Heterogenous Competition in a Differentiated Duopoly with Behavioral Uncertainty*

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Abstract

This study examines how uncertainty about rival firm's behavior affects the equilibrium and its dynamics under heterogenous competition. A differentiated mixed duopoly is considered in which one firm sets a quantity and the other firm charges a price. It is demonstrated that behavioral uncertainty stabilizes the unstable dynamic process in comparison to behavioral certainty and generates Pareto-improvement in the sense that the firms can earn larger profits.

Keywords: heterogenous strategy, asymmetric product differentiation, behavioral uncertainty, mixed duopoly.

1 Introduction

We study a mixed duopoly model with incomplete or partial information in which one firm adjusts its quantity and the other firm its price but each firm lacks knowledge of the rival firm's strategic behavior (i.e., quantity or price strategy). The mixed duopoly is an intermediate model between a pure Cournot duopoly and a pure Bertrand model, which was first considered by Bylka and Komar (1976) and then extended in various directions. Using a symmetric differentiated duopoly proposed by Dixit (1979), Singh and Vives (1984) showed, among others, that the quantity-adjusting strategy is more (less) profitable than the price-adjusting strategy if the goods are substitutes (complements). Recently Matsumoto and Szidarovszky (2008) constructed a n-firm differentiated mixed oligopoly model with $n > 2$ showing that their result is sensitive to the duopoly assumption. Szidarovszky and Molnár (1992) proved the equivalence of the equilibrium to the non-linear complementarity problem and also proved the existence and uniqueness of the Nash equilibrium in a general N-firm model. These, only to name a few, are static studies. Matsumoto and Onozaki (2005) and Yousefi and Szidarovszky (2005) modeled the dynamic process of mixed duopolies with nonlinear demands and showed the birth of complicated fluctuations.

In this paper we return to the duopoly model and assume uncertainty about the type of strategic variable chosen by the rival firm. We call a situation in which each firm has complete information about the behavior of the rival firm and the market demand *fully-informed* and the one in which each firm has incomplete information about the behavior of the rival firms but complete information about the market demand *partially-informed*. In the recent literature it has been demonstrated that various phenomena may occur as a result of the lack of information about the market demand. Bischi *et al.* (2004) construct a Cournot duopoly in which Nash equilibrium is unique under perfect information and find the emergence of self-confirming steady state and present a situation in which global bifurcation of basin of attraction occurs when firms choose their actions based on a misspecified market demand. Bischi *et al.* (2007) examine a quantity adjustment process of a nonlinear duopoly model and show its stability under incomplete information about the market demand, despite the fact that it can generate complicated dynamics under complete information when the non-linearities become stronger. In the existing literature, however, very little has been done with respect to the effects caused by the behavioral uncertainty on the equilibrium of the mixed duopoly and its dynamic process.

The main purpose of this study is to shed lights on these unknown effects. In particular, we demonstrate three main results:

1) Incomplete information is chosen endogenously in the presence of exogenous uncertainty about the rival's behavior.

2) Incomplete information stabilizes the market which is unstable under full information.

3) Incomplete information makes Parete-improvement by increasing the profits of both firms.

The remainder of the paper is organized as follows. In Section 2, we present our mixed duopoly model. In the first half of Section 3, we analyze the optimal behavior of the firms with full information in which the behavioral strategy of the firms is common knowledge. Then, in the second half, we reexamine this model with partial information in which the behavioral strategy of the rival firm is unknown. In Section 4, we compare the two types of duopoly equilibria with full information and with partial information. In Section 5, we construct discrete dynamic models in the fully-informed and in the partially informed cases and then show the main conclusions mentioned just above. Section 6 concludes the paper.

2 Cournot-Bertrand Competitions

2.1 Mixed Duopoly Model

There are two firms in a market in which firm i produces good x_i with a constant unit production cost c_i for $i = 1, 2$. The inverse demand function of firm i is

$$P_i = \alpha_i - \beta_i x_i - \gamma_i x_j, \tag{1}$$

where $\alpha_i > 0$ and $\beta_i > 0$.[1] By introducing notation

$$p_i = \frac{P_i}{\beta_i}, \ A_i = \frac{\alpha_i}{\beta_i} \text{ and } \theta_i = \frac{\gamma_i}{\beta_i},$$

we obtain the simplified inverse demand functions,

$$p_i = A_i - x_i - \theta_i x_j. \tag{2}$$

Here A_i denotes the maximum price to be attained when demand is zero and is assumed to satisfy the followin conditions :

Assumption 1. (a) $A_1 = A_2 = A$; (b) $A > \max(c_1, c_2)$.

Assumption 1(a) is imposed on only for analytical simplicity and Assumption 1(b) is imposed to guarantee interior optimum in the profit maximization of the firms.

The parameter θ_i in (2) denotes the degree of product differentiation. It is the ratio of γ_i over β_i where β_i indicates the direct effect on price p_i caused

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[1]It is shown in Singh and Vives (1984) that this linear structure of inverse demand is obtained by maximizing the quadratic and strictly concave utility function

$$U(x_1, x_2) = \alpha_1 x_1 + \alpha_2 x_2 - (\beta_1 x_1^2 + 2\gamma_i x_1 x_2 + \beta_2 x_2^2) - \sum_{i=1}^{2} P_i x_i.$$

by a change in x_i and γ_i shows the indirect effect caused by a change in x_j. If $\theta_i = 1$, then both effects are the same and hence the inverse demand functions for $i = 1, 2$ become identical, which imply demands for homogeneous goods. If $\theta_i = 0$, then there is no indirect effect. Thus the two goods are independent and the duopoly market is divided into two monopoly markets. A positive or negative θ_i implies that the two goods are substitutes or complements. In this study we confine our analysis to the case in which goods are substitutes and the direct effect dominates the indirect effect.[2] To this end, we make the following assumption:

Assumption 2. $0 < \theta_i < 1$.

Solving (2) with $i = 1, 2$ for x_1 and x_2 yields the direct demand of firm i,

$$x_i = \frac{1}{1 - \theta_1 \theta_2} \left\{ (1 - \theta_i)A - p_i + \theta_i p_j \right\}. \tag{3}$$

Each firm is either quantity-adjusting or price-adjusting and maximizes its profit by taking its rival's behavior as given. In particular, firm i maximizes its profit

$$\pi_i = (p_i - c_i)x_i \tag{4}$$

subject to either (2) if it is quantity adjusting or (3) if price-adjusting. With two firms and two strategies, there are four possible types of duopoly competition according to the strategy selection of the firms. In this study, we focus on *heterogeneous competition* in which different firms take different strategies: in *Cournot-Bertrand* (*CB*) competition, firm 1 is quantity-adjusting, and firm 2 is price-adjusting and in *Bertand-Cournot* (*BC*) competition, firm 1 is price-adjusting and firm 2 is quantity-adjusting.[3]

Since Bertrand-Cournot competition is dual to Cournot-Bertrand competition, we restrict our study only to the optimal behavior of firms in *CB* competition. As a benchmark, we start with the case with full information and then proceed to the case of partial information. The optimal values of outputs, prices and profits of the firms under full information are obtained in Section 2.2 and those under partial information will be derived in Section 2.3.

2.2 Full Information Case

Under full information, firm 1 knows that its rival is price-adjusting, sets its quantity x_1, taking its rival's strategic variable p_2 as given. Setting $i = 1$ and substituting (2) into (4) and then substituting x_2 from the solution of (2) give the profit of firm 1 in terms of x_1 and p_2 :

$$\pi_1 = ((1 - \theta_1)A - (1 - \theta_1 \theta_2)x_1 + \theta_1 p_2 - c_1)x_1. \tag{5}$$

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[2]Changing the sign of θ_i from positive to negative, we can go from substitutes to complements. Yousefi and Szidarovszky(2005) exhibits emergence of complex dynamics in a nonlinear differentiated duopoly when the indirect effect dominates the direct effect.

[3]The remaining two types are Cournot-Cournot competition in which the two firms are quantity-adjusting and Bertrant-Bertrand competition in which the two firms are price adjusting.

Firm 2 also knows that firm 1 is quantity-adjusting, charges its price p_2 by taking its rival's strategic variable x_1 as given. Solving (2) with $i = 2$ for x_2 and then substituting it into (4) give the profit of firm 2 in terms of x_1 and p_2:

$$\pi_2 = (p_2 - c_2)(A - \theta_2 x_1 - p_2). \tag{6}$$

Assuming interior optimum and solving the first-order conditions for profit maximization of π_i yields the following forms of the best reply functions:

$$R_1^{CB}(p_2) = \frac{\theta_1}{2(1 - \theta_1 \theta_2)} p_2 + \frac{(1 - \theta_1)A - c_1}{2(1 - \theta_1 \theta_2)} \tag{7}$$

and

$$R_2^{CB}(x_1) = -\frac{\theta_2}{2} x_1 + \frac{A + c_2}{2}. \tag{8}$$

The superscript "CB" attached to functions and variables indicates that they are given in a CB competition.

An intersection of these best reply functions is a solution of the simultaneous equations, $x_1 = R_1^{CB}(p_2)$ and $p_2 = R_2^{CB}(x_1)$. In the (p_2, x_1) plane, the $x_1 = R_1^{CB}(p_2)$ curve is positive-sloping and the $p_2 = R_2^{CB}(x_1)$ curve is negative-sloping. Since both curves are linear, there is a unique solution. We call the solution a CB equilibrium. The outputs at the equilibrium are

$$x_1^{CB} = \frac{2(A - c_1) - \theta_1(A - c_2)}{4 - 3\theta_1 \theta_2} \tag{9}$$

and

$$x_2^{CB} = \frac{(2 - \theta_1 \theta_2)(A - c_2) - \theta_2(A - c_1)}{4 - 3\theta_1 \theta_2}. \tag{10}$$

The corresponding equilibrium prices are

$$p_1^{CB} = \frac{-(2 - \theta_1 \theta_2)(A - c_1) - \theta_1(1 - \theta_1 \theta_2)(A - c_2) + (4 - 3\theta_1 \theta_2)A}{4 - 3\theta_1 \theta_2} \tag{11}$$

and

$$p_2^{CB} = \frac{-\theta_2(A - c_1) - 2(1 - \theta_1 \theta_2)(A - c_2) + (4 - 3\theta_1 \theta_2)A}{4 - 3\theta_1 \theta_2}. \tag{12}$$

Note that both are positive due to Assumption 2.[4] Substituting the CB outputs and prices into the profit functions presents the CB equilibrium profits:

$$\pi_1^{CB} = (1 - \theta_1 \theta_2) \left(x_1^{CB} \right)^2 \tag{13}$$

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[4] The numerators of (11) and (12) are rewritten, respectively, as

$$c_1(2 - \theta_1 \theta_2) + c_2 \theta_1(1 - \theta_1 \theta_2) + A(2 - \theta_1)(1 - \theta_1 \theta_2) > 0$$

and

$$c_1 \theta_2 + 2c_2(1 - \theta_1 \theta_2) + A(2 - \theta_2(1 + \theta_1)) > 0.$$

and
$$\pi_2^{CB} = \left(x_2^{CB}\right)^2, \tag{14}$$

showing that the profits are positive at the CB equilibrium.

Before proceeding further, we introduce an alternative expression of the cost ratio,
$$c = \frac{A - c_2}{A - c_1},$$

which is positive by Assumption 1(b). Solving this definition for c_2 gives
$$c_2 = cc_1 + (1 - c)A$$

which implies that $c_2 = c_1$ if $c = 1$, $c_2 > c_1$ if $c < 1$ and $c_2 < c_1$ if $c > 1$. Clearly c is closely related to the cost ratio. From (9) and (10), the nonnegativity conditions for CB outputs are given as
$$\frac{\theta_2}{2 - \theta_1\theta_2} \le c \le \frac{2}{\theta_1}. \tag{15}$$

If the marginal cost, c_1, of firm 1 is large enough to make the cost ratio exceed the threshold value $2/\theta_1$, then firm 1 chooses zero-production or exits the market to avoid negative profit. By the same token, if the marginal cost, c_2, of firm 2 is large enough to make the cost ratio smaller than the threshold value $\theta_2/(2 - \theta_1\theta_2)$, then firm 2 chooses zero-production or exists the market. In either case, duopoly competition turns into monopoly. Although it seems to be interesting to consider the birth of monopoly through the duopoly competition with the extreme cost ratio, we confine our analysis to the case when both firms stay in the market.

2.3 Partial Information Case

In CB competition with partial information (CB_p competition henceforth), it is assumed that none of the firms have enough information to observe its rival's strategic behavior and therefore each firm determines its best choice, presuming that its rival takes the same strategy. That is, firm 1 thinks that it is in a Cournot competition and firm 2 believes that it is in a Bertrand competition. The profit functions of the firms are therefore defined as follows. Substituting (2) for $i = 1$ into (4) gives the profit of firm in terms of its strategic variable x_1 and the believed strategic variable x_2 of its competitor:
$$\pi_1 = (A - x_1 - \theta_1 x_2 - c_1)x_1.$$

Differentiating π_1 with respect to x_1 gives the first-order condition for the profit maximization, which is then solved for x_1 to obtain the best response of firm 1:
$$x_1 = \frac{A - c_1 - \theta_1 x_2}{2}. \tag{16}$$

Firm 2 thinks that it is in a Bertrand competition so its strategic variable is p_2 and believes that its rival sets a price, p_1. Substituting (3) for $i = 2$ into (4) gives its profit:

$$\pi_2 = (p_2 - c_2)\frac{(1 - \theta_2)A + \theta_2 p_1 - p_2}{1 - \theta_1 \theta_2}.$$

Differentiating π_2 with respect to p_2 gives the first-order condition, which is solved for p_2 to derive the best response of firm 2:

$$p_2 = \frac{(1 - \theta_2)A + c_2 + \theta_2 p_1}{2}. \tag{17}$$

Due to the partial information, each of expressions (16) and (17) depends on the variable that the rival firm is supposed to choose. To express both best response functions in terms of the decision variables, x_1 and p_2, of the firms, we solve (2) with $i = 2$ for x_2 and (3) with $i = 1$ for p_1 to obtain

$$x_2 = A - \theta_2 x_1 - p_2$$

and

$$p_1 = (1 - \theta_1)A - (1 - \theta_1 \theta_2)x_1 + \theta_1 p_2.$$

Substituting them into (16) and (17) provides the best response functions in terms of the decision variables, x_1 and p_2:

$$R_1^{CB_p}(x_1, p_2) \equiv \frac{(1 - \theta_1)A - c_1 + \theta_1 \theta_2 x_1 + \theta_1 p_2}{2} \tag{18}$$

and

$$R_2^{CB_p}(x_1, p_2) \equiv \frac{(1 - \theta_1 \theta_2)A + c_2 - \theta_2(1 - \theta_1 \theta_2)x_1 + \theta_1 \theta_2 p_2}{2}. \tag{19}$$

The superscript "CB_p" is attached to functions and variables to indicate that they are given in CB_p competition.

The equilibrium point is defined by a pair of (x_1, p_2) such that $x_1 = R_1^{CB_p}(x_1, p_2)$ and $p_2 = R_2^{CB_p}(x_1, p_2)$. Solving these equations simultaneously and using the demand function expressions yield the equilibrium outputs

$$x_1^{CB_p} = \frac{(2 - \theta_1 \theta_2)(A - c_1) - \theta_1(A - c_2)}{4 - 3\theta_1 \theta_2} \tag{20}$$

and

$$x_2^{CB_p} = \frac{2(A - c_2) - \theta_2(A - c_1)}{4 - 3\theta_1 \theta_2}, \tag{21}$$

where the nonnegativity conditions for the outputs are given by

$$\frac{\theta_2}{2} \leq c \leq \frac{2 - \theta_1 \theta_2}{\theta_1}. \tag{22}$$

Using (2) with $i = 1$ and $i = 2$, we have the equilibrium prices,

$$p_1^{CB_p} = \frac{-2(1 - \theta_1 \theta_2)(A - c_1) - \theta_1(A - c_2) + (4 - 3\theta_1 \theta_2)A}{4 - 3\theta_1 \theta_2} \tag{23}$$

and

$$p_2^{CB_p} = \frac{-\theta_2(1 - \theta_1\theta_2)(A - c_1) - (2 - \theta_2\theta_2)(A - c_2) + (4 - 3\theta_1\theta_2)A}{4 - 3\theta_1\theta_2}, \quad (24)$$

both of which are positive. The equilibrium profits are given as

$$\pi_1^{CB_p} = \left(x_1^{CB_p}\right)^2 \qquad (25)$$

and

$$\pi_2^{CB_p} = (1 - \theta_1\theta_2)\left(x_2^{CB_p}\right)^2. \qquad (26)$$

Notice that both profits are positive at the equilibrium.

3   Information Difference

In this section, we compare the equilibrium strategies under full information with those under partial information to show the effects caused by information asymmetry.

We start with firm 1. The output difference is obtained by subtracting (20) from (9),

$$x_1^{CB} - x_1^{CB_p} = \frac{(A - c_1)\theta_1\theta_2}{4 - 3\theta_1\theta_2} > 0$$

and the price difference by subtracting (11) from (23),

$$p_1^{CB} - p_1^{CB_p} = \frac{(A - c_1)\theta_1\theta_2}{4 - 3\theta_1\theta_2}(c\theta_1 - 1),$$

where $c\theta_1 - 1$ may be positive, negative or even zero.[5] The profit difference is given by subtracting (13) from (25),

$$\pi_1^{CB} - \pi_1^{CB_p} = \left(\sqrt{1 - \theta_1\theta_2}\,x_1^{CB} + x_1^{CB_p}\right)\left(\sqrt{1 - \theta_1\theta_2}\,x_1^{CB} - x_1^{CB_p}\right),$$

in which the first factor is positive and the second factor can be rewritten as

$$\frac{(A - c_1)\left(1 - \sqrt{1 - \theta_1\theta_2}\right)\left(\sqrt{1 - \theta_1\theta_2} - (1 - c\theta_1)\right)}{4 - 3\theta_1\theta_2}.$$

Since $A - c_1$, $1 - \sqrt{1 - \theta_1\theta_2}$ and the denominator are positive,

$$sign\left[\pi_1^{CB} - \pi_1^{CB_p}\right] = sign[\sqrt{1 - \theta_1\theta_2} - (1 - c\theta_1)]. \qquad (27)$$

We will next show that $\pi_1^{CB} > \pi_1^{CB_p}$ under economically meaningful conditions.

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[5] If $c \leq 1$ (i.e., firm 1 has a more efficient cost function), then $p_1^{CB} < p_1^{CB_p}$.

To determine the sign of the right hand side of (27), we first suppose $c \geq 1$. Then

$$\sqrt{1 - \theta_1 \theta_2} - (1 - c\theta_1) \geq \sqrt{1 - \theta_1 \theta_2} - (1 - \theta_1) > 0,$$

where the last inequality is due to relation $\sqrt{1 - \theta_1 \theta_2} > \sqrt{1 - \theta_1} > 1 - \theta_1$ for $0 < \theta_i < 1$. Therefore we have $\pi_1^{CB} > \pi_1^{CB_p}$ for $c \geq 1$. Next suppose that $c < 1$. The $\pi_1^{CB} = \pi_1^{CB_p}$ locus is negative-sloping in θ_1 and defined by

$$\theta_2 = c(2 - c\theta_1),$$

which implies that $\pi_1^{CB} > \pi_1^{CB_p}$ if $\theta_2 < c(2 - c\theta_1)$ and $\pi_1^{CB} < \pi_1^{CB_p}$ if $\theta_2 > c(2 - c\theta_1)$. Returning to the nonnegativity condition of x_2^{CB} and solving it for θ_2 yields that the $x_2^{CB} = 0$ locus is

$$\theta_2 = \frac{2c}{1 + c\theta_1}.$$

Subtracting this equation from the equal-profit locus yields

$$c(2 - c\theta_1) - \frac{2c}{1 + c\theta_1} = \frac{c^2 \theta_1 (1 - c\theta_1)}{1 + c\theta_1} > 0,$$

since $0 < \theta_1 < 1$ and $0 < c < 1$. This shows that the $\pi_1^{CB} = \pi_1^{CB_p}$ locus is located in the region in which $x_2^{CB} < 0$. Alternatively, $\pi_1^{CB} > \pi_1^{CB_p}$ in the feasible region in which $x_2^{CB} \geq 0$ for $c < 1$. Combining this observation with the last result implies that $\pi_1^{CB} > \pi_1^{CB_p}$ in the feasible region of Θ in which $x_2^{CB} > 0$.

We now compare the equilibrium strategies of firm 2 in CB competitions with both full and partial informations. As in the case of firm 1, the output difference and price difference are obtained as follows:

$$x_2^{CB} - x_2^{CB_p} = -\frac{(A - c_2)\theta_1 \theta_2}{4 - 3\theta_1 \theta_2} < 0$$

and

$$p_2^{CB} - p_2^{CB_p} = \frac{(A - c_1)\theta_1 \theta_2}{4 - 3\theta_1 \theta_2}(c - \theta_2),$$

where $c - \theta_2$ may be positive, negative or zero.[6] The profit difference is given by

$$\pi_2^{CB} - \pi_2^{CB_p} = \left(x_2^{CB} + \sqrt{1 - \theta_1 \theta_2} x_2^{CB_p} \right) \left(x_2^{CB} - \sqrt{1 - \theta_1 \theta_2} x_2^{CB_p} \right),$$

where the first factor is positive and the second factor can be rewritten as

$$\frac{(A - c_2)\left(1 - \sqrt{1 - \theta_1 \theta_2}\right)\left(c(1 - \sqrt{1 - \theta_1 \theta_2}) - \theta_2\right)}{4 - 3\theta_1 \theta_2}.$$

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[6] If $c > 1$ (i.e., firm 2 has a more efficient cost function), then $p_2^{CB} > p_2^{CB_p}$.

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Since $A - c_2$, $1 - \sqrt{1 - \theta_1 \theta_2}$ and the denominator are positive,

$$sign\left[\pi_2^{CB} - \pi_2^{CB_p}\right] = sign[c(1 - \sqrt{1 - \theta_1 \theta_2}) - \theta_2]. \qquad (28)$$

We can also show that $\pi_2^{CB} < \pi_2^{CB_p}$ under economically meaningful conditions.

To determine the sign of the right hand side of (28), we first suppose that $c \leq 1$. Then

$$c(1 - \sqrt{1 - \theta_1 \theta_2}) - \theta_2 \leq 1 - \theta_2 - \sqrt{1 - \theta_1 \theta_2} < 0,$$

since $\sqrt{1 - \theta_1 \theta_2} > \sqrt{1 - \theta_2} > 1 - \theta_2$ for $0 < \theta_i < 1$. Therefore, $\pi_2^{CB} < \pi_2^{CB_p}$ as $c \leq 1$. Next suppose that $c > 1$. The $\pi_2^{CB} = \pi_2^{CB_p}$ locus is negative-sloping in θ_1 and defined by

$$\theta_2 = c(2 - c\theta_1),$$

which implies that $\pi_2^{CB} < \pi_2^{CB_p}$ if $\theta_2 < c(2 - c\theta_1)$, and $\pi_2^{CB} > \pi_2^{CB_p}$ if $\theta_2 > c(2 - c\theta_1)$. Returning to the nonnegativity condition of $x_1^{CB_p}$ and solving it for θ_2 implies that the $x_1^{CB_p} = 0$ locus is

$$\theta_2 = \frac{2 - c\theta_1}{\theta_1}.$$

Subtracting it from the equal profit locus yields

$$c(2 - c\theta_1) - \frac{2 - c\theta_1}{\theta_1} = \frac{(2 - c\theta_1)(c\theta_1 - 1)}{1 + c\theta_1} > 0.$$

The last inequality is due to $c\theta_1 - 1 > 0$ which is shown as follows. The $x_1^{CB_p} = 0$ locus crosses the horizontal line $\theta_2 = 1$ at $\theta_1^1 = \frac{2}{1+c}$, which implies that the feasible domain of the $x_1^{CB_p} = 0$ locus is an interval $[\theta_1^1, 1)$. For $\theta_1 > \theta_1^1$,

$$1 < \frac{2c}{1 + c} < c\theta_1.$$

The direction of the inequality implies that the $\pi_2^{CB} = \pi_2^{CB_p}$ locus is located in the region in which $x_1^{CB_p} < 0$. Alternatively, $\pi_2^{CB} < \pi_2^{CB_p}$ in the feasible region in which $x_1^{CB} \geq 0$ for $c > 1$. Combining this observation with the last result implies that $\pi_{12}^{CB} < \pi_2^{CB_p}$ in the feasible region of θ_1 and θ_2, in which $x_1^{CB_p} > 0$. We can summarize the above result as follows:

Theorem 1 *In the feasible parameter region in which $x_1^{CB_p} > 0$ and $x_2^{CB} > 0$, the quantity-adjusting firm earns more profit in the fully informed case than in the partially informed case (i.e., $\pi_1^{CB} > \pi_1^{CB_p}$) while the price-adjusting firm earns more profit in the partially informed case than in the fully informed case(i.e., $\pi_2^{CB} < \pi_2^{CB_p}$).*

In comparing the equilibrium strategies under full information with those under partial information, an interesting phenomenon can be noticed: the equilibrium strategies under partial information are the duals of the equilibrium strategies under full information. Comparing the equilibrium output levels (9) and (10) with (20) and (21), the equilibrium prices (11) and (12) with (23) and (24), and the equilibrium profits (13) and (14) with (25) and (26), we can see that they are identical if we interchange the two firms (i.e., $x_i^{CB} = x_j^{CB_p}$, $p_i^{CB} = p_j^{CB_p}$ and $\pi_i^{CB} = \pi_j^{CB_p}$ for $i, j = 1, 2$ and $i \neq j$). Since the CB_p competition is dual to the BC competition ($\pi_1^{CB_p} = \pi_1^{BC}$ and $\pi_2^{CB_p} = \pi_2^{BC}$), Theorem 1 can be alternatively summarized as $\pi_1^{CB} > \pi_1^{BC}$ and $\pi_2^{CB} < \pi_2^{BC}$, which is the result shown by Singh and Vives (1984). Although the equilibrium values are the same, it does not necessarily imply that the dynamic processes are also identical. We will now turn our attention to the analysis of the dynamic processes.

4 Stability Analysis

In this section, we investigate the stability of the CB equilibrium and that of the CB_p equilibrium. If the firms have naive expectations and the time scale is discrete, then the best reply discrete dynamics under full information is constructed as follows:

$$
\begin{cases}
x_1(t+1) = \dfrac{\theta_1}{2(1 - \theta_1 \theta_2)} p_2(t) + \dfrac{(1 - \theta_1)A - c_1}{2(1 - \theta_1 \theta_2)} \\[3mm]
p_2(t+1) = -\dfrac{\theta_2}{2} x_1(t) + \dfrac{A + c_2}{2}.
\end{cases}
\tag{29}
$$

This is a two dimensional linear system with Jacobi matrix

$$
J^{CB} = \begin{pmatrix}
0 & \dfrac{\theta_1}{2(1 - \theta_1 \theta_2)} \\[4mm]
-\dfrac{\theta_2}{2} & 0
\end{pmatrix}.
$$

The eigenvalues of the associated characteristic equation are

$$
\lambda_{1,2} = \pm i \frac{1}{2} \sqrt{\frac{\theta_1 \theta_2}{1 - \theta_1 \theta_2}},
$$

which are inside the unit circle if and only if

$$
\theta_1 \theta_2 < \frac{4}{5}.
\tag{30}
$$

In addition to this stability condition, we need to take the nonnegativity constraints (15) on the outputs, x_1^{CB} and x_2^{CB}, into account, to obtain the proper condition for the stability of the CB equilibrium point:

Theorem 2 *In a CB competition with full information, the unique CB equilibrium is feasible and stable if the degrees of the product differentiation, θ_1 and θ_2, and the cost ratio, c, are in the following set:*

$$\Theta^{CB} = \left\{ (\theta_1, \theta_2, c) \in \Theta \mid \theta_1 \theta_2 < \frac{4}{5\theta_2}, \ \frac{\theta_2}{2 - \theta_1 \theta_2} \leq c \leq \frac{2}{\theta_1} \right\}.$$

We now continue with examining the stability of the CB_p competition. Assuming naive expectations again, the best reply discrete dynamics under partial information is as follows:

$$\begin{cases} x_1(t+1) = \dfrac{\theta_1 \theta_2}{2} x_1(t) + \dfrac{\theta_1}{2} p_2(t) + \dfrac{(1 - \theta_1)A - c_1}{2} \\[3mm] p_2(t+1) = -\dfrac{\theta_2(1 - \theta_1 \theta_2)}{2} x_1(t) + \dfrac{\theta_1 \theta_2}{2} p_2(t) + \dfrac{(1 - \theta_1 \theta_2)A + c_2}{2}. \end{cases} \tag{31}$$

This dynamic system is also two-dimensional and linear with Jacobi matrix

$$J^{CB_p} = \begin{pmatrix} \dfrac{\theta_1 \theta_2}{2} & \dfrac{\theta_1}{2} \\[3mm] -\dfrac{\theta_2(1 - \theta_1 \theta_2)}{2} & \dfrac{\theta_1 \theta_2}{2} \end{pmatrix}.$$

The characteristic equation is quadratic in λ,

$$\lambda^2 - \theta_1 \theta_2 \lambda + \frac{\theta_1 \theta_2}{4} = 0.$$

It is easy to verify that its eigenvalues are inside the unit circle since the stability conditions of this two dimensional system are satisfied under Assumption 2:

$$1 - \det J^{CB_p} = 1 - \frac{\theta_1 \theta_2}{4} > 0$$

and

$$1 \pm tr J^{CB_p} + \det J^{CB_p} = 1 \pm \theta_1 \theta_2 + \frac{\theta_1 \theta_2}{4} > 0.$$

In addition to these conditions, the outputs $x_1^{CB_p}$ and $x_2^{CB_p}$ are subject to the nonnegativity conditions (22). Hence our second stability result can be summarized as follows:

Theorem 3 *In a CB competition with partial information, the unique CB_p equilibrium is feasible and stable if the degrees of the product differentiation, θ_1 and θ_2, and c are in the following set:*

$$\Theta^{CB_p} = \left\{ (\theta_1, \theta_2, c) \mid \frac{\theta_2}{2} \leq c \leq \frac{2 - \theta_1 \theta_2}{\theta_1} \right\}.$$

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The parameter region in which equilibrium outputs are non-negative under CB and CB_p competitions is defined by

$$\Theta = \left\{ (\theta_1, \theta_2, c) \mid \frac{\theta_2}{2 - \theta_1\theta_2} \leq c \leq \frac{2 - \theta_1\theta_2}{\theta_1} \right\}.$$

It can be divided into two parts by the $\theta_1\theta_2 = 4/5$ locus,

$$\Theta = \Theta_s \cup \Theta_U$$

with

$$\Theta_s = \left\{ (\theta_1, \theta_2, c) \in \Theta \mid \theta_1\theta_2 < \frac{4}{5} \right\} \text{ and } \Theta_U = \left\{ (\theta_1, \theta_2, c) \in \Theta \mid \theta_1\theta_2 \geq \frac{4}{5} \right\}.$$

As Figure 1 shows, the lighter-gray region is Θ_s, the darker-gray region is Θ_U and one or two non-negativity conditions are violated in the white regions.

Comparing these stability conditions with those under partial information, we first notice that less information improves stability since the fully-informed dynamic process is unstable while the partially-informed process is stable if $(\theta_1, \theta_2, c) \in \Theta_U$. We then notice that collecting information on the rival's strategy by both firms to change the partially informed competition to fully informed competition is advantageous for firm 1 since $\pi_1^{CB} > \pi_1^{CB_p}$ and disadvantageous for firm 2 since $\pi_2^{CB} < \pi_2^{CB_p}$, regardless of whether the CB equilibrium is stable or unstable.
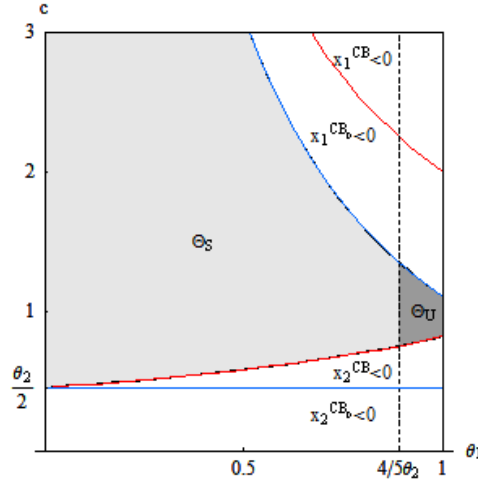

Figure 1. Division of the feasible region by the stability condition

As a result of profit maximizing behavior of the firms, it may be reasonable to assume that firm 1 will collect information about the bahavioral strategy of its rival, make it known to firm 2 as well as fully its own behavioral strategy

to the competitor in order to change a partially informed oligopoly into the fully informed case. However there is no guarantee that firm 2 will believe and trust firm 1 in this message, so the price adjusting frim will continue believing that it is in a partially informed mixed oligopoly. Hence CB competition with asymmetric information endogenously emerges. In this casae, firm 1 has the best reply function $R_1^{CB}(p_2)$ and firm 2 has $R_2^{CB_p}(x_1, p_2)$. An intersection of these best reply functions is an equilibrium state under asymmetric informatition and is denoted by $(x_1^{CB_a}, p_2^{CB_a})$ with

$$x_1^{CB_a} = \frac{(2 - \theta_1\theta_2)(A - c_1) - \theta_1(A - c_2)}{(4 - \theta_1\theta_2)(1 - \theta_1\theta_2)} \tag{32}$$

and

$$p_2^{CB_a} = A - \frac{\theta_2(A - c_1) + 2(A - c_2)}{4 - \theta_1\theta_2}. \tag{33}$$

Similarly to (29) and (31), the discrete dynamic system with naive expectations is given by

$$\begin{cases} x_1(t+1) = \dfrac{\theta_1}{2(1 - \theta_1\theta_2)}p_2(t) + \dfrac{(1 - \theta_1)A - c_1}{2(1 - \theta_1\theta_2)} \\[3mm] p_2(t+1) = -\dfrac{\theta_2(1 - \theta_1\theta_2)}{2}x_1(t) + \dfrac{\theta_1\theta_2}{2}p_2(t) + \dfrac{(1 - \theta_1\theta_2)A + c_2}{2} \end{cases}$$

with Jacobi matrix

$$J^{CB_a} = \begin{pmatrix} 0 & \dfrac{\theta_1}{2(1 - \theta_1\theta_2)} \\[4mm] -\dfrac{\theta_2(1 - \theta_1\theta_2)}{2} & \dfrac{\theta_1\theta_2}{2} \end{pmatrix}.$$

The CB_a equilibrium is asymptotically stable, since the stability conditions are satisfied:

$$1 - \det J^{CB_a} > 0 \text{ and } 1 \pm tr J^{CB_a} + \det J^{CB_a} > 0$$

where

$$\det J^{CB_a} = \frac{\theta_1\theta_2}{4} \text{ and } tr J^{CB_a} = \frac{\theta_1\theta_2}{2}.$$

We summarize this result as follows:

Theorem 4 *In a CB competition with asymmetric information, the unique CB_a equilibrium is feasible and stable if $(\theta_1, \theta_2, c) \in \Theta$.*

This stability result implies that even if firm 1 has full information and firm 2 has only partial information, they can still arrive at the CB_a equilibrium. Since this holds for $(\theta_1, \theta_2, c) \in \Theta_U$, we find again that less information stabilizes the otherwise unstable market under full information.

The equilibrium values of the other variables are

$$p_1^{CB_a} = \frac{-2(A - c_1) - \theta_1(A - c_2) + (4 - \theta_1\theta_2)A}{4 - \theta_1\theta_2} \tag{34}$$

and

$$x_2^{CB_a} = \frac{-\theta_2(A - c_1) + (2 - \theta_1\theta_2)(A - c_2)}{(4 - \theta_1\theta_2)(1 - \theta_1\theta_2)}. \tag{35}$$

We can also obtain the equilibrium profits of the two firms under asymmetric information:

$$\pi_1^{CB_a} = \frac{1}{1 - \theta_1\theta_2}(x_1^{CB_a})^2 \tag{36}$$

and

$$\pi_2^{CB_a} = \frac{1}{1 - \theta_1\theta_2}(x_2^{CB_a})^2. \tag{37}$$

Then we have the following results concerning profit comparisons:

Theorem 5 *Equilibrium profits obtained at the CB_a equilibrium are larger than the profits obtained at any other equilibria:*

$$\pi_1^{CB_a} > \pi_1^{CB} \text{ and } \pi_2^{CB_a} > \pi_2^{CB_p}.$$

Proof. Using (9), (10), (32) and (35), it can be shown that

$$x_1^{CB} - x_1^{CB_a} = -\frac{\theta_1^2\theta_2}{(4 - 3\theta_1\theta_2)(4 - \theta_1\theta_2)(1 - \theta_1\theta_2)}x_2^{CB} \leq 0$$

and

$$x_2^{CB_p} - x_2^{CB_a} = -\frac{\theta_1^2\theta_2}{(4 - 3\theta_1\theta_2)(4 - \theta_1\theta_2)(1 - \theta_1\theta_2)}x_1^{CB_p} \leq 0,$$

where the inequalities are due to the non-negativity conditions of x_2^{CB} and $x_1^{CB_p}$. Based on Assumption 2, the following profit ratios are greater than unity

$$\frac{\pi_1^{CB_a}}{\pi_1^{CB}} = \frac{1}{(1 - \theta_1\theta_2)}\left(\frac{x_1^{CB_p}}{x_1^{CB}}\right)^2 > 1 \text{ and } \frac{\pi_2^{CB_a}}{\pi_2^{CB_p}} = \frac{1}{(1 - \theta_1\theta_2)}\left(\frac{x_2^{CB_a}}{x_2^{CB_p}}\right)^2 > 1.$$

Since $\pi_1^{CB} > \pi_1^{CB_p}$ and $\pi_2^{CB_p} > \pi_2^{CB}$ have been already shown in Theorem 1, this completes the proof. ∎

5   Concluding Remarks

In this study we constructed mixed duopoly models with full information and with partial information in which one firm is quantity-adjusting and the other price-adjusting but each firm has uncertainty about the rival firm's behavioral strategy. If the competition starts with partial information, firm 1 finds it

profitable to collect information on the rival's strategic behavior (i.e., quantity-adjusting or price-adjusting), tells this knowledge to the competitor as well as informs it about its own strategic behavior to change the partially informed oligopoly into a fully informed competition. If firm 2 does not trust its rival in this message, it may continue believing that the oligopoly is still partially informed. In consequence a CB competition with asymmetric information emerges.

Our main conclusions are summarized as follows:

1) CB competition with asymmetric information is a natural consequence in the presence of exogenous uncertainty about the rival firm's behavior.

2) Asymmetric information stabilizes the market in the sense that the CB_a equilibrium is stable even if $(\theta_1, \theta_2, c) \in \Theta_U$.

3) Asymmetric information makes Pareto-improvement since both firms can make larger profits at the CB_a equilibrium than at any other equilibria.

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