} \frac{A^2 - 2c_jA\bar{Q} - \alpha_j\sigma_j^2A\bar{Q}^2}{(A + \alpha_j\sigma_j^2\bar{Q}^2)^2} - 1 \\ &< \sum_{j \in \mathbb{K}} \frac{(A - c_j\bar{Q})^2}{(A + \alpha_j\sigma_j^2\bar{Q}^2)^2} - 1 \\ &\leq \left(\sum_{j \in \mathbb{K}} \frac{A - c_j\bar{Q}}{A + \alpha_j\sigma_j^2\bar{Q}^2}\right)^2 - 1 \leq 0 \end{aligned} \tag{19}$$

which is an obvious contradiction. Hence the uniqueness of the equilibrium is proved.

(3) From the unique value of Q, the individual equilibrium output of firm j is $\tilde{R}_j(Q)$. ■

We note that the proof of this proposition suggests a simple computer method to find the equilibrium by solving the single-variable equation (18) for Q, and then the equilibrium outputs of the firms are $q_j = \tilde{R}_j(Q)$ for $j = 1, 2, \ldots, n$ by using equation (15).

5 Best Response Dynamics

We have already seen that the best response $R_j(Q_j)$ of each firm j is unique. If $Q_j \geq A/c_j$, then it is zero, otherwise it is the unique positive solution of equation (6) for q_j. By implicit differentiation of (6) we have

$$A - \alpha_j\sigma_j^2 Q^2 R_j' - (c_j + \alpha_j\sigma_j^2 q_j)2Q(1 + R_j') = 0$$

implying that

$$R_j' = \frac{A - 2(c_j + \alpha_j q_j \sigma_j^2)Q}{\alpha_j\sigma_j^2 Q^2 + 2(c_j + \alpha_j q_j \sigma_j^2)Q}.$$

By using equation (6) again, this can be simplified to

$$R_j' = \frac{A(q_j - Q_j)}{\alpha_j\sigma_j^2 Q^3 + 2AQ_j}.$$

This relation has two important consequences:

1. Since $q_j - Q_j = 2q_j - Q$, the derivative R_j' is nonpositive if and only if $q_j \leq \frac{Q}{2}$, that is, firm j is not producing more than half of the industry output.
2. Clearly,
$$R_j' \geq \frac{-0.5\alpha_j\sigma_j^2 Q^3 - AQ_j}{\alpha_j\sigma_j^2 Q^3 + 2AQ_j} = -\frac{1}{2}.$$

In the absence of a dominant firm all derivatives R_j' are nonpositive and greater than or equal to -1/2 at the equilibrium. Therefore the local asymptotic stability conditions of discrete and continuous best response dynamics (see Theorems 2.1 and 2.2 of [1]) can be directly applied. However in the presence of a dominant firm there is the possibility of complex eigenvalues and so the general results cannot be applied. The asymptotic properties of the equilibrium might become more complex.

6 The Semi-Symmetric Case

Assume that $n \geq 2$ and consider now an oligopoly with $n-1$ identical firms with parameter values α, σ^2 and c_x, and assume that firm n has different marginal cost c_y with the same α and σ^2 values. From equation (16) we have

$$q_j(Q) = \frac{AQ - c_x Q^2}{A + \alpha\sigma^2 Q^2}$$

for $j = 1, 2, \ldots, n\text{-}1$, and

$$q_n(Q) = \frac{AQ - c_y Q^2}{A + \alpha\sigma^2 Q^2}$$

by assuming interior best responses. Then equation (18) has the form

$$\frac{(n-1)AQ-(n-1)c_xQ^2}{A+\alpha\sigma^2Q^2}+\frac{AQ-c_yQ^2}{A+\alpha\sigma^2Q^2}-Q=0$$

which can be rewritten as

$$\alpha\sigma^2Q^2+Q((n-1)c_x+c_y)-(n-1)A=0. \tag{20}$$

Firm n is dominant when $q_n(Q)>\frac{Q}{2}$ which is equivalent to (17) with c_j replaced by c_y, which occurs if and only if $Q<Q_n^{(P)}$. Since the left hand side of (20) is a convex parabola with a positive and a negative root, this is the case if and only if

$$\alpha\sigma^2Q_n^{(P)2}+Q_n^{(P)}((n-1)c_x+c_y)-(n-1)A>0.$$

That is, when

$$\alpha\sigma^2\left(\frac{-c_y+\sqrt{c_y^2+A\alpha\sigma^2}}{\alpha\sigma^2}\right)^2+((n-1)c_x+c_y)\frac{-c_y+\sqrt{c_y^2+A\alpha\sigma^2}}{\alpha\sigma^2}-(n-1)A>0$$

which can be reduced to the simple relation

$$c_x>\frac{(n-2)\alpha\sigma^2A+c_y(-c_y+\sqrt{c_y^2+A\alpha\sigma^2})}{(n-1)(-c_y+\sqrt{c_y^2+A\alpha\sigma^2})}=\frac{(n-2)A}{(n-1)Q_n^{(P)}}+\frac{c_y}{n-1}. \tag{21}$$

Assume that the first $n-1$ firms select identical speeds of adjustment and identical initial outputs. Then their entire trajectories are also identical. Let K_x and K_y denote the speeds of adjustment of the first $n-1$ firms and firm n, respectively, and $x(t)$ and $y(t)$ the corresponding outputs.

Continuous-time Model

With continuous time scales we have the best response dynamics

$$\dot{x}(t)=K_x[R_x((n-2)x(t)+y(t))-x(t)], \tag{22}$$

$$\dot{y}(t)=K_y(R_y((n-1)x(t))-y(t)). \tag{23}$$

The Jacobian of this system has the form

$$J_c=\begin{pmatrix} K_x[(n-2)R_x'-1] & K_xR_x' \\ K_y(n-1)R_y' & -K_y \end{pmatrix}$$

where R_x' and R_y' are the derivatives of the best response functions at the equilibrium. The characteristic polynomial, after rearranging terms, is

$$\det(J_c-\lambda I)=\lambda^2+\bar{p}\lambda+K_xK_y\bar{q}$$

where

$$\bar{p}=K_x+K_y-K_x(n-2)R_x',$$

$$\bar{q}=-R_x'\left[(n-2)+(n-1)R_y'\right]+1.$$

We have that $R'_x < 0$, and only R'_y can be positive if the single last firm is dominant. So $\bar{p} > 0$. Since R'_x is negative, we have

$$\begin{aligned}\bar{q} &\geq -R'_x \left[(n-2) - \frac{n-1}{2}\right] + 1 \\ &= -R'_x \frac{n-3}{2} + 1.\end{aligned}$$

This is positive if $n \geq 3$. If $n = 2$, then

$$\bar{q} \geq 1 + \frac{R'_x}{2} \geq 1 - \frac{1}{4} > 0.$$

So $\bar{q}$ is always positive. Therefore we have the following result.

Proposition 2 *The equilibrium is locally asymptotically stable in the continuous time system (22) and (23).*

Discrete-time Model

In the case of discrete time scales the dynamic equations (22) and (23) are modified to read

$$x(t+1) = x(t) + K_x \left[R_x((n-2)x(t) + y(t)) - x(t)\right], \tag{24}$$

$$y(t+1) = y(t) + K_y \left[R_y((n-1)x(t)) - y(t)\right]. \tag{25}$$

The Jacobian of this system has the form

$$J_d = \begin{pmatrix} 1 + K_x[(n-2)R'_x - 1] & K_x R'_x \\ K_y(n-1)R'_y & 1 - K_y \end{pmatrix}$$

with characteristic polynomial

$$\det(J_d - \lambda I) = \lambda^2 + p\lambda + q, \tag{26}$$

where the coefficients are given by

$$p = K_x + K_y - 2 - K_x(n-2)R'_x$$

and

$$q \;= K_x K_y \left[-R'_x(n-2) - R'_x R'_y(n-1) + 1\right] - K_x - K_y + K_x R'_x(n-2) + 1.$$

It is well known that the eigenvalues are inside the unit circle if and only if

$$q < 1 \text{ and } \pm p + q + 1 > 0.$$

1. *Consider first the condition* $q < 1$; this occurs when

$$K_x K_y \left[-R'_x(n-2) - R'_x R'_y(n-1) + 1\right] - K_x - K_y + K_x R'_x(n-2) < 0. \tag{27}$$

The multiplier of K_xK_y is $\bar{q}$, which is positive, so this relation can be rewritten as

$$K_y(-1 + K_x\bar{q}) < K_x(1 - R'_x(n-2)).$$

The right hand side is positive. If $K_x \le 1/\bar{q}$, then this inequality must hold, otherwise it is valid if

$$K_y < \frac{K_xB}{-1 + K_x\bar{q}} \tag{28}$$

where $B = 1 - R'_x(n-2)$. Notice that (28) is a hyperbola. Figure 2 illustrates the set of points (K_x, K_y) such that $q < 1$.

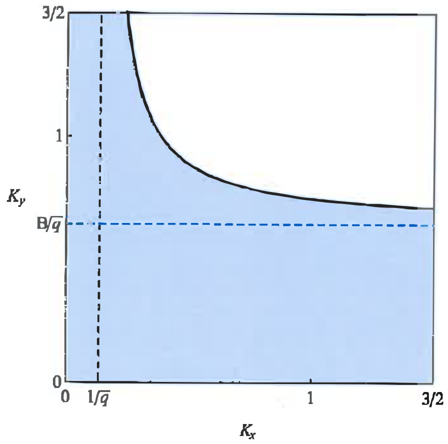


Figure 2: The feasible region for $q < 1$ is shaded

2. *Consider next the condition* $p + q + 1 > 0$; this is the case when

$$K_xK_y\left[-R'_x(n-2) - R'_xR'_y(n-1) + 1\right] > 0. \tag{29}$$

We have already seen that the multiplier of K_xK_y is $\bar{q}$, so this relation is always satisfied.

3. *The last condition is* $-p + q + 1 > 0$; this now has the form

$$K_xK_y\bar{q} - 2K_y + 4 - 2K_xB > 0. \tag{30}$$

This relation can be rewritten as

$$K_y(K_x\bar{q} - 2) > -4 + 2K_xB. \tag{31}$$

Notice that $\bar{q} = B - R'_xR'_y(n-1)$.

Assume first that $R'_y > 0$. Then $\bar{q} > B$, so

$$\frac{2}{\bar{q}} < \frac{2}{B}.$$

If $K_x < 2/\bar{q}$, then both the coefficient of K_y and the right hand side of (31) are negative, so (31) holds if

$$K_y < \frac{-2BK_x + 4}{2 - K_x\bar{q}}. \tag{32}$$

If $K_x = 2/\bar{q}$, then (31) holds with arbitrary $K_y > 0$. If $2/\bar{q} < K_x \leq 2/B$, then (31) holds since the left hand side is positive and the right hand side is nonpositive. If $2/B < K_x$, then (31) holds if

$$K_y > \frac{2BK_x - 4}{-2 + K_x\bar{q}}. \tag{33}$$

Figure 3(a) shows the region satisfying (32) and (33).

If $R'_y = 0$, then $B = \bar{q}$, so (31) has the form

$$K_y(K_x\bar{q} - 2) > 2(K_x\bar{q} - 2),$$

which holds if either

$$K_x < \frac{2}{\bar{q}} \text{ and } K_y < 2$$

or

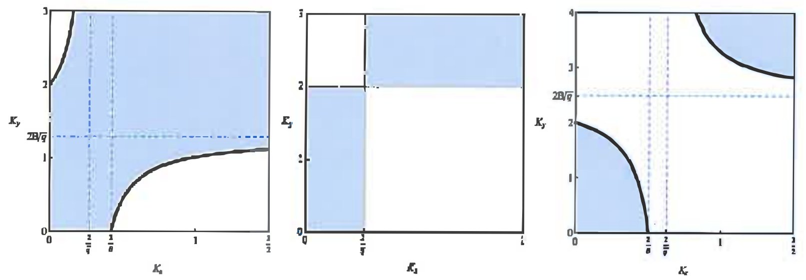
$$K_x > \frac{2}{\bar{q}} \text{ and } K_y > 2.$$

Figure 3(b) shows the stability region in this case.

Assume next that $R'_y < 0$. Then $B > \bar{q}$, so

$$\frac{2}{\bar{q}} > \frac{2}{B}.$$

If $K_x < 2/B$, then (32) is the condition. If $2/B \leq K_x \leq 2/\bar{q}$, then no solution exists, since the left hand side of (31) is negative and the right hand side is nonnegative or the left hand side is nonpositive and the right hand side is positive. If $K_x > 2/\bar{q}$, then the condition is (33). Figure 3(c) shows the stability region in the case of $R'_y < 0$.



(a) $R'_y > 0$ (b) $R'_y = 0$ (c) $R'_y < 0$

Figure 3: The feasible regions of $-p + q + 1 > 0$ are shaded

In summary, the system (24) and (25) is locally asymptotically stable if point $(K_x,\ K_y)$ is inside the feasible regions of Figure 2 and Figure 3(a) (for $R'_y > 0$), Figure 3(b) (for $R'_y = 0$) or Figure 3(c) (for $R'_y < 0$). The equilibrium is locally asymptotically stable if the speeds of adjustment K_x and K_y are sufficiently small. We can formally state this result as follows.

Proposition 3 *The equilibrium is locally asymptotically stable in the discrete time system (24) and (25) if*

$$0 < K_x \le \frac{1}{q} \text{ or (26) is satisfied}$$

and any one of the following conditions holds:

(i) $R'_y > 0$ *and either* $K_x < \frac{2}{q}$ *and (32) holds or* $\frac{2}{q} \le K_x \le \frac{2}{B}$ *or* $K_x > \frac{2}{B}$ *and (33) holds,*

(ii) $R'_y = 0$ *and either* $K_x < \frac{2}{q}$ *and* $K_y < 2$ *or* $K_x > \frac{2}{q}$ *and* $K_y > 2$,

(iii) $R'_y < 0$ *and either* $K_x < \frac{2}{B}$ *and (32) holds or* $K_x > \frac{2}{q}$ *and (33) holds.*

Example

For notational simplicity we assume that $\alpha = \sigma = 1$. Let $x(t)$ and $y(t)$ be the output of firm j $(1 \le j \le n-1)$ and firm n, respectively. Then the outputs of the rest of the industry are $Q_x = (n-2)x + y$ and $Q_y = (n-1)x$, respectively. The marginal costs are denoted by c_x and c_y. The best response of firm x is given by

$$R_x(Q_x) = \begin{cases} 0 & Q_x \ge \dfrac{A}{c_x} \\ x^* & \text{otherwise}, \end{cases} \tag{34}$$

where x^* is the unique positive solution of the equation $AQ_x - (c_x + x)(Q_x + x)^2 = 0$ and by the Cardano formula it has the form

$$x^* = \left(-\frac{b_x}{2} + \sqrt{r_x}\right)^{\frac{1}{3}} + \left(-\frac{b_x}{2} - \sqrt{r_x}\right)^{\frac{1}{3}} - \frac{c_x + 2Q_x}{3}$$

with

$$a_x = -\frac{1}{3}(c_x - Q_x)^2,$$

$$b_x = \frac{2}{27}(c_x - Q_x)^3 - AQ_x,$$

$$r_x = \left(\frac{b_x}{2}\right)^2 + \left(\frac{a_x}{3}\right)^3.$$

Furthermore the best response of firm y is

$$R_y(Q_y) = \begin{cases} 0 & Q_y \ge \dfrac{A}{c_y} \\ y^* & \text{otherwise}, \end{cases} \tag{35}$$

where y^* is the unique positive solution of equation $AQ_y - (c_y + y)(Q_y + y)^2 = 0$ and has the form

$$y^* = \left(-\frac{b_y}{2} + \sqrt{r_y}\right)^{\frac{1}{3}} + \left(-\frac{b_y}{2} - \sqrt{r_y}\right)^{\frac{1}{3}} - \frac{c_y + 2Q_y}{3}$$

with

$$a_y = -\frac{1}{3}(c_y - Q_y)^2,$$

$$b_y = \frac{2}{27}(c_y - Q_y)^3 - AQ_y,$$

$$r_y = \left(\frac{b_y}{2}\right)^2 + \left(\frac{a_y}{3}\right)^3.$$

The equilibrium can be obtained as follows. In this case equation (20) simplifies to

$$Q^2 + Q((n-1)c_x + c_y) - (n-1)A = 0.$$

To proceed further, we specify the parameter values, $n = 10$, $A = 5$, $c_x = 8$ and $c_y = 2$. The unique positive solution of the industrial output is $Q \simeq 0.6032$ with the equilibrium outputs of the firms $x \simeq 0.01962$ and $y \simeq 0.42661$. It is clear that firm y is dominant. We can see this by using relation (21) as well. Its right hand side equals

$$\frac{5(n-2)}{n-1} + \frac{2}{n-1} = \frac{5n-8}{n-1},$$

since

$$Q_y^{(P)} = \frac{-2+\sqrt{4+5}}{1} = 1.$$

So firm y is dominant if $c_x > (5n-8)/(n-1) \simeq 4.67$.

The stability region is shown as the gray region in Figure 4 where the upper decreasing curve (i.e., the $q = 1$ locus) is the Neimark-Sacker bifurcation line, and the increasing curve is the Period-doubling bifurcation line (i.e., the $-p + q + 1 = 0$ locus).

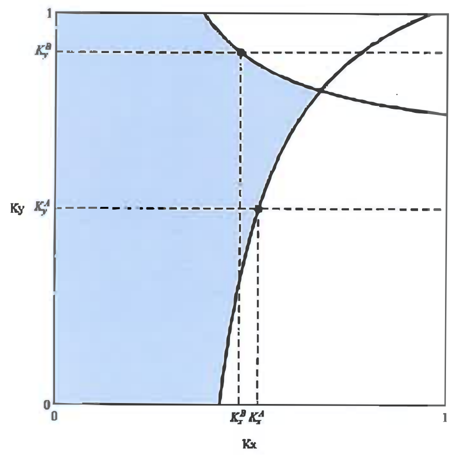
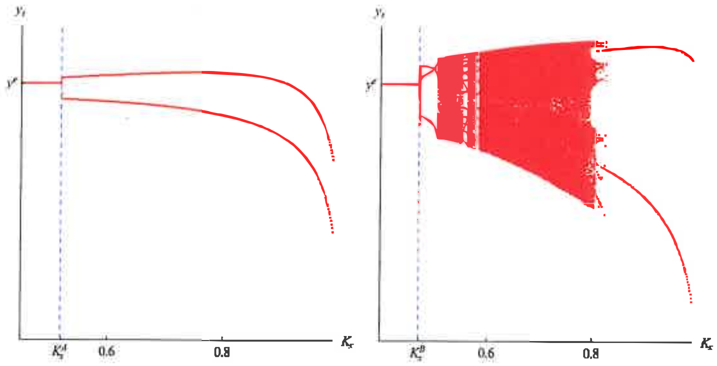


Figure 4: The feasible stability region is in gray

Figure 5 shows the corresponding bifurcation diagrams: Figure 5(a) shows the Period-doubling bifurcation with $K_y = K_y^A$ with changing values of K_x, while Figure 5(b) illustrates the Neimark-Sacker bifurcation with $K_y = K_y^B$

and changing values of K_x. The equilibrium loses stability at the dotted point either on the Niemark-Sacker bifurcation line or the Period-doubling bifurcation line in Figure 4. Under $K_y^A = 0.5$ and $K_y^A = 0.9$, the corresponding abscissas of the dotted points are $K_x^A \simeq 0.52$ and $K_x^B \simeq 0.47$. It is also seen that the equilibrium bifurcates to a period-2 cycle at K_x^A in Figure 5(a) while it bifurcates to a period-3 cycle at K_x^B and then turns into aperiodic cycles for larger values of K_x and finally converges to a period-2 cycle for even larger values of K_x in Figure 5(b).



(a) $K_y = 0.5$ (b) $K_y = 0.9$

Figure 5: Bifurcation diagrams with $n = 10$, $A = 5$, $c_x = 2$ and $c_y = 8$

7 Conclusions

Isoelastic oligopolies with linear cost functions were examined under uncertainty in the price function. After the best response functions were determined the existence of a unique equilibrium was proved and simple algorithm was introduced to compute the equilibrium output levels of the firms.

Complete stability analysis was performed in the semi-symmetric case and the occurrence of Hopf and Neimark-Sacker bifurcations was shown. The theoretical results were also illustrated by computer simulation, where the stability regions and the two types of bifurcation diagrams were presented.

In future research we will examine the more general case with nonlinear cost functions, a case that might result in more complex dynamics.

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