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Delay Kaldor-Kalecki Model Revisited

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Abstract

This paper studies dynamics of the Kaldor-Kalecki model of national income and capital stock. The investment function is assumed to have not only a Kaldorian characteristics, namely, a S-shaped form but also a Kaleckian characteristics, that is, a gestation delay between "investment decision" and "investment implementation." We divide the analysis into two parts. In the first part, we assume that the time period under consideration is short enough so that the capital stock is not affected by the flow of investment and then examine the delay effect on dynamics of national income. In the second part, taking the capital accumulation into account, we draw attention to how the delay affects cyclic dynamics observed in the non-delay Kaldor-Kalecki model. It is demonstrated that the investment delay quantitatively affects the dynamic behavior but not qualitatively.

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1 Introduction

In real economy, macroeconomic variables such as national income, capital accumulation, interest rate, etc. exhibit cyclic fluctuations. As a natural consequence, it has been the main interest in studying macro economic dynamics to detect endogenous sources of such cyclic behavior. Since investment is considered to be the key factor to cyclic dynamics for evolution of national economy, a lot of efforts have been devoted to studying investment determinations. As early as in 1930s, a few years before the publication of the Keynes' General Theory, Kalecki (1935) introduces an idea of the consumption function and the multiplier in his statistic analysis of macro economy. Furthermore, regarding investment in the dynamic analysis, he assumes a lag between "investment order" and "investment installation" and call it a gestation lag of investment.[1] Adopting a linear investment function, he constructs a macro dynamic model as a delay differential equation of retarded type. It had been already shown that Kalecki's model gives rise to a cycle due to the delay. Kaldor (1940), on the other hand, studies the evolution of production and capital formation and believes that nonlinearities of behavioral equations could be a clincher in such cyclic oscillations. He builts a 2D model of national income and captial with nonlinear investment and saving functions. Its basic movements consist of two sorts. One is the movements along the nonlinear functions and the other is shifts of these functions according to capital accumulations. So far, it has also been confirmed in various ways that the Kaldor model is capable of generating cyclic behavior when nonlinearities become strong enough. Indeed, Ichimura (1955) reduces the model to the Liénard equation, Chang and Smyth (1971) rigorously show the existence of a limit cycle by applying the Poincaré-Bendixson theorem and so does Lorenz (1993) by the Hopf bifurcation theorem. Furthermore, Grasman and Wentzel (1994) show multi-stability in the Kaldor model, that is, the coexistence of stable and unstable cycles when the equilibrium is locally stable. "Nonlinearity" and "delay" are now treated as two of the main ingredients for endogenous cycles. More than a half century after Kaldor (1940), Krawiec and Szydlowski (1999) combine the Kaldor model with the Kaleckian time delay in investment to built a delay business cycle model. Their model is often called (delay) Kaldor-Kalecki model. To emphasize the role of delay, they assume a linear investment function as in the Kalecki model and then show the occurrence of a limit cycle with respect to time delay.

We will revisit the Kaldor-Kalecki model and concern the roles of nonlinearity and delay for the birth of limit cycles in order to complement Krawiec and Szydlowski (1999). For this purpose, we first recapitulate the 2D Kaldor-Kalecki model and specify the investment function. Then we proceed to the analysis of delay dynamics with two steps. At the first step, we investigate dynamics of national income, keeping the stock of capital fixed. As is well known, no cyclic behavior arises in the Kaldor model without captial accumulation. We focus attention on the effect caused by delay on such stable dynamics. Since invest-

[1] Since "lag" and "delay" do not have distinctive different meanings, we use these words interchangeably. In particular, we mainly use "delay" in this study.

ment and savings are short term, eliminating captial or fixing capital implies short-run dynamic analysis. At the second step we consider delay dynamics of national income and capital accumulation. As is seen above, the nonlinear Kaldor model without delay can generate not only one limit cycle but also multiple limit cycles while the linear delay Kaldor-Kalecki model also give rise to a limit cycle. A natural question we rise is *how the delay affects these cyclic behavior in the long-run*.

This paper is organized as follows. In Section 2, the basic elements of the Kaldor-Kalecki model is rebuilt. In Section 3, short-run dynamics is examined under the investment delay. In Section 4, after reviewing the Kaldor model without delay quickly, we analytically and numerically detect the delay effect on cyclic dynamics of the Kaldor-Kalecki model from the long-run point of view in the sense that capital accumulation is explicitly taken into account. Section 5 contains some concluding remarks.

2 Delay Dynamic Model

Kaldor (1940) relates investment to the level in income (i.e., profit principle) and extends it by proposing a sigmoidal, instead of linear, investment function. A brief description of his model is given by two equations,

$$\begin{aligned} \dot{Y}(t) &= \alpha\left[\Phi(Y(t), K(t)) - S(Y(t))\right] \\ \dot{K}(t) &= \Phi(Y(t), K(t)) - \delta K(t). \end{aligned} \tag{1}$$

where $Y(t)$ is national income at time t, $K(t)$ denotes capital, $\Phi(Y(t), K(t))$ is an investment function, $S(Y(t))$ is a savings function and the parameters α and δ denote the adjustment coefficient and the rate of depreciation. The first equation describes the national income adjustment process and the second describes capital accumulation process. As will be reviewed soon, the Kaldor model is able to generate endogenous limit cycles. There are two ingredients for the birth of the cycles, the nonlinearity of investment in Y and the dependency of investment in K. These two prevent Y and K for global divergence when the equilibrium is locally unstable. Using our notation and following his spirit, a Kaleckian investment function can be presented by

$$I(t) = \Phi(Y(t-\theta), K(t))$$

where θ denotes the gestation delay.[2] Replacing the investment function in the second equation of system (2) with Kaleckian function yields the Kaldor-Kalecki

[2] There are several extensions of this delay investment function. Kadder and Talibi Alaudi (2008) introduce time delay also in capital stock in capital accumulation equation (i.e., $\Phi(Y(t-\theta), K(t-\theta))$). Zhou and Li (2009) assume that the investment function in the capital accumulation depends on the income and the captial stock at differrent gestation periods (i.e., $\Phi(Y(t-\theta_1), K(t-\theta_2))$ with $\theta_1 \neq \theta_2$).

model of the income and captial stock,

$$\begin{aligned}\dot{Y}(t) &= \alpha\left[\Phi(Y(t), K(t)) - S(Y(t))\right], \\ \dot{K}(t) &= \Phi(Y(t-\theta), K(t)) - \delta K(t).\end{aligned} \tag{2}$$

We plan to analyze dynamics generated by (2) with two steps. At first step, we confine attention to short-run dynamics. To this end, we impose two assumptions:

Assumption 1. The time period under consideration is short enough so that the capital stock is not affected by the flow of investment.

Assumption 2. The investment function in the first equation of system (2) contains the delay.

As a consequence of these modifications, the second equation concerning the evolution of capital is eliminated and the variable K in the investment function also disappears. Thus the modified Kaldor-Kalecki model can be presented by a one dimensional nonlinear differential equation with one time delay,

$$\dot{Y}(t) = \alpha\left\{\varphi[Y(t-\theta)] - S[Y(t)]\right\} \tag{3}$$

After examining short-run dynamics, we proceed to the second step in which the dynamic analysis of $Y(t)$ and $K(t)$ generated by system (2) is investigated. It is considered to be long-run dynamics in the sense that the national income as well as the capital stocks evolve over time. We will numerically confirm the existing analytical results and then consider the effects of the delay on them.

3 Short-run Dynamics

We first make the following two assumptions for the sake of analytical simplicity.

Assumption 3. The saving function is linear and has no autonomous savings,

$$S(Y) = sY,\ 0 < s < 1.$$

Assumption 4. The investment function has the S-shaped form,

$$\varphi(Y) = A * 2^{-\frac{1}{(CY+D)^2}} + BY - \bar{K}.$$

Parameter $\bar{K}$ in $\varphi(Y)$ is positive implying the fixed level of the capital stock.[3] At a stationary state of equation (3), two conditions, $\dot{Y}(t) = 0$ and $Y(t) = Y(t-\theta) = Y^e$ for all $t \geq 0$, are satisfied. Economically, these conditions can be restated as investment is equal to saving at the equilibrium. In Figure 1, we

[3] This is a simplified version of the investment function adopted in Lorenz (1987). It is replaced with the full version in the latter half of this paper.

superimpose the linear saving function, sY with $s = 0.282$, on the three sigmoid investment functions with $A = 35$, $B = 0.02$, $C = 0.01$, $D = 0.00001$ and three different values of $\bar{K}$ (i.e., $\bar{K} = 5, 10, 15$). The equilibrium state is determined by the intersection of these curves.

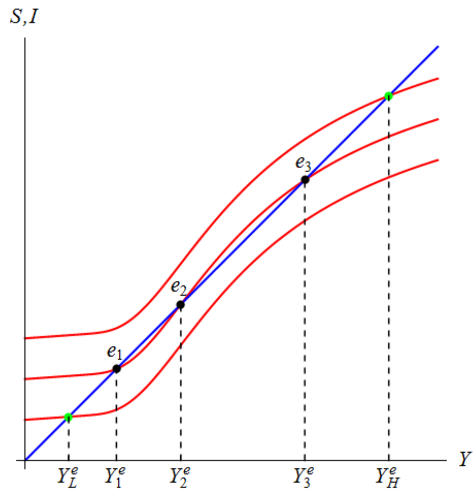


Figure 1. Determination of equilibrium with different values of $\bar{K}$

When the fixed value of $\bar{K}$ is small, we will have a high level of investment and thus a high short-run equilibrium level Y_H^e of national income. As K increases, the investment curve shifts downward. Due to the S-shaped form, there is a case where the saving curve crosses the investment curve three times. These intersections are denoted by the black dotes and their x-coordinates are the corresponding equilibrium levels of national income, $Y_1^e < Y_2^e < Y_3^e$. A further increase of $\bar{K}$ shifts the investment curve downward enough resulting in only one intersection denoted by the lower green dotted point and the corresponding national income is Y_L^e. Investment is small here and thus the equilibrium level is also small.

We draw attention to stability of the three equilibrium points obtained under the middle value of $\bar{K}$. Let $Y_{\delta i} = Y - Y_i^e$ for $i = 1, 2, 3$. The linear approximation of equation (3) is

$$\dot{Y}_{\delta i}(t) = \alpha \eta_i Y_{\delta i}(t - \theta) - \alpha s Y_{\delta i}(t) \tag{4}$$

where $\eta_i = \varphi'(Y_i^e) > 0$ denotes the slope of the investment curve at the equilibrium income Y_i^e. Substituting an exponential solution, $e^{-\lambda t}u$, yields the characteristic equation,

$$\lambda + \alpha s - \alpha \eta_i e^{-\lambda \theta} = 0. \tag{5}$$

In the absence of time delay (i.e., $\theta = 0$), we simply have,

$$\lambda_i = \alpha(\eta_i - s). \tag{6}$$

If $\eta_i < s$ or $\lambda_i < 0$, then the equilibrium income Y_i^e is stable. Since investment is greater or less than savings in the left or right of the intersection, any non-equilibrium Y approaches its equilibrium level by the multiplier. Similarly it is locally unstable if $\eta_i > s$ or $\lambda_i > 0$. The sign of $\eta_i - s$ depends on the slopes of the investment and saving curves evaluated at the equilibrium point. Apparently, among three equilibrium values, $\eta_1 - s < 0$ at e_1, $\eta_2 - s > 0$ at e_2 and $\eta_3 - s < 0$ at e_3. The middle one (i.e., Y_2^e) is locally unstable and the remaining high and low ones (i.e., Y_1^e and Y_3^e) are locally stable. It is also clear that the unique equilibrium Y_L^e or Y_H^e is locally stable. As a benchmark, we summarize these results as follows.

Theorem 1 *When equation (4) with $\theta = 0$ has three equilibria, then the middle one is locally unstable while both of the larger and smaller ones are locally asymptotically stable.*

Kaldor (1940) focuses on the unstable equilibrium. The main key to it is the captial accumulation that shifts the short-term investment function over time which gives rise to a cycle. On the other hand, we focus on the stable equilibrium and consider whether the delay affects its stability. Hence we return to equation (5) with $\theta > 0$. It should be noticed that $\lambda = 0$ does not solve this equation unless $s = \eta_i$. So if stability of Y_i^e switches at $\theta = \bar{\theta}$, then equation (5) must have a pair of pure imaginary roots there. Since roots of a real function always come in conjugate pairs, we assume $\lambda = i\omega$ with $\omega > 0$. Substitution of this root divides equation (5) into the real and imaginary parts,

$$\begin{aligned} \alpha s - \alpha\eta_i \cos\theta\omega = 0, \\ \omega + \alpha\eta_i \sin\theta\omega = 0. \end{aligned} \tag{7}$$

Moving the first term in each equation to the right, squaring the resultant equations and adding them together yield

$$\omega^2 = \alpha^2(\eta_i + s)(\eta_i - s) \tag{8}$$

where the first two factors on the right hand side are positive. If $\eta_i - s > 0$, then there is a positive ω and stability switch can occur. The condition, $\eta_i > s$, could be possible only at the middle equilibrium point Y_2^e. However since it is already shown to be locally unstable for $\theta = 0$, Y_2^e still remains unstable for any $\theta > 0$. On the other hand, $\eta_i < s$ holds at the other equilibrium points. If $\eta_i - s \leq 0$, then there is no $\omega > 0$ implying that stability switch does not occur. Summarizing these results yields the following:

Theorem 2 *For any equilibria of the delay equation (3), no stability switch occurs for any positive value of the delay.*

Theorem 2 implies that the delay does not affect asymptotic dynamics of the delay model (3). In spite of this result, we show that it really matters in transient dynamics. In particular, following Beddington and May (1975), we show that the delay increases the real parts of the eigenvalues for a stable equilibrium point and decreases the magnitude for an unstable equilibrium point. We proceed to illustrate these effects of the investment delay in the three equilibria case. We assume that $\lambda = x + iy$ with $y \geq 0$ and substitute it into equation (5), After arranging the terms, we obtain

$$x + iy = -\alpha s + \alpha\eta_i e^{-x\theta} \cos y\theta + i\left(-\alpha\eta_i e^{-x\theta} \sin y\theta\right).$$

Comparing both sides finds that the real and imaginary parts are

$$\begin{aligned} x &= -\alpha s + \alpha\eta_i e^{-x\theta} \cos y\theta, \\ y &= -\alpha\eta_i e^{-x\theta} \sin y\theta, \end{aligned} \tag{9}$$

from which we derive the form of the real part depending on y and θ,

$$x = -\alpha s - y \cot y\theta. \tag{10}$$

Moving the first term in the right hand side of the first equation in (9) to the left hand side, squaring both sides of the resultant equation and adding the square of the second equations to it yield the imaginary part depending on x and θ,

$$y = \sqrt{(\alpha\eta_i)^2 e^{-2x\theta} - (x + \alpha s)^2}. \tag{11}$$

Solving equation (9) with $y = 0$ for x yields the real part functional for θ. The equation is

$$x + \alpha s = \alpha\eta_i e^{-x\theta} \tag{12}$$

which clearly has a unique real solution for x that is denoted by $x(\theta)$. For $\theta = 0$,

$$x_i(0) = \alpha(\eta_i - s)$$

which is identical with equation (6). By implicitly differentiating this equation (12), we have that

$$\frac{dx(\theta)}{d\theta} = \alpha\eta_i e^{-\theta x(\theta)} \left(-\frac{dx(\theta)}{d\theta}\theta - x(\theta)\right)$$

implying that

$$\frac{dx(\theta)}{d\theta} = -x(\theta)\frac{\alpha\eta_i e^{-\theta x(\theta)}}{1 + \alpha\eta_i\theta e^{-\theta x(\theta)}}.$$

So if $x(\theta) > 0$, then $dx(\theta)/d\theta < 0$ implying that $x(\theta)$ decreases, and if $x(\theta) < 0$, then $dx(\theta)/d\theta > 0$ implying that $x(\theta)$ increases. In both cases $|x(\theta)|$ decreases. Hence we have Theorem 3.

Theorem 3 *Larger delays result in the smaller absolute value of the real parts and thus slow down convergence speed to stable equilibrium.*

4 Long-run Dynamics

4.1 Kaldor Model

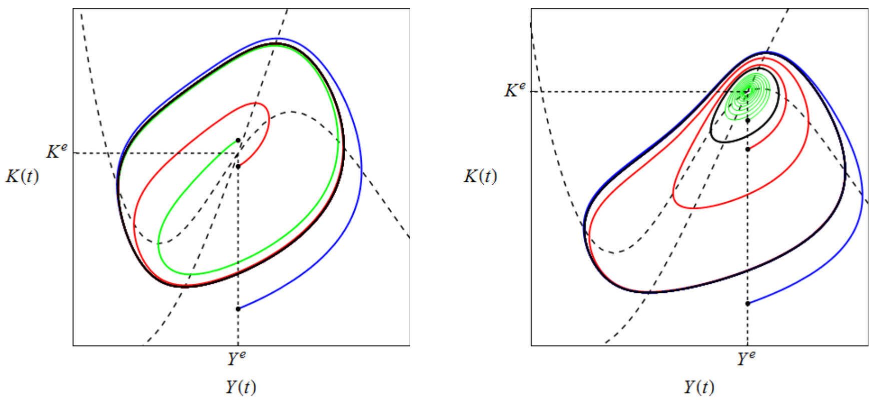
We review the original Kaldor model (1). So far, in this model, it is demonstrated that the nonlinearity of investment functions leads to the two remarkable results. One is the existence of a stable limit cycle shown by Chang and Smyth (1971) with applying the Poincáre-Bendixson theorem when the equilibrium is locally unstable equilibrium and the other is the coexistence of a stable limit cycle and a unstable limit cycle by Grasman and Wentzel (1994) with the use of a Hopf bifurcation theorem when the equilibrium is locally stable. Figure 2 graphically confirms these results with the following form of the separable investment function,[4]

$$\Phi(Y, K) = 25 * 2^{-\frac{1}{(0.015Y+0.00001)^2}} + 0.05Y + 5\left(\frac{320}{K}\right)^3 .$$

and $\alpha = 3$. The positive sloping dashed curve is the $\dot{K} = 0$ locus and the convex-concave dashed curve is the $\dot{Y} = 0$ locus. The intersection of these curves determines the stationary equilibrium point denoted by (Y^e, K^e). In Figure 2(A) where we take $s = 0.3$, the stationary point $(Y^e, K^e) \simeq (54.05,\ 324.32)$, is locally unstable as we will see shortly and two trajectories starting in a neighborhood of the stationary point explosively oscillate and approach the limit cycle. In Figure 2(B), s is decreased to 0.282 and any other parameters are kept to be fixed. The equilibrium point $(Y^e, K^e) \simeq (67.72, 381.94, 54.05)$ becomes stable, which is enclosed by an unstable inner limit cycle which is, in turn, enclosed by an outer stable limit cycle. A green trajectory starting inside of the inner limit cycle converges to the equilibrium point while both the red trajectory starting outside of the inner limit cycle and the blue trajectory starting outside of the outer limit cycle approaches the outer cycle. The black dots denote the initial

[4] Lorenz (1987) uses this form of the function to show the occurrence of chaotic motion in a multisector Kaldorian business cycle model. Grasman and Wentzel (1994) also use this form.

points in simulations.



(A) One limit cycle (B) Two limit cycles

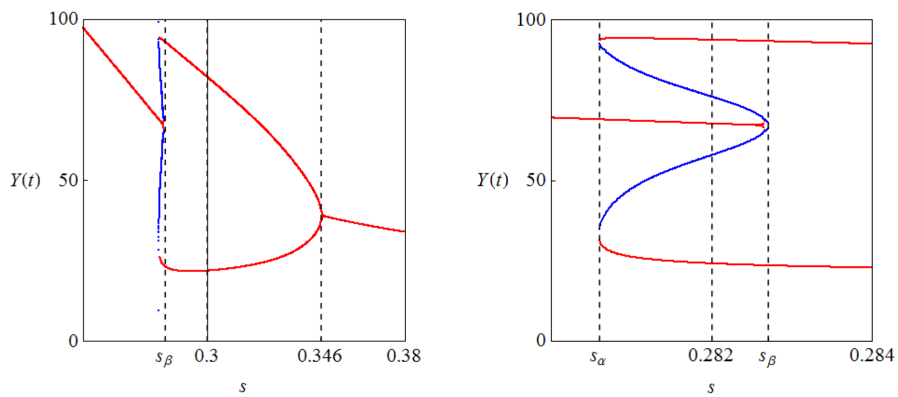
Figure 2. Limit cycles in the Kaldor model

Kaldorian nonlinear dynamics is often examined with respect to the value of the adjustment coefficient α.[5] Figure 2 indicates that Kaldorian dynamics has also strong sensitivity to the value of s. Two different dynamics illustrated in Figure 2 imply that there is a threshold value of s and the changing of the parameter through this value causes a qualitative change in the nature of dynamics. A bifurcation diagram gives good insights into what is happening to evolution of the equilibrium point as the value of the parameter is changed. In Figure 3(A), the value of s is increased from 0.25 to 0.38 with an increment 1/10000. For each value of s, the dynamic system (1) runs for $t \in [0, 500]$ and the local maximum and minimum values of the specified trajectory for $400 \leq t \leq 500$ are plotted. If the diagram shows one point against s, it implies that the equilibrium is stable and the trajectory converges to it. If it shows two points, then the trajectory has one maximum and one minimum point, implying an emergence of a limit cycle. Figure 3(A) roughly indicates that the equilibrium point is locally stable for smaller values of s, loses its stability bifurcating to a limit cycle for medium values and then gains stability for larger values. Figure 3(B) is an enlargement of Figure 3(A) around the two threshold values, s_α and s_β. If $s < s_\alpha$, the Kaldor system generates a stable equilibrium. As s passes s_β, the trajectories of the system converges to a stable limit cycle with the radius equal to the distance between the upper and lower red branches. On the other hand, for $s \in [s_\alpha, s_\beta] \simeq [0.281, 0, 283]$,[6] cyclic dynamics can emerge. Notice that the dotted vertical lime at $s = 0.3$ crosses the bifurcation diagram twice in Figure 3(B) and the corresponding limit cycle is depicted in Figure 2(A). Similarly, the

[5] See, for example, Lorenz (1993)

[6] The value of s_a is numerically obtained and the value of s_β is analytically determined as will be seen.

dotted vertical line at $s = 0.282$ crosses the blue curve twice and the red curve twice in Figure 3(B) and the subcritical Hopf bifurcation leads to multi-stability, that is, the coexistence of a stable equilibrium, an unstable limit cycle and a stable limit cycle as illustrated in Figure 2(B). A distance between the upper and lower blue branches corresponds to the radius of the unstable limit cycle.



(A) $0.25 \leq s \leq 0.38$

(B) $0.28 \leq s \leq 0.284$

Figure 3. Bifurcation diagrams with respect to s

4.2 Kaldor-Kalecki model

We now turn attention to the Kaldor-Kalecki model (2) that has the same stationary equilibrium point, (Y^e, K^e), as the Kaldor model (1). As is already mentioned, Krawiec and Szydlowski (1999) show the existence of limit cycle in the Kaldor-Kalecki model. In this section we revisit this property and compare the results in the non-delay model with the results in the delay model to find how the delay affect dynamics.

Let $Y_\delta = Y - Y^e$ and $K_\delta = K - K^e$. By linearizing (2) at the equilibrium point, we have

$$\begin{aligned} \dot{Y}_\delta(t) &= \alpha\left[(\eta - s)Y_\delta(t) - \beta K_\delta(t)\right], \\ \dot{K}_\delta(t) &= \eta Y_\delta(t-\theta) - (\beta + \delta)K_\delta(t), \end{aligned} \tag{13}$$

where

$$\eta = \frac{\partial \Phi}{\partial Y} = 0.05 + \frac{0.52 \times 2^{-\frac{1}{(0.015Y^e + 0.00001)^2}}}{(0.015Y^e + 0.00001)^3}$$

and

$$\beta = -\frac{\partial \Phi}{\partial K} = -\frac{15 \times (320)^3}{(K^e)^4}.$$

Notice that the values of β and η depend on the the point where they are evaluated, although their dependency is not explicitly expressed in the following. Suppose exponential solutions, $Y_\delta(t) = e^{\lambda t}u$ and $K_\delta(t) = e^{\lambda t}v$. Then the characteristic equation is written as

$$\lambda^2 + a\lambda + b + ce^{-\lambda\theta} = 0 \tag{14}$$

where

$$a = (\beta + \delta) - \alpha(\eta - s),$$
$$b = -\alpha(\eta - s)(\beta + \delta)$$

and

$$c = \alpha\beta\eta.$$

We first examine the non-delay case (i.e., $\theta = 0$) in which the corresponding characteristic equation is reduced to

$$\lambda^2 + a\lambda + b + c = 0.$$

The necessary and sufficient conditions for the roots of the quadratic equation to be negative if real and to have negative real parts if complex are $a > 0$ and $b + c > 0$. Under the specified values of the parameters, $b + c > 0$ always. In Figure 4, the negative sloping black curve is the locus of (s, Y^e) and the closed red curve is the locus of $a = 0$. The black locus crosses the red curve four times at $s = s_i$ for $i = 1, 2, 3, 4$.[7] It is verified that $a < 0$ inside and $a > 0$ outside.[8] Hence, the stationary equilibrium obtained along the black curve is locally stable for $s < s_2$ and $s > s_3$ and locally unstable for $s_2 < s < s_3$ where

$$s_1 \simeq 0.265,\ s_2 \simeq 0.283,\ s_3 \simeq 0.346 \text{ and } s_4 \simeq 0.416.$$

[7] Notice that s_2 is identical with s_β. See footnote 6.

[8] We use the green curve later when the delay model is examined.

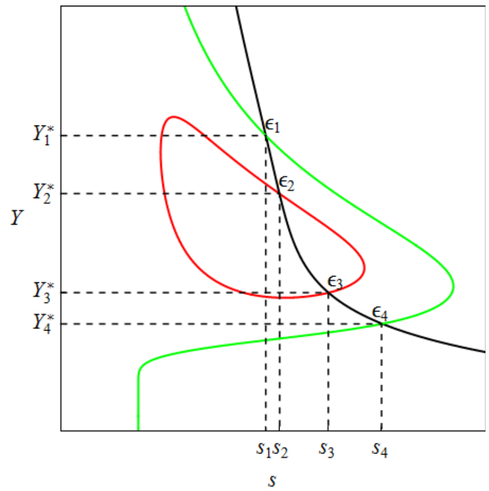


Figure 4. Stability and instability regions

We now return to equation (14) and examine the delay case (i.e., $\theta > 0$). Suppose that $\lambda = i\omega$ with $\omega > 0$ is a solution of the equation for some $\theta > 0$. Substituting it into the equation and separating the real and imaginary parts present

$$\begin{aligned} -\omega^2 + b + c\cos\omega\theta = 0, \\ \omega a - c\sin\omega\theta = 0. \end{aligned} \tag{15}$$

Thus

$$c^2 = (\omega^2 - b)^2 + (\omega a)^2.$$

Hence

$$\omega^4 - (2b - a^2)\omega^2 + b^2 - c^2 = 0$$

and its roots are

$$\omega_\pm^2 = \frac{2b - a^2 \pm \sqrt{(2b - a^2)^2 - 4(b^2 - c^2)}}{2}$$

where

$$2b - a^2 = -\left[(\beta + \delta)^2 + \alpha^2(\eta - s)^2\right] < 0.$$

If $b^2 - c^2 < 0$, then $\omega_+ > 0$ and there is only one imaginary solution, $\lambda = i\omega_+$. On the other hand, if $b^2 - c^2 \geq 0$, then both roots are negative or complex so no imaginary solution exist. In Figure 4, the green curve is the locus of $b^2 - c^2 = 0$. It is verified that $b^2 - c^2 > 0$ in the left side of the green curve and $b^2 - c^2 < 0$ in the right side. Thus stability of Y^e along the black curve is affected by the value of s. Along the black curve, stability of Y^e (as well as stability of K^e) can be switched to instability for $s_1 < s < s_2$ and $s_3 < s < s_4$

for some value of θ while Y^e is locally unstable for $s_2 < s < s_3$ regardless of the value of θ. In case of stability switch, solving the first equation of equation (15) for θ yields the partition curve[9]

$$\theta = \frac{1}{\omega_+} \cos^{-1}\left(\frac{\omega_+^2 - b}{c}\right) \tag{16}$$

that divides the parameter region into two subregions, stability is reserved in one subregion and stability is lost in the other subregion. Figure 5 presents the division of the (s, θ) plane. For $\theta = 0$ (i.e., no-delay case), as it is already confirmed, stability is lost for $s_2 < s < s_3$. As θ increases and becomes positive, we have two results. One is that as far as $s \in [s_2, s_3]$, the equilibrium is locally unstable regardless of the values of θ (i.e., in the white region of Figure 5). The other is that the instability interval of s becomes larger. Stability is preserved in the yellow region and lost in the blue region. The boundary of these regions is the partition curve described by equation (16) that is downward-sloping for $s \in [s_1, s_2]$ and upward-sloping for $s \in [s_3, s_4]$. The blue regions are the enlarged instability regions due to the positive delay.

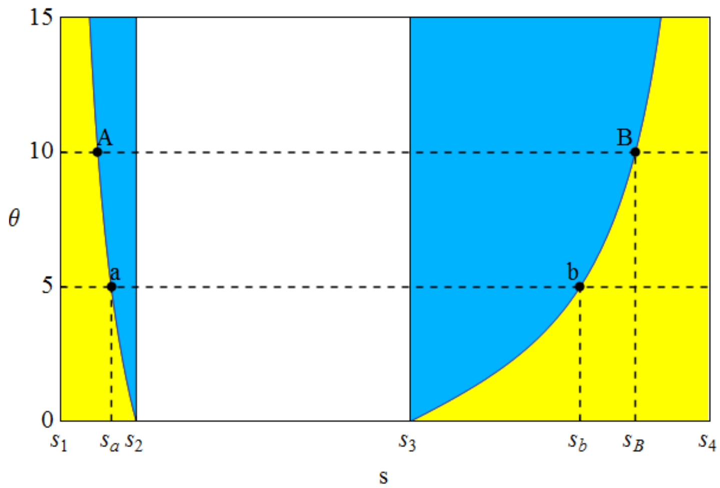


Figure 5. Partition of the (s, θ) plane

Figure 5 gives the bifurcation diagrams with respect to s and reveals the effects caused by increasing θ on dynamics with respect to s from a different view point. The red curve is for $\theta = 0$ and is identical with the one given in Figure 3(A) although multi-stability phenomenon occurred around s_2 is omitted for the sake of graphical simplicity. The blue curve is for $\theta = 5$ where s is increased along the dotted line at $\theta = 5$ in Figure 5 where the dotted line crosses the partition curves at points a and b. Let us denote the s-values of the

[9]Solving the second equation yields the same partition curve in a different form.

intersections by $s_a \simeq 0.277$ and $s_b \simeq 0.386$. Stability is lost for $s = s_a$ and regained for $s = s_b$. The similar bifurcation cascade is obtained for $\theta = 10$ and described by the green curve. The dotted line at $\theta = 10$ crosses the partition curves at points A and B whose s-values are $s_A \simeq 0.274$ and $s_B \simeq 0.399$.[10] Stability is lost for $s = s_A$ and regained for $s = s_B$. Since qualitatively different dynamics arises according to $s < s_2$ or $s > s_2$, we first consider dynamics for $s > s_2$. In Figure 6(A) where the bifurcation diagrams are expanding as θ increases, we observe the following,

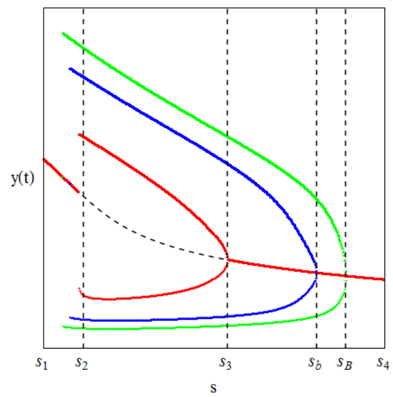
(i) the equilibrium point bifurcates to a limit cycle when s crosses the downward-sloping partition curve;

(ii) the amplitude (or ups and downs) of the cycle becomes larger as delay becomes larger as illustrated by the expansion of the bifurcation diagrams;

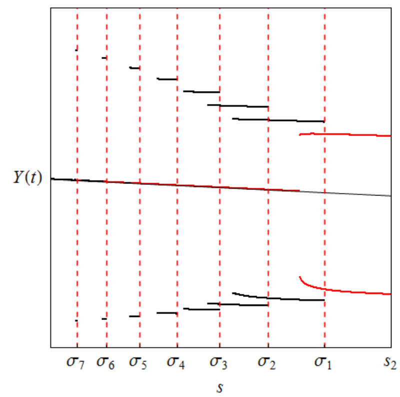
(iii) the stability-regain value of s increases as θ becomes larger, implying that the larger delay has the stronger destabilizing effect by expanding the instability region more.

We turn attention to dynamics for $s < s_2$ to find that it is harder to generate multi-stability as θ becomes larger. In Figure 6(B) the red curve describes the bifurcation diagrams for $s < s_2$, which is the same as Figure 3(B) without the blue curves. We increase the value of θ from 1 to 7 and denote the corresponding threshold values of s determined by equation (16) as σ_k for $k = 1, 2, ..., 7$. Notice that stability is lost for $s = \sigma_k$ for $\theta = k$ since the real parts of the eigenvalues are zero. The black curves ending at the red dotted line at $s = \sigma_k$ imply that multi-stability occurs for s between the starting point of the black curves and the ending point for $\theta = k$. It is observed that the lengths of the black curves become shorter with larger length of delay. Furthermore, for $\theta = 7$, the black curves almost disappear. Therefore it becomes harder to have multi-stability as θ increases.

[10] In Figure 5, s_A is not labelled to avoid confusion.



(A) Shift due to delay



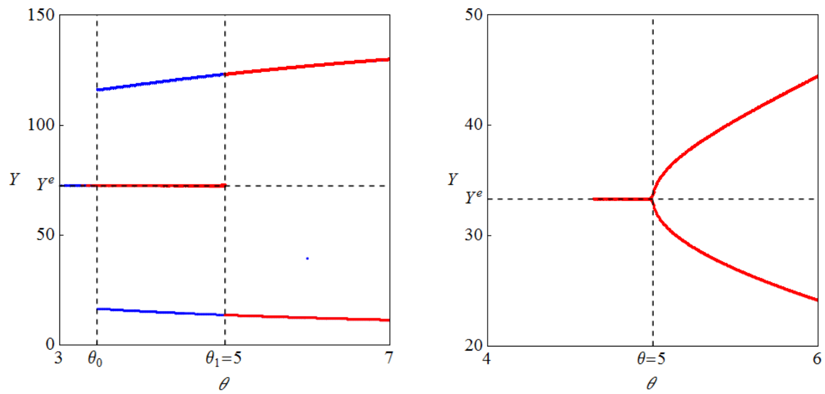
(B) Multi-stability

Figure 6. Bifurcation diagrams with respect to s

We perform two more numerical simulations to see how the stable equilibrium is destabilized via increasing value of θ (i.e., the delay effect). In the above simulations we change the value of s with fixed value of θ. In these simulations, we increase the value of θ, taking the value of s fixed. In the first example, we take $\theta = 5$ and $s \simeq 0.277 \in [s_1, s_2]$ that is obtained via equation (16). For $\theta = 0$, the equilibrium is locally asymptotically stable. When θ arrives at the downward-sloping partition curve at $\theta = 5$, stability is lost and further increased θ induces the equilibrium to bifurcate to a limit cycle. Figure 7(A) shows a bifurcation diagram with respect to θ. It is seen first that the red curve jumps to a limit cycle at $\theta_1 = 5$ via a subcritical Hopf bifurcation. It is further seen that the blue curve appears for $\theta_0 \simeq 3.45$, implying the occurrence of multi-stability for $\theta \in [\theta_0, \theta_1]$. It is also verified that the occurrence of multi-stability becomes harder as the value of θ increases. In the second example, we change the value of s to $s \simeq 0.386 \in [s_3, s_4]$. The increasing θ arrives at the upward-sloping partition curve for $\theta_1 = 5$. Figure 7(B) indicates that the stable equilibrium loses stability at $\theta = \theta_1$ and bifurcates to a limit cycle via supercritical Hopf bifurcation. Summarizing the delay effects obtained above gives the following results:

(i) The equilibrium point bifurcates to a limit cycle via supercritical Hopf bifurcation when increasing θ crosses the upward-sloping partition curve and via subcritical Hopf bifurcation when it crosses the downward-sloping partition curve;

(ii) Multi-stability can happen with respect to delay.



(A) $s \simeq 0.277$ (B) $s \simeq 0.386$

Figure 7. Bifurcation diagrams with respect to θ

5 Concluding Remark

We investigate the Kaldor-Kalecki model in which the investment function has a S-shaped form and a gestation lag of investment between "investment decision" and "investment installation." The main analysis can be divided into two parts. In the first part, with a constant level of the capital stock, short-run dynamics of national income is examined and two results are obtained. First, the delay does not affect asymptotical dynamics in the sense that no stability switch occurs for any values of the delay. Second, the convergence speed gets faster as the delay becomes larger. In the second part, evolution of national income and the capital accumulation are simultaneously examined. Two nonlinear phenomenon, the birth of a limit cycle and coexistence of stable and unstable limit cycles around the stabile equilibrium point, both of which can emerge without delay, are preserved even if the delay is introduced. However it is numerically confirmed that the amplitude of trajectory's fluctuations becomes larger as the delay becomes larger. It can be concluded that the delay affects the long-run as well as short-run dynamics quantitatively but not qualitatively.

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