

A Little Help from My Friend: International Subsidy Games with Hyperbolic Demands*

Akio Matsumoto[†]
Chuo University

Ferenc Szidarovszky[‡]
University of Arizona

Abstract

In this paper, we assume a hyperbolic price function and construct a three-country model with two active governments and two firms. The purpose of this study is to consider dynamic behavior of the sequential subsidy game in which the governments determine their optimal trade policies and, accordingly, the firms choose their optimal outputs. We first show the existence of an optimal trade policy under realistic conditions. Our main results are summarized as follows: (1) when the production costs are identical, then a trade policy and the corresponding optimal output are stable if the demand is elastic while multistability (i.e., coexistence of multiple attractors) and complex dynamics occur if the demand is inelastic; (2) when the production costs are different, then a stable trade policy induces chaotic output fluctuations regardless of demand elasticity; (3) policy dynamics can be chaotic if demand is elastic while multistability still occurs if the demand is inelastic.

Keywords: Three-country model, Two-stage game, Policy dynamics, Chaos, Multistability, Hyperbolic price function

JEL Classification: F13

*This paper was prepared when the first author visited the Department of Systems and Industrial Engineering of the University of Arizona. He appreciated its hospitality over his stay. The authors are grateful to Junichi Minagawa for his supporting computational study. The usual disclaimer applies.

[†]Department of Economics, Chuo University, 742-1, Higashi-Nakano, Hachioji, Tokyo, 192-0393, Japan. akiom@tamacc.chuo-u.ac.jp

[‡]Department of Systems and Industrial Engineering, University of Arizona, Tucson, AZ, 85721-0020, USA. szidar@sie.arizona.edu

1 Introduction

Market is imperfectly competitive since the number of firms is small, the goods are differentiated or there is some kind of economies of scale. In real world, we see many imperfectly competitive industries in which the firms compete fiercely both domestically and internationally. In such a imperfectly competitive international market, governments may have incentives to introduce trade policies like tariff, export subsidy and tax. If governments can affect, or more precisely, strategically alter market structure by introducing policies, then it is very important to examine how behavior of the firms would be affected or what the impact it would have on the market. To address these issues, a lot of efforts have been devoted since the 1980s. Brander and Spencer (1985) demonstrate that an increase in a domestic export subsidy raises the domestic profit when the firms compete in a Cournot way. Eaton and Grossman (1986) show that an export tax can be optimal when the firms compete in a Bertrand way. It is now well-known that the source of this sharp difference comes from the difference in the assumption on the firms' strategic behavior (that is, the goods are strategic substitutes or strategic complements, the definitions of which are due to Bulow *et al.* (1985)). It is also well-known that this behavioral difference relates to the curvature and the elasticity of the demand function. Recently, constructing a simplified three-country model with two firms and two governments, Bandyopadhyay (1997) shows the following clear-cut results on the trade policy when demand is hyperbolic:

- (1) When the production costs of the firms are identical, then a subsidy is optimal if demand is elastic while a tax is optimal if demand is inelastic.
- (2) When the production costs are different, then the lower-cost firm enjoys the higher subsidy and such a trade policy is locally stable in policy space if demand is elastic while the higher-cost firm enjoys the higher subsidy and such a policy is locally unstable if demand is inelastic.

We move one step forward to consider local as well as global dynamics of trade policy and corresponding outputs in the three-country model. This study complements the results of Bandyopadhyay from a dynamic point of view. It is also an extension of the work of Matsumoto and Serizawa (2007) who focus mainly on the comparative static analysis of a similar three-country model. The price function is assumed to be hyperbolic so that the dynamic model of outputs to be considered in this paper resembles nonlinear dynamic duopoly models, which has been extensively studied during the last twenty years. Comprehensive summary of the earlier works has been presented in Puu and Sushko (2002) and Puu (2003). More recent developments on this field are given in Bischi *et al.* (2009). The third aim of this study is to apply the theoretical results obtained there to the dynamic analysis in the framework of international economics.

This paper is organized as follows. Section 2 presents a variant of the three-country model in which both governments take active roles. Section 3 considers policy dynamics and the corresponding output dynamics when the demand is

elastic. Section 4 examines the same dynamic issue when the demand is inelastic. Section 5 gives concluding remarks.

2 The Model

The model presented below is a variant of the three-country model. Two countries are called *Home* and *Foreign* and governed by the Home government and the Foreign government (henceforth referred to as *H*-government and *F*-government). There is one firm in each country, the firm in the Home country is named firm 1 and the one in the Foreign country firm 2. They produce indifferent outputs, x and y , with constant marginal costs, c_1 and c_2 , respectively, and export all of the outputs they produce to a third country. Competition in the third country is modeled through a two-stage game. At the first stage, the governments hosting their firms choose subsidy rates, s_i for $i = 1, 2$, so as to maximize their welfare, taking the optimal behavior of the firms as given. At the second stage, the firms employ the quantity competition in a Cournot way and choose outputs so as to maximize their profits, taking their governments' trade policies as given. Optimal subsidies and optimal outputs are backwardly determined.

This section is divided into four parts. We first solve the profit maximization problems of the firms in Section 2.1 and then in Section 2.2 we consider stability of the output equilibrium, taking the trade policies of the governments as given. We solve the welfare maximization problems of the governments in Section 2.3 and finally determine the optimal trade policy in Section 2.4.

2.1 Profit Maximization

Let the inverse demand function be hyperbolic,

$$P = \frac{1}{Q^\lambda},$$

where Q is the total output, $Q = x + y$, λ is the reciprocal of demand elasticity and $\lambda > 0$. At the second stage in which governments' subsidies are given, firm 1 and firm 2 choose outputs to maximize their profits

$$\pi_1 = (P - (c_1 - s_1))x,$$

and

$$\pi_2 = (P - (c_2 - s_2))y.$$

The first-order conditions of the profit maximization are given by

$$\frac{\partial \pi_1}{\partial x} = \frac{(1 - \lambda)x + y}{(x + y)^{\lambda+1}} - c_x = 0,$$

and

$$\frac{\partial \pi_2}{\partial y} = \frac{(1 - \lambda)y + x}{(x + y)^{\lambda+1}} - c_y = 0,$$

where $c_x = c_1 - s_1$ and $c_y = c_2 - s_2$ for notational simplicity.¹ We call the production cost including the subsidy an *actual cost*. Although we will show later that the actual costs with the optimal subsidies are non-negative, we suppose for a time being that $c_x > 0$ and $c_y > 0$. From the first-order conditions, the implicit forms of firms' best responses are derived as

$$(1 - \lambda)x + y = c_x(x + y)^{1+\lambda}, \quad (1)$$

and

$$(1 - \lambda)y + x = c_y(x + y)^{1+\lambda}. \quad (2)$$

Dividing (1) by (2) yields

$$y = \frac{k - (1 - \lambda)}{1 - k(1 - \lambda)}x, \quad x + y = \frac{(1 + k)\lambda}{1 - k(1 - \lambda)}x \quad \text{where } k = \frac{c_x}{c_y}.$$

The first two equations are substituted into (1) and (2), and we solve the resultant equations for the corresponding outputs to obtain

$$x^C = \frac{(2 - \lambda)^{\frac{1}{\lambda}} (c_y - (1 - \lambda)c_x)}{\lambda(c_x + c_y)^{\frac{\lambda+1}{\lambda}}}, \quad (3)$$

and

$$y^C = \frac{(2 - \lambda)^{\frac{1}{\lambda}} (c_x - (1 - \lambda)c_y)}{\lambda(c_x + c_y)^{\frac{\lambda+1}{\lambda}}}, \quad (4)$$

where superscript C is attached to variables associated with the Cournot point. In order to assure the nonnegativity of Cournot outputs, we make the following assumptions.

Assumption 1. (1) $0 < \lambda < 2$ and (2) $1 - \lambda < k < \frac{1}{1 - \lambda}$ when $\lambda < 1$.

For the convenience of latter considerations, we define by Ω the feasible set of demand elasticity and the actual cost ratio that satisfy Assumption 1,

$$\Omega = \{(\lambda, k) \mid 0 < \lambda < 2, \text{ and } 1 - \lambda < k < \frac{1}{1 - \lambda} \text{ for } \lambda < 1\}.$$

The Cournot outputs in (3) and (4) are substituted into the profit functions to obtain the Cournot profits:

$$\pi_1^C = \frac{(2 - \lambda)^{\frac{1-\lambda}{\lambda}} (c_y - (1 - \lambda)c_x)^2}{\lambda (c_x + c_y)^{\frac{1+\lambda}{\lambda}}}, \quad (5)$$

and

$$\pi_2^C = \frac{(2 - \lambda)^{\frac{1-\lambda}{\lambda}} (c_x - (1 - \lambda)c_y)^2}{\lambda (c_x + c_y)^{\frac{1+\lambda}{\lambda}}}. \quad (6)$$

¹It can be checked that the second-order conditions are satisfied for any x and y that solve the first-order conditions.

For later analysis, we point out that

$$\begin{cases} \text{sign} \left[\frac{\partial}{\partial y} \left(\frac{\partial \pi_1}{\partial x} \right) \Big|_{(x,y)=(x^C,y^C)} \right] = \text{sign} \left[\frac{\partial x^C}{\partial s_2} \right], \\ \text{sign} \left[\frac{\partial}{\partial x} \left(\frac{\partial \pi_2}{\partial y} \right) \Big|_{(x,y)=(x^C,y^C)} \right] = \text{sign} \left[\frac{\partial y^C}{\partial s_1} \right], \end{cases} \quad (7)$$

where the left hand sides are the signs of the cross derivatives of the marginal profit functions evaluated at the Cournot point and the right hand sides are the cross effects of the Cournot outputs caused by a change in the subsidy of the rival government, for instance,

$$\frac{\partial}{\partial y} \left(\frac{\partial \pi_1}{\partial x} \right) \Big|_{(x,y)=(x^C,y^C)} = \frac{c_y}{(x^C + y^C)^{2+\lambda}} \left(\frac{2-\lambda}{(c_x + c_y)^{1+\lambda}} \right)^{\frac{1}{\lambda}} (1 - (1 + \lambda - \lambda^2)k)$$

and,

$$\frac{\partial x^C}{\partial s_2} = \frac{c_y}{\lambda^2} \left(\frac{2-\lambda}{(c_x + c_y)^{1+2\lambda}} \right)^{\frac{1}{\lambda}} (1 - (1 + \lambda - \lambda^2)k).$$

The terms in the square brackets in the left hand side of (7) refers to the definitions of a *strategic substitute* and a *strategic complement*. The output of firm 1 is strategic substitutes or strategic complements according to whether $1 - (1 + \lambda - \lambda^2)k$ is negative or positive. In the similar way, it can be shown that the output of firm 2 is a strategic substitute and a strategic complement according whether $k - (1 + \lambda - \lambda^2)$ is negative or positive. It is not difficult to show that an increase of the subsidy of one government increases the output of its firm. Hence the equality in (7) implies that if the output of firm 1 is a strategic substitute to the output of firm 2, then an increase of the subsidy given by F -government decreases the output of firm 1 via increasing the output of firm 2. In the same way, if the output of firm 1 is a strategic complement to the output of firm 2, then an increase of the subsidy given by the F -government increases the output of firm 1 through increasing the output of firm 2. Figure 1 shows the deviations of the feasible set Ω by the strategic characteristics of the outputs.² The set Ω is shaded in gray and is divided into the four parts by two curves, $k = 1 + \lambda - \lambda^2$ and $k(1 + \lambda - \lambda^2) = 1$. There, "S" and "C" stand for strategic substitute and strategic complement. Thus the region denoted by (S, C) means that firm 1 treats its output as a strategic substitute whereas firm 2 considers its output as a strategic complement for (λ, k) belonging to the region, and another

²We refer to λ^* and the points denoted by A and a later.

combinations are defined in the same way.

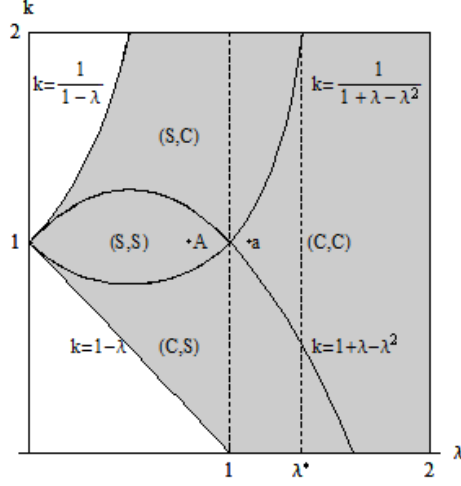


Figure 1. Division of feasible (λ, k) region Ω

2.2 Output Dynamics

We turn our attention to an output adjustment process. We first derive local stability conditions and then consider the nonnegativity conditions for the output trajectories. From (1) and (2), the adjustment process with native expectations can be written as the implicit equations,

$$\begin{cases} (1 - \lambda)x(t + 1) + y(t) = c_x(x(t + 1) + y(t))^{1+\lambda}, \\ x(t) + (1 - \lambda)y(t + 1) = c_y(x(t) + y(t + 1))^{1+\lambda}. \end{cases} \quad (8)$$

The first equation will be denoted by $\gamma_1(x(t + 1), y(t)) = 0$ and the second equation by $\gamma_2(x(t), y(t + 1)) = 0$ as it is impossible to solve explicitly (8) for $x(t + 1)$ and $y(t + 1)$ unless $\lambda = 1$.³

The fixed point of this process has been already obtained in (3) and (4). To find local stability conditions, we derive the Jacobi matrix by linearizing equations of (8) in the neighborhood of the Cournot point and locate the eigenvalues. Notice that the Jacobi matrix has the special form,

$$J = \begin{pmatrix} 0 & \frac{(\lambda^2 - \lambda - 1)k + 1}{(1 + 2\lambda - \lambda^2)k - (1 - \lambda)} \\ \frac{(\lambda^2 - \lambda - 1) + k}{(1 + 2\lambda - \lambda^2) - (1 - \lambda)k} & 0 \end{pmatrix}.$$

³A two-stage game with unit-elastic demand (i.e., $\lambda = 1$) is considered in Matsumoto and Szidarovszky (2009).

Since trace of the Jacobian matrix is zero, stability is confirmed if the absolute value of the product of the eigenvalues is less than unity:

$$|\Psi(\lambda, k)| = \left| -\frac{(\lambda^2 - \lambda - 1)k + 1}{(1 + 2\lambda - \lambda^2)k - (1 - \lambda)} \frac{(\lambda^2 - \lambda - 1) + k}{(1 + 2\lambda - \lambda^2) - (1 - \lambda)k} \right| < 1. \quad (9)$$

Solving $\Psi(\lambda, k) = -1$ yields $k = -1$, $\lambda = 0$ or $\lambda = 2$. They contradict to Assumption 1 and to the fact that the actual cost ratio is positive. We omit this case for further considerations. Solving $\Psi(\lambda, k) = 1$ for k yields two solutions for which loss of stability occurs:

$$\begin{cases} \psi_1(\lambda) = \frac{2 + \lambda \left(2 + \lambda - 3\lambda^2 + \lambda^3 - (\lambda + 1)(\lambda - 2)\sqrt{\lambda^2 - 4\lambda + 5} \right)}{2 + \lambda(2 - 4\lambda + \lambda^2)}, \\ \psi_2(\lambda) = \frac{2 + \lambda \left(2 + \lambda - 3\lambda^2 + \lambda^3 + (\lambda + 1)(\lambda - 2)\sqrt{\lambda^2 - 4\lambda + 5} \right)}{2 + \lambda(2 - 4\lambda + \lambda^2)}. \end{cases} \quad (10)$$

For all k above the $k = \psi_1(\lambda)$ curve and below the $k = \psi_2(\lambda)$ curve, the absolute value $|\Psi(\lambda, k)|$ is greater than one. so the Cournot point is locally unstable. In the same way, for all k between these curves, the absolute value $|\Psi(\lambda, k)|$ is less than unity implying the local asymptotic stability of the Cournot point.

Next we examine the nonnegativity of trajectories generated by (8). From the first equation of (8), it can be found that $x(t+1) = 0$ for $y(t) = y_{\max}$ where

$$y_{\max} = \left(\frac{1}{c_x} \right)^{\frac{1}{\lambda}}.$$

It is also found that the output $x(t+1)$ arrives at its maximum x_{\max} for $y(t) = y_m$ where

$$x_{\max} = \left(\frac{1}{c_x(1 + \lambda)^{1+\lambda}} \right)^{\frac{1}{\lambda}} \quad \text{and} \quad y_m = \lambda x_{\max}$$

The relation $x_{\max} + (1 - \lambda)y_{\max} = c_y (x_{\max} + y_{\max})^{1+\lambda}$ describes that firm 2 chooses to produce y_{\max} when it expects that the competition will produce x_{\max} . Arranging this equation yields

$$k = \phi_1(\lambda) = \frac{1 - \lambda + \left(\frac{1}{(1 + \lambda)^{1+\lambda}} \right)^{\frac{1}{\lambda}}}{\left(1 + \left(\frac{1}{(1 + \lambda)^{1+\lambda}} \right)^{\frac{1}{\lambda}} \right)^{1+\lambda}}. \quad (11)$$

For $k < \phi_1(\lambda)$, a trajectory of output x can be negative. So $k \geq \phi_1(\lambda)$ is the nonnegativity condition for the trajectories when the actual cost ratio is less

than unity. In the same way we can obtain the nonnegativity condition for the trajectories when the actual cost ratio is greater than unity as $k \leq \phi_2(\lambda)$ where

$$\phi_2(\lambda) = \frac{\left(1 + \left(\frac{1}{(1+\lambda)^{1+\lambda}}\right)^{\frac{1}{\lambda}}\right)^{1+\lambda}}{1 - \lambda + \left(\frac{1}{(1+\lambda)^{1+\lambda}}\right)^{\frac{1}{\lambda}}} \left(= \frac{1}{\phi_1(\lambda)}\right). \quad (12)$$

We denote the unstable and nonnegativity region by

$$\Omega_U = \{(\lambda, k) \in \Omega \mid \phi_1(\lambda) < k < \psi_1(\lambda) \text{ or } \psi_2(\lambda) < k < \phi_2(\lambda)\}.$$

Whenever a pair of (λ, k) falls inside region Ω_U , the Cournot point becomes locally unstable. Although the local instability means global instability in a linear dynamic model, this is not a case in the nonlinear case. We examine what kind of dynamics the output adjustment process (8) can generate in the case of local instability. However, equation (8) is given in implicit forms, so it is difficult to examine dynamics analytically. At the expense of generality, we perform numerical simulations after determining the optimal trade policy of the governments.

2.3 Welfare Maximization

At the first stage of the two-stage game, the governments determine the optimal subsidy levels to maximize the national welfare

$$W_1(s_1, s_2) = \pi_1^C(s_1, s_2) - s_1 x^C(s_1, s_2), \quad (13)$$

and

$$W_2(s_2, s_1) = \pi_2^C(s_2, s_1) - s_2 y^C(s_2, s_1), \quad (14)$$

where for notational simplicity, H - and F -governments will be also indexed by "1" and "2". Our first interest is on the condition under which the government decides to give a subsidy or to charge a tax. Assuming an interior optimum and solving the first-order conditions of the welfare maximization for s_i yield

$$s_1 = \underbrace{x^C \frac{dP}{dQ}}_{(-)} \underbrace{\frac{\partial y^C / \partial s_1}{\partial x^C / \partial s_1}}_{(+)} \geq 0 \text{ according to } \frac{\partial y^C}{\partial s_1} \leq 0,$$

and

$$s_2 = \underbrace{y^C \frac{dP}{dQ}}_{(-)} \underbrace{\frac{\partial x^C / \partial s_2}{\partial y^C / \partial s_2}}_{(+)} \geq 0 \text{ according to } \frac{\partial x^C}{\partial s_2} \leq 0$$

where P is the price function and $Q = x + y$ the outputs of the industry.⁴ Since the sign of the cross derivative depends on the strategic characteristic of the output as shown in (7), the optimal trade policy of the governments are summarized as follows.

Lemma 1 *A government pays a subsidy to its firm if the rival firm considers its own output as strategic substitute and levies an export tax to its firm if the rival firm considers its own output as strategic complement.*

Our second interest is on subsidy differential. Neary (1994) shows that the subsidy differential can be expressed as

$$s_1 - s_2 = -\frac{\xi}{\psi}(c_1 - c_2)$$

where, in his notation⁵, ξ and ψ are given by

$$\xi = 2 + R + \alpha\alpha^*R^2$$

and

$$\psi = 2 + R.$$

Using c_x and c_y , we can express the subsidy differential in terms of the actual cost differential,

$$s_1 - s_2 = -\frac{\xi}{\psi + \xi}(c_x - c_y)$$

where $\psi + \xi = 2(2 + R) + \alpha\alpha^*R^2$. Neary (1994) has already shown that $\xi > 0$ and $\psi + \xi > 0$ due to the second-order conditions of the welfare maximization problems. Thus we can summarize the result on the subsidy differential as follows.

Lemma 2 *The government gives a subsidy in such a way that the firm with the lower actual cost receives the higher subsidy than the firm with the higher actual cost.*

Notice that Lemma 1 is concerned with the sign of subsidy and Lemma 2 is concerned with the subsidy differential. Based on these lemmas, we can arrive at the following optimal subsidy policy.

⁴For instance, the first-order condition for the H -government is

$$\frac{\partial \pi_1^C}{\partial x} \frac{\partial x^C}{\partial s_1} + \frac{\partial \pi_1^C}{\partial y} \frac{\partial y^C}{\partial s_1} + \frac{\partial \pi_1^C}{\partial s_1} - x_1 - s_1 \frac{\partial x^C}{\partial s_1} = 0$$

where

$$\frac{\partial \pi_1^C}{\partial x} = 0, \quad \frac{\partial \pi_1^C}{\partial y} = x^C \frac{\partial P}{\partial Q} \frac{\partial Q}{\partial y}, \quad \frac{\partial Q}{\partial y} = 1 \quad \text{and} \quad \frac{\partial \pi_1^C}{\partial s_1} = x^C.$$

⁵ R is a measure of the concavity of demand curve defined by $Q \frac{P''}{P}$, α and α^* are the market share of firm 1 and firm 2 at the Cournot point, respectively, $\alpha = \frac{x^C}{Q^C}$ and $\alpha^* = \frac{y^C}{Q^C}$ where $Q^C = x^C + y^C$.

Theorem 1 *The interior optimal subsidies, s_1^e and s_2^e , depend on the actual cost ratio and the strategic characteristics of the outputs in the following way:*

- (1) *If both outputs are strategic substitutes, then the governments give subsidies to their firms such that*

$$s_1^e \geq s_2^e > 0 \text{ if } c_y \geq c_x \text{ and } s_2^e > s_1^e > 0 \text{ if } c_x > c_y.$$

- (2) *If both outputs are strategic complements, then the governments charge export taxes to their firms such that*

$$0 > s_1^e \geq s_2^e \text{ if } c_y \geq c_x \text{ and } 0 > s_2^e > s_1^e \text{ if } c_x > c_y.$$

- (3) *In the mixed case when one firm treats its output as a strategic substitute and the other firm treats its output as a strategic complement, then the firm with the higher actual cost receives an export tax while the firm with the lower cost receives an export subsidy;*

$$s_1^e > 0 > s_2^e \text{ if } c_y > c_x \text{ and } s_2^e > 0 > s_1^e \text{ if } c_x > c_y.$$

When $c_1 = c_2$, as seen in Figure 1, the outputs are strategic substitutes if demand is elastic and are strategic complements if demand is inelastic. Thus parts (1) and (2) of Theorem 1 can be restated as follows⁶:

Corollary 1 *When the firms are symmetric, an export subsidy is optimal if demand is elastic and an export tax is optimal if demand is inelastic*

In order to get a complete description of the dynamics in the international subsidy game, we have to specify the welfare functions, derive the explicit forms of the best reply functions of the governments and consider their characteristics in the policy space. Substituting x^C , y^C , $Q^C = x^C + y^C$ and $P^C = (Q^C)^{-\lambda}$ into (13) and (14) yields the explicit forms of H -government's welfare function,

$$W_1(s_1, s_2) = \frac{(2 - \lambda)^{\frac{1}{\lambda} - 1}}{\lambda(c_x + c_y)^{\frac{1}{\lambda} + 1}} (c_x + c_y - c_1(2 - \lambda)) (c_y - (1 - \lambda)c_x), \quad (15)$$

and F -government's welfare function,

$$W_2(s_2, s_1) = \frac{(2 - \lambda)^{\frac{1}{\lambda} - 1}}{\lambda(c_x + c_y)^{\frac{1}{\lambda} + 1}} (c_x + c_y - c_2(2 - \lambda)) (c_x - (1 - \lambda)c_y). \quad (16)$$

Differentiating $W_1(s_1, s_2)$ and $W_2(s_2, s_1)$ with respect to s_1 and s_2 , respectively, yields the first-order conditions,

$$\frac{\partial W_1}{\partial s_1} = \beta f(s_1, s_2) = 0 \implies f(s_1, s_2) = 0, \quad (17)$$

⁶These are the same as Case 2a and Case 3a of Bandyopadhyay (1997).

and

$$\frac{\partial W_2}{\partial s_2} = \beta g(s_2, s_1) = 0 \implies g(s_2, s_1) = 0, \quad (18)$$

where β is a positive constant,

$$\beta = \frac{(2 - \lambda)^{\frac{1}{\lambda} - 1}}{\lambda^2 (c_x + c_y)^{\frac{1}{\lambda} + 2}} > 0,$$

and $f(s_1, s_2)$ and $g(s_2, s_1)$ are defined, respectively, by

$$\begin{aligned} f(s_1, s_2) &= (c_x + c_y)(c_y - s_1 - (1 - \lambda)c_1)(1 - \lambda)\lambda \\ &\quad + (c_y - (1 - \lambda)c_x)(c_y - s_1 - (1 + \lambda - \lambda^2)c_1), \end{aligned}$$

and

$$\begin{aligned} g(s_2, s_1) &= (c_x + c_y)(c_x - s_2 - (1 - \lambda)c_2)(1 - \lambda)\lambda \\ &\quad + (c_x - (1 - \lambda)c_y)(c_x - s_2 - (1 + \lambda - \lambda^2)c_2). \end{aligned}$$

The second-order conditions are

$$\frac{\partial^2 W_1}{\partial s_1^2} = \beta \frac{df(s_1, s_2)}{ds_1} < 0 \Leftrightarrow \frac{df(s_1, s_2)}{ds_1} < 0, \quad (19)$$

and

$$\frac{\partial^2 W_2}{\partial s_2^2} = \beta \frac{dg(s_2, s_1)}{ds_2} < 0 \Leftrightarrow \frac{dg(s_2, s_1)}{ds_2} < 0, \quad (20)$$

where the derivatives are

$$\frac{df(s_1, s_2)}{ds_1} = - \left\{ 2s_1(1 - \lambda)^2 + \lambda \left[(3 - 2\lambda)c_y + (1 - \lambda)c_1 \right] \right\},$$

and

$$\frac{dg(s_2, s_1)}{ds_2} = - \left\{ 2s_2(1 - \lambda)^2 + \lambda \left[(3 - 2\lambda)c_x + (1 - \lambda)c_2 \right] \right\}.$$

A best reply function of the H -government is derived first. Solving $f(s_1, s_2) = 0$ with respect to s_1 yields a pair of roots,

$$s_{1s,L} = -\frac{1}{2(1 - \lambda)^2} \left\{ \lambda \left[(1 - \lambda)c_1 + (3 - 2\lambda)c_y \right] \pm (2 - \lambda)\sqrt{D_1(s_2)} \right\}, \quad (21)$$

where s_{1L} is the larger root, s_{1s} is the smaller root and $D_1(s_2)$ denotes the discriminant defined by

$$D_1(s_2) = (c_y - (1 - \lambda)c_1)^2 + 4(1 - \lambda)\lambda^2 c_1 c_y.$$

It is clear that $D_1(s_2) > 0$ if $\lambda < 1$. In the case of $\lambda > 1$, it is possible to determine two threshold values of s_2 that makes $D_1(s_2) = 0$,

$$s_2^{(\pm)} = c_2 + (\lambda - 1) \left\{ (1 - 2\lambda^2) \pm 2\lambda\sqrt{\lambda^2 - 1} \right\} c_1.$$

It then follows that $D_1(s_2) > 0$ for $s_2 > s_2^{(+)}$ or $s_2 < s_2^{(-)}$ if $\lambda > 1$.

Next we examine which root of (21) satisfies the second-order condition. Supposing $D_1(s_2) > 0$, multiplying both sides of (21) by $2(1 - \lambda)^2$ and then moving the first two terms of the right hand side to the left, we obtain the alternative expressions,

$$2(1 - \lambda)^2 s_{1L} + \lambda((1 - \lambda)c_1 + (3 - 2\lambda)c_y) = (2 - \lambda)\sqrt{D_1(s_2)},$$

and

$$2(1 - \lambda)^2 s_{1S} + \lambda((1 - \lambda)c_1 + (3 - 2\lambda)c_y) = -(2 - \lambda)\sqrt{D_1(s_2)}.$$

The left hand sides of both equations are equal to $-df(s_{1L}, s_2)/ds_1$ and $-df(s_{1S}, s_2)/ds_1$, which should be positive if the root satisfies the second-order condition. The right hand side of the first equation is positive and the one of the second equation is negative under the assumptions of $\lambda < 2$ and $D_1(s_2) > 0$. Hence, given s_2 , s_{1L} is the root that satisfies the first-order and the second-order conditions and therefore it is the best reply function of the H -government:

$$r_1(s_2) = -\frac{\lambda[(1 - \lambda)c_1 + (3 - 2\lambda)c_y] - (2 - \lambda)\sqrt{D_1(s_2)}}{2(1 - \lambda)^2}. \quad (22)$$

In the same way, we can show that the following solution of $g(s_2, s_1) = 0$ with respect to s_2 is the best reply function of the F -government:

$$r_2(s_1) = -\frac{\lambda[(1 - \lambda)c_2 + (3 - 2\lambda)c_x] - (2 - \lambda)\sqrt{D_2(s_1)}}{2(1 - \lambda)^2} \quad (23)$$

where $D_2(s_1)$ is the discriminant defined by

$$D_2(s_1) = (c_x - (1 - \lambda)c_2)^2 + 4(1 - \lambda)\lambda^2 c_2 c_x.$$

Similarity to the previous case, $D_2(s_1) > 0$ if $\lambda < 1$. In the case of $\lambda > 1$, the critical values of s_1 making $D_2(s_1) = 0$ are obtained as

$$s_1^{(\pm)} = c_1 + (\lambda - 1) \left\{ (1 - 2\lambda^2) \pm 2\lambda\sqrt{\lambda^2 - 1} \right\} c_2$$

where $D_2(s_1) > 0$ for $s_1 > s_1^{(+)}$ or $s_1 < s_1^{(-)}$ if $\lambda > 1$.

We will next impose the following two external constraints on the subsidy levels, s_i , since the governments must behave with control. The first constraint is the lower bounds s_i^L of the subsidy, which is the upper bound of the export tax and is negative. Its specific value will be determined later. The second constraint requires an upper bound s_i^U of the subsidy, which may be due to the limited amount of governments' budgets, and is assumed to be equal to the production cost. Intuitively speaking, in determining their export subsidies, the governments will not give more than the production costs of their firms.

Assumption 2 $s_i^L \leq s_i \leq s_i^U$ for $i = 1, 2$ where $s_i^L < 0$ and $s_i^U = c_i > 0$.

If demand is elastic, then $r_1(s_2)$ restricted to the interval $[s_2^L, s_2^U]$ and $r_2(s_2)$ restricted to the interval $[s_1^L, s_1^U]$ are the constrained best reply functions. They are continuous on these intervals. If demand is inelastic, then the best reply functions restricted these domains are modified to be piecewise continuous as it will be shown later.

Let us examine in more detail the shapes of the best reply functions in the case of $\lambda > 1$. $r_1(s_2)$ is defined for only $s_2 \geq s_2^{(+)}$ and $s_2 \leq s_2^{(-)}$ so it is not defined for $s_2^{(-)} < s_2 < s_2^{(+)}$. In Figure 2, the downward-sloping dotted-solid bold curve is shown in the lower-left and the slightly upward-sloping bold curve in the upper-right part of the figure. They are two pieces of the $s_1 = r_1(s_2)$ curve. The upward-sloping solid line is the $df_1(s_1, s_2)/ds = 0$ locus. Let s_1^M be the solution of $df_1(s_1, s_2^{(+)})/ds = 0$. Then $s_2^{(+)} < c_2$ and $c_1 < s_1^M$ because for $1 < \lambda < 2$,

$$s_2^{(+)} = c_2 + \underbrace{(\lambda - 1)\{(1 - 2\lambda^2) + 2\lambda\sqrt{\lambda^2 - 1}\}}_{(-)} c_1 < c_2$$

and

$$s_1^M = \lambda \underbrace{\left\{ 2\lambda^2 - \lambda - 2 + \lambda(3 - 2\lambda)\sqrt{\frac{\lambda + 1}{\lambda - 1}} \right\}}_{\geq 1} c_1 > c_1.$$

The direct substitution of $s_2^{(+)}$ into $r_1(s_2)$ shows that $r_1(s_2^{(+)}) = s_1^M$ while $s_1^U = c_1$ and $s_2^U = c_2$ by definition. It then follows that $r_1(s_2) > s_1^U$ for $s_2^{(+)} \leq s_2 \leq s_2^U$. This imply that the best reply function violates Assumption 2 and thus is not defined for $s_2^{(+)} \leq s_2 \leq s_2^U$ either. To remedy this unfavorable property of $r_1(s_2)$, we extend it to the interval $[s_2^L, s_2^U]$ in the following way. On the upward-sloping solid curve of Figure 2, $0 = df(s_1, s_2)/ds_1$ holds and the second order condition is satisfied under this curve. Let s_1^m be the solution of $0 = df(s_1, s_2^{(-)})/ds_1$.⁷ Under the assumption that $s_1^m \leq s_1^L$, s_2^ℓ is defined so that $r_1(s_2^\ell) = s_1^L$ and $s_2^\ell \leq s_2^{(-)}$. Then define the piecewise best reply function of the H -government by

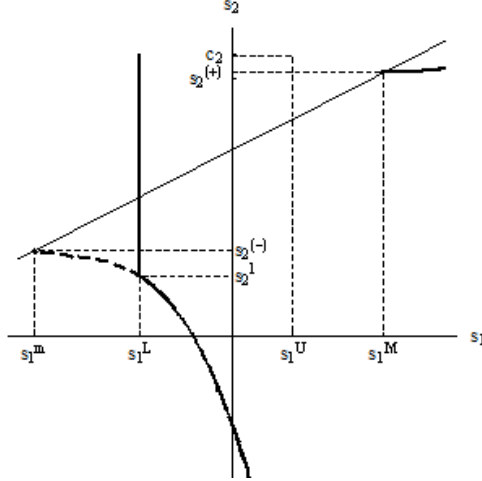
$$R_1(s_2) = \begin{cases} r_1(s_2) & \text{for } s_2^L \leq s_2 \leq s_2^\ell, \\ s_1^L & \text{for } s_2^\ell < s_2 \leq s_2^U. \end{cases} \quad (24)$$

In Figure 2, the bold curve kinked at (s_1^L, s_2^ℓ) is the best reply curve defined by (24). In the same way, the piecewise continuous best reply function of the F -government is also defined under the assumption that $s_2^m \leq s_2^L$:

$$R_2(s_1) = \begin{cases} r_2(s_2) & \text{for } s_1^L \leq s_1 \leq s_1^\ell, \\ s_2^L & \text{for } s_1^\ell < s_1 \leq s_1^U \end{cases} \quad (25)$$

⁷ $s_1^m = \lambda \left(2\lambda^2 - \lambda - 2 - \lambda(3 - 2\lambda)\sqrt{\frac{\lambda + 1}{\lambda - 1}} \right) c_1$. It also holds that $r_H(s_2^{(-)}) = s_1^m$.

where $s_1^{(-)}$ is the smaller root that solves $D_2(s_1) = 0$, s_2^m is the solution of equation $df(s_2, s_1^{(-)})/ds_2 = 0$ and s_1^ℓ satisfies $r_2(s_1^\ell) = s_2^L$.



Piecewise continuous best reply function of the H -government

2.4 Determination of the Optimal Subsidy

In this section, we look for an explicit solution of the trade policy that is determined by an intersection of the best response curves of the two governments. The nonlinearity of the best reply functions may lead to multiple optimal points. Since the specific forms of the best reply functions depend on the cost structure and demand elasticity, we first consider the special case with symmetric firms (i.e., $c_1 = c_2$) and then proceed to the general case.

Substitute $c_1 = c_2 = c$ into (22) and (23) and suppose that $\lambda < 1$. In Figure 3 in which we take $\lambda = 0.8$ and $c = 1$,⁸ the reaction curve of the H -government takes a U -shaped profile with respect to the s_2 axis and so does the reaction curve of the F -government with respect to the s_1 axis.⁹ The area surrounded by the dotted rectangle is the feasible region defined by $[s_1^L, s_1^U] \times [s_2^L, s_2^U]$. Since the two governments are symmetric with respect to the diagonal of the (s_1, s_2) space under the assumption of identical production costs, their intersection occurs at a point on the diagonal. Solving $r_1(s_2) = r_2(s_1)$ with the condition $s_1 = s_2$ yields two positive solutions, which are denoted by E^e and E^A in Figure 3. Two solutions imply that the governments have two choices, the lower (interior) subsidies,

$$E^e = (s_1^e, s_2^e) \text{ with } s_1^e = s_2^e = \frac{\lambda(1-\lambda)c}{2(3-2\lambda)} > 0, \quad (26)$$

⁸Point $(0.8, 1)$ corresponds to point A in Figure 1.

⁹To keep notational consistency, we denote the best reply functions in the case of $\lambda < 1$ by $R_i(s_i)$ for $i, j = 1, 2$ and $i \neq j$.

and higher (cornered) subsidies,

$$E^A = (s_1^A, s_2^A) \text{ with } s_1^A = c_1 (= s_1^U > 0) \text{ and } s_2^A = c_2 (= s_2^U > 0). \quad (27)$$

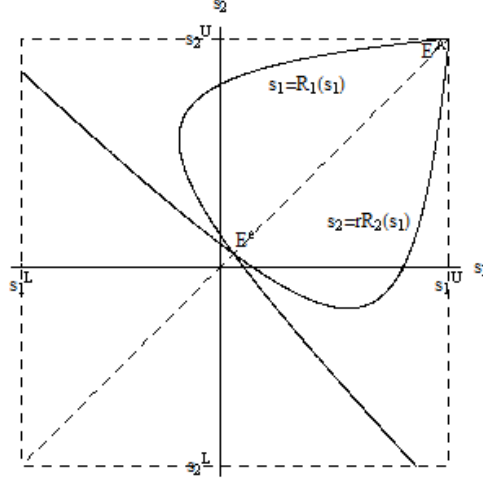


Figure 3. Determination of the optimal subsidy in the case of $\lambda < 1$

If demand is inelastic (i.e., $\lambda > 1$), then there are three optimal points denoted by E^e , E^a and E^b , as shown in Figure 4 in which we take $\lambda = 1.1$ and $c = 1$.¹⁰ The solid curve and the dotted curve are the best reply curves of the H -government and the F -government, respectively. It can be seen that at an interior point, E^e , the governments charge export taxes,

$$E^e = (s_1^e, s_2^e) \text{ with } s_1^e = s_2^e = \frac{\lambda(1-\lambda)}{2(3-2\lambda)}c < 0. \quad (28)$$

The piecewise-continuous curves intersect at two other points, E^a and E^b , which are born because we modified the best reply functions. The two points describe a mixed trade policy in a sense that one government chooses to give an export subsidy and the other government to levy an export tax,

$$E^a = (s_1^a, s_2^a) \text{ with } s_1^a = s_1^L < 0 \text{ and } s_2^a = r_F(s_1^L) > 0 \quad (29)$$

and

$$E^b = (s_1^b, s_2^b) \text{ with } s_1^b = r_H(s_2^L) > 0 \text{ and } s_2^b = s_2^L < 0. \quad (30)$$

¹⁰Point (1.1, 1) corresponds to point a in Figure 1.

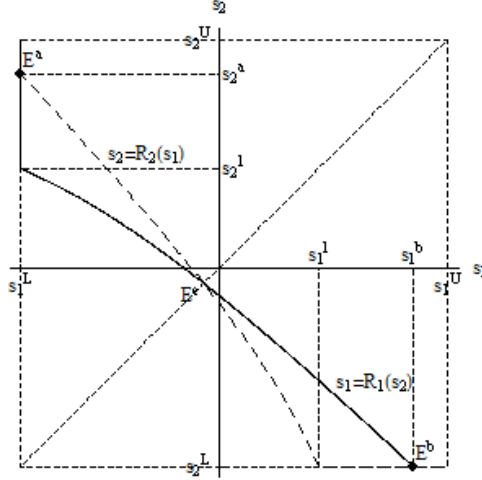


Figure 4. Determination of the optimal subsidy in the case of $\lambda > 1$

We now determine the appropriate value of s_i^L . The best reply functions are obtained under the assumptions that $s_1^m \leq s_1^L$ and $s_2^m \leq s_2^L$, which can be spelled out as

$$\lambda \left(2\lambda^2 - \lambda - 2 - (3 - 2\lambda)\lambda \sqrt{\frac{\lambda + 1}{\lambda - 1}} \right) c \leq s_i^L \quad (31)$$

The equilibrium subsidies are constrained by the two conditions: first they must be greater than the lower bound,

$$\frac{\lambda(1 - \lambda)}{2(3 - 2\lambda)} c \geq s_i^L \quad (32)$$

second, the interior equilibrium point is located in the region where the second-order conditions are satisfied. To this end, the following condition is necessary,

$$\frac{\lambda(3\lambda - 4)c}{2 - 7\lambda + 4\lambda^2} \geq \frac{\lambda(1 - \lambda)c}{2(3 - 2\lambda)}, \quad (33)$$

where the value of the left hand side corresponds to the coordinate of an intercept of the two loci, $df(s_1, s_2)/ds_1 = 0$ and $dg(s_1, s_2)/ds_2 = 0$. These three constraint curves cross at the point $(\frac{15 - \sqrt{17}}{8}, \frac{61 - 11\sqrt{17}}{16(3 - \sqrt{17})}c) \simeq (1.36, -0.87c)$ and (33) is always true for $\lambda \leq \frac{15 - \sqrt{17}}{8}$. If we assume the following assumption, then these three conditions are satisfied:

Assumption 3. $s_i^m \leq s_i^L \leq \frac{\lambda(1 - \lambda)}{2(3 - 2\lambda)} c$ and $\lambda \leq \lambda^* = \frac{15 - \sqrt{17}}{8}$.

We summarize the above derivation in the following theorem:

Theorem 2 *Suppose that the firms are symmetric and Assumption 3 holds. The governments have two distinctive policies of an export subsidy if demand is elastic and three distinctive policies (one is an export tax and the other two are mixed policies) if demand is inelastic.*

Now we proceed to the determination of the optimal trade policy when the firms are asymmetric (i.e., $c_1 \neq c_2$). In principle, the optimal subsidy is determined by an intersection of the best reply curves of the governments. By continuity, the intersection is located in the first quadrant of the (s_1, s_2) space if the cost difference is small and in the second or fourth quadrant if it is large. However, due to the complicated expressions of (22) and (23), we cannot derive general explicit solutions so we specify the parameters' value and numerically obtain the intersections when the cost difference is large. We start with the case of elastic demand. With $c_1 = 1$ and $\lambda = 0.8$ we take $c_2 = 1.25$ in Figure 5(A) where $s_1^e \simeq 0.49$ and $s_2^e \simeq -0.22$. Similarly we select $c_2 = 0.75$ in Figure 5(B) where $s_1^e \simeq -0.16$ and $s_2^e \simeq 0.42$. The actual cost ratios are

$$k = \frac{1 - 0.49}{1.25 - (-0.22)} \simeq 0.34 \text{ and } k = \frac{1 - (-0.16)}{1.25 - 0.42} \simeq 3.45.$$

When $(\lambda, k) = (0.8, 0.34)$ or $(\lambda, k) = (0.8, 3.45)$, the government adopt the mixed trade policy: one government gives an export subsidy to its firm and the other government levies an export tax on its firm. Notice that the lower-cost firm enjoys the higher subsidy.

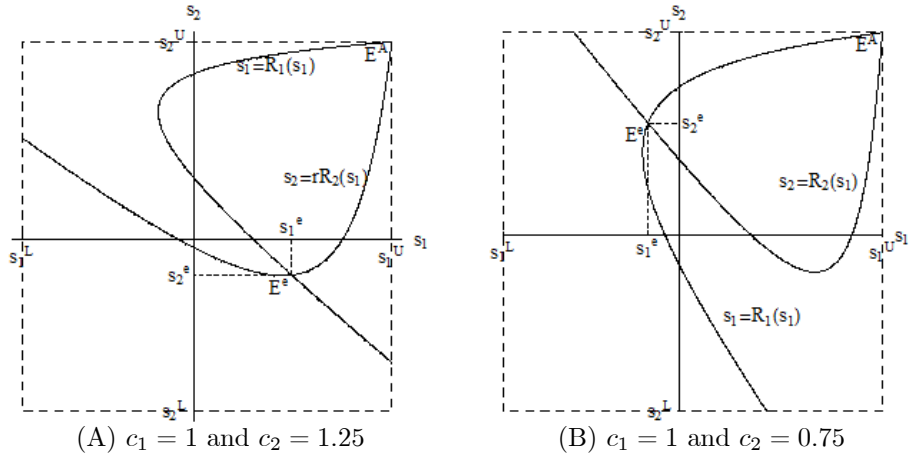


Figure 5. Asymmetric firms and elastic demand ($\lambda = 0.8$)

We turn to the case of inelastic demand. With $c_1 = 1$ and $\lambda = 1.1$, we take $c_2 = 1.05$ in Figure 6(A) where $s_1^e \simeq -0.32$ and $s_2^e \simeq 0.17$ and $c_2 = 0.95$ in Figure 6(B) where $s_1^e \simeq 0.17$ and $s_2^e \simeq -0.31$. The actual cost ratios become

$$k = \frac{1 - (-0.32)}{1.05 - 0.17} \simeq 1.50 \text{ and } k = \frac{1 - 0.17}{1.05 - (-0.31)} \simeq 0.66.$$

When $(\lambda, k) = (1.1, 1.5)$, The H -government levies an export tax and the F -government gives an export subsidy. On the other hand when $(\lambda, k) = (1.1, 0.66)$, the policy is reversed. Notice that the higher-cost firm enjoys the higher subsidy, which is different from the results obtained by de Meza (1986) and Neary (1994). This contradiction is pointed out and called a "perverse" case by Bandyopadhyay (1997). However, since $k > 1$ in Figure 6(A) and $k < 1$ in Figure 6(B), it can be observed that the firm with the lower actual cost receives higher subsidy, as a result of the optimal trade policy.

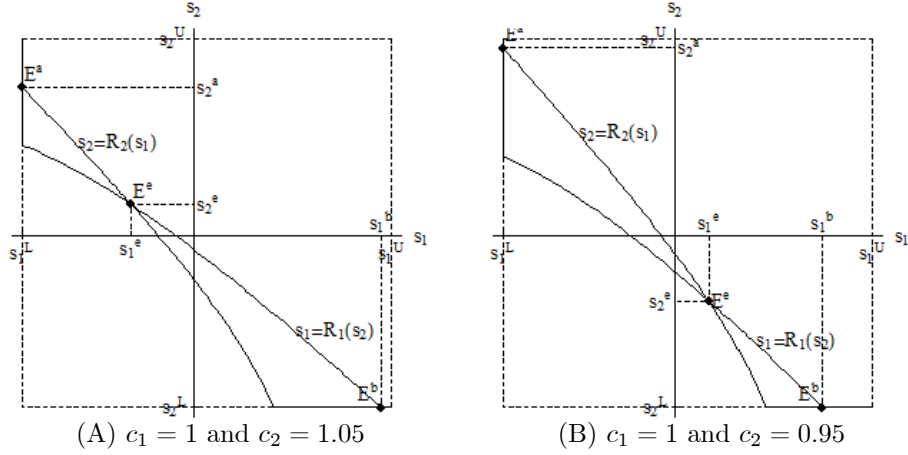


Figure 6. Asymmetric firms and inelastic demand ($\lambda = 1.1$)

3 Dynamics with Elastic Demand

The total dynamic system of the two-stage game consists of the policy dynamic system with adaptive expectations,

$$\begin{cases} s_1(t+1) = (1 - \alpha_1)s_1(t) + \alpha_1 R_1(s_2(t)), \\ s_2(t+1) = (1 - \alpha_2)s_2(t) + \alpha_2 R_2(s_1(t)), \end{cases} \quad (34)$$

where α_i is the adjustment coefficient with $0 < \alpha_i \leq 1$, and the output dynamic system with naive expectations,

$$\begin{cases} (1 - \lambda)x(t+1) + y(t) = (c_1 - s_1(t))(x(t+1) + y(t))^{1+\lambda}, \\ x(t) + (1 - \lambda)y(t+1) = (c_2 - s_2(t))(x(t) + y(t+1))^{1+\lambda}. \end{cases} \quad (35)$$

Notice that the output dynamic system depends on the variables s_1 and s_2 of the policy dynamic system but not *vice versa*. In this section we consider the total dynamics when demand is elastic.

We first examine the local stability of the dynamic process of the optimal policy selection. Its Jacobian matrix is

$$J = \begin{pmatrix} 1 - \alpha_1 & \alpha_1 \frac{\partial R_H}{\partial s_2} \\ \alpha_2 \frac{\partial R_F}{\partial s_1} & 1 - \alpha_2 \end{pmatrix}$$

with trace

$$tr J = 2 - (\alpha_1 + \alpha_2)$$

and determinant

$$\det J = (1 - \alpha_1)(1 - \alpha_2) - \alpha_1 \alpha_2 \frac{\partial R_H}{\partial s_2} \frac{\partial R_F}{\partial s_1}.$$

We can recall that the necessary and sufficient conditions for the local asymptotic stability of a two-dimensional system are as follows:

$$1 - tr J + \det J = \alpha_1 \alpha_2 (1 - \gamma) > 0,$$

$$1 + tr J + \det J = 2(2 - (\alpha_1 + \alpha_2)) + \alpha_1 \alpha_2 (1 - \gamma) > 0, \quad (36)$$

$$1 - \det J = \alpha_1 + \alpha_2 - \alpha_1 \alpha_2 (1 - \gamma) > 0,$$

with

$$\gamma = \frac{\partial R_H}{\partial s_2} \frac{\partial R_F}{\partial s_1}.$$

The first and second conditions are satisfied if $\gamma < 1$ and so is the third condition if $\gamma > -1$. Therefore the stability condition is $|\gamma| < 1$. We will next check whether or not this stability condition is fulfilled in the cases of symmetric and asymmetric firms.

3.1 Symmetric firms: $c_1 = c_2$

The derivatives of the best reply functions evaluated at the steady state are symmetric if the production costs are identical. Furthermore, in the case of elastic demand, these are obtained as

$$0 > \frac{\partial r_H(s_2)}{\partial s_2} = \frac{\partial r_F(s_1)}{\partial s_1} = \frac{-5 + 13\lambda - 11\lambda^2 + 3\lambda^3}{(\lambda - 1)^2(13 - 15\lambda + 4\lambda^2)} > -1 \text{ at } E^e,$$

and

$$\frac{\partial r_H(s_2)}{\partial s_2} = \frac{\partial r_F(s_1)}{\partial s_1} = \frac{\lambda^2 - 2\lambda - 1}{\lambda - 1} > 1 \text{ at } E^A.$$

All derivatives are independent of the value of the common production cost. Since γ is the product of these derivatives,

$$\gamma = \left(\frac{-5 + 13\lambda - 11\lambda^2 + 3\lambda^3}{(\lambda - 1)^2(13 - 15\lambda + 4\lambda^2)} \right)^2 < 1 \text{ at } E^e,$$

and

$$\gamma = \left(\frac{1 + 2\lambda - \lambda^2}{1 - \lambda} \right)^2 > 1 \text{ at } E^A.$$

The stability conditions (36) confirm that E^e is locally stable and E^A is locally unstable.

By (26), the optimal export subsidies of the two governments are identical. In consequence, the actual cost ratio is unity when the firms are symmetric. Returning to the stability condition of the output dynamics, (9), we see that the output equilibrium is locally stable, since

$$|\Psi(\lambda, k)|_{k=1} = \left| - \left(\frac{\lambda - 1}{\lambda - 3} \right)^2 \right| < 1 \text{ for } 0 < \lambda < 1.$$

It is possible to numerically confirm that the stationary points E^e in the policy space and C in the output space are also stable. We summarize these results:

Theorem 3 *If the production costs are identical and demand is elastic, then (i) the inner stationary point E^e in the policy space is locally asymptotically stable while the corner point E^A is locally unstable; (ii) the output stationary point associated with E^e is locally asymptotically stable.*

3.2 Asymmetric firms: $c_1 \neq c_2$

In this section we consider policy and output dynamics when the firms are asymmetric ($c_1 \neq c_2$) and will see that more exotic phenomena emerge. The policy dynamic system (34) determines the feedback effect through the export subsidy policy on the subsequent behavior of the firms and (35) determines the output adjustment, aiming to arrive at the stationary point. The dynamic structure of (35) resembles to that of the nonlinear duopoly model studies by Puu (2003), in which it is shown that the cost difference is a source of complex dynamics. The cost difference also increases nonlinearities involved in (34). One drawback of introducing the cost difference is to make the best reply function more complicated and derivations of analytical solutions very difficult in not impossible. In such cases, it is a natural way to specify the model and use numerical simulations to examine how this second-stage game works.

We take $c_1 = 1$, $c_2 = 1.0567$, $\alpha_1 = \alpha_2 = 0.8$ and $\lambda = 0.98$. In Figure 7(A) in which two trajectories, one starting at point a and the other at point b , are illustrated, the adaptive adjustment process of the traded policy leads to an asymptotically stable equilibrium point. On the other hand, in Figure 7(B), given the optimal trade policy, the unstable Cournot point gives rise to chaotic fluctuations. This is a typical example of the situation with $\lambda < 1$ in which chaotic output dynamics are born although the trade policy is stable. The actual cost ratio plays a crucial role for the birth of complicated dynamics. The optimal trade policy is obtained as $s_1^e \simeq 0.703518$, $s_2^e \simeq -0.64385$, and the actual cost ratio is

$$k = \frac{c_1 - s_1^e}{c_2 - s_2^e} \simeq 0.174345.$$

Due to (10) and (11), the threshold values of the stability and feasibility are

$$\psi_2(\lambda)|_{\lambda=0.98} \simeq 0.18124 \text{ and } \phi_1(\lambda)|_{\lambda=0.98} \simeq 0.17414.$$

Therefore we have

$$\psi_2(\lambda)|_{\lambda=0.98} > k > \phi_1(\lambda)|_{\lambda=0.98}.$$

The first inequality implies instability of the output stationary point and the second inequality guarantees the nonnegativity of a trajectory. If k gets closer to $\phi_1(\lambda)|_{\lambda=0.98}$, then the output fluctuates more aperiodically.

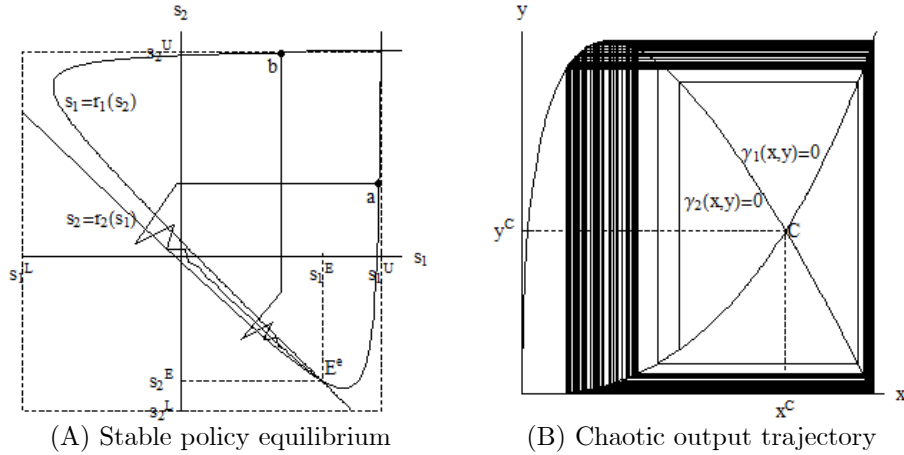


Figure 7. Stable trade policy and chaotic output dynamics

In Figure 8(A) the production cost of firm 2 is increased to $c_2 = 1.2$ with all other parameters kept fixed. As a result, the stationary point of the policy equilibrium is destabilized. In spite of this instability, the nonlinearities of the system prevent the dynamics from diverging and therefore generate bounded fluctuations around the stationary state (s_1^e, s_2^e) . In Figure 8(B) c_2 is further increased to $c_2 = 1.34$ and the degree of elasticity is decreased to 0.96 from 0.98. The policy dynamic system is simulated for 20,000 iterations. The first 5,000 are discarded and the remaining data are plotted in the (s_1, s_2) space. It shows the birth of chaotic attractor. This shows that trajectories aperiodically fluctuate in the long-run. Since the output dynamics depends on the policy dynamics, it also fluctuates aperiodically in the output space. We summarize these numerical results:

Theorem 4 *If the firms are asymmetric and the demand is elastic, then (i) the output dynamics can exhibit chaotic fluctuations even if the policy dynamics is stable; (ii) the policy dynamics can become chaotic and so can the output dynamics.*

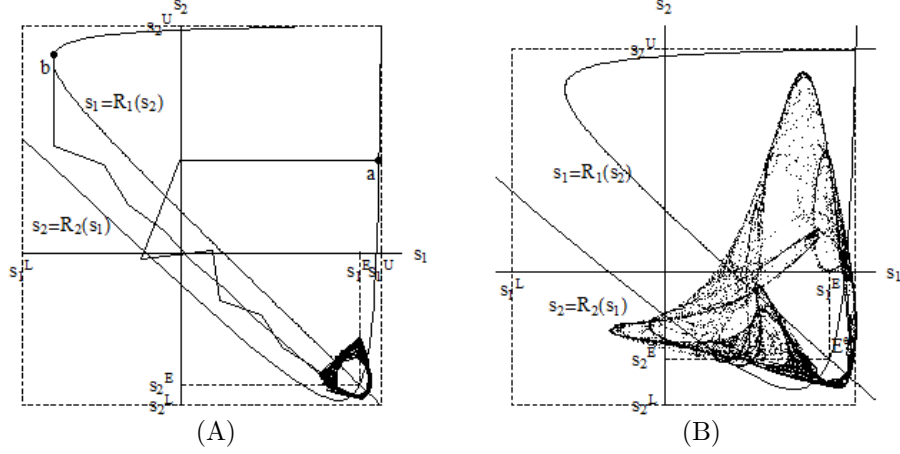


Figure 8. Birth of chaotic attractors

4 Dynamics with Inelastic Demand

4.1 Symmetric firms: $c_1 = c_2$

In the case of inelastic demand, the derivatives evaluated at the inner stationary point E^e have the same form as the ones in the case of elastic demand. Due to Assumption 3, it is defined for $1 < \lambda < \lambda^* = \frac{15 - \sqrt{17}}{8}$, which is the smaller root of equation $13 - 15\lambda + 4\lambda^2 = 0$. The derivatives evaluated at the stationary point are

$$\frac{\partial r_H(s_2)}{\partial s_2} = \frac{\partial r_F(s_1)}{\partial s_1} = \frac{-5 + 13\lambda - 11\lambda^2 + 3\lambda^3}{(\lambda - 1)^2(13 - 15\lambda + 4\lambda^2)} < -1 \text{ at } E^e.$$

It follows that the product of the derivatives,

$$\gamma = \left(\frac{-5 + 13\lambda - 11\lambda^2 + 3\lambda^3}{(1 - \lambda)^2(13 - 15\lambda + 4\lambda^2)} \right)^2 > 1 \text{ at } E^e.$$

Hence E^e is locally unstable. Since the product of the derivatives is zero at the other two equilibria, E^a and E^b , both points are locally stable. In addition to the stable stationary points, there exists a stable period-2 cycle along the diagonal $s_1 = s_2$. The periodic points are (s_1^L, s_2^L) and (s_1^b, s_2^b) with

$$(s_1^b, s_2^b) = (R_1(s_2^L), R_2(s_1^L)) \text{ and } (s_1^L, s_2^L) = (R_1(s_2^b), R_2(s_1^b)).$$

Thus the dynamics system (34) with $\lambda > 1$ is characterized by *multistability*, i.e., coexistence of two attractors and the stable period-2 cycle. In consequence,

we can construct their basins of attraction to see how the asymptotic behavior of the trajectories depends on the choice of the initial point.

Figure 9 illustrates the basins of attraction when the policies are naively adjusted (i.e., $\alpha_1 = \alpha_2 = 1$). The sets of point in the policy space such that the initial points chosen in the green region converges to the point E^a , those in the blue region evolve to the point E^b . Two trajectories, one starting at point a in the green region and the other starting at point b in the blue region are shown in Figure 9(A). The red region represents the basin of attraction of the period-2 cycle and consists of points which generate trajectories converging to the period-2 cycle. One stable trajectory is depicted in Figure 9(B) in which an initial point denoted my c is selected in the upper-right red region. The trajectory starting at point c repeatedly jumps from one red region to the other red region and gradually approaches the periodic point on the diagonal. The following theorem provides the summary of the optimal trade policy:

Theorem 5 *If the firms are symmetric and the demand is inelastic, then three attractors coexist: the two stable fixed points, $E^a = (s_1^a, s_2^a)$ and $E^b = (s_1^b, s_2^b)$, and the stable period-2 cycle $E^{cA} = (s_1^{cA}, s_2^{cA})$ and $E^{cB} = (s_1^{cB}, s_2^{cB})$ where*

$$s_1^a = s_1^L, \quad s_2^a = R_F(s_1^L), \quad s_1^b = R_H(s_2^L), \quad s_2^b = s_2^L$$

and

$$(s_1^{cA}, s_2^{cA}) = (s_1^a, s_2^b) \text{ and } (s_1^{cB}, s_2^{cB}) = (s_1^b, s_2^a).$$

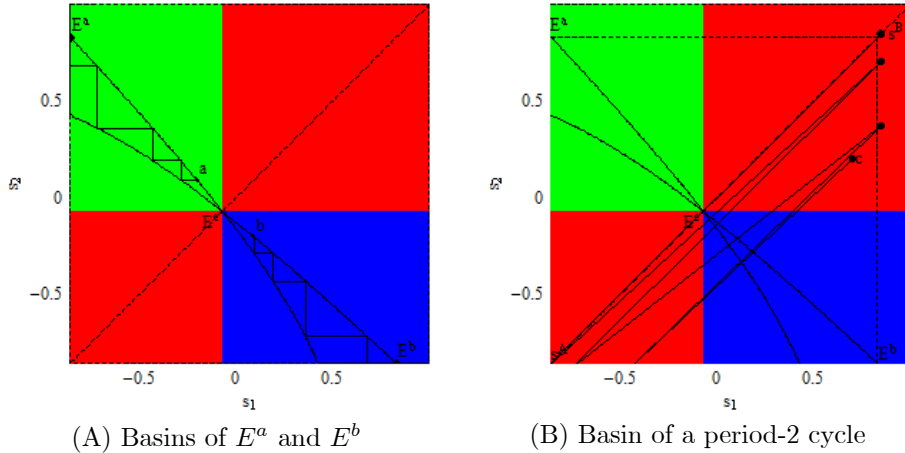


Figure 9. Basins of attraction when $\lambda > 1$

4.1.1 Mixed Trade Policy: E^a or E^b

We now turn our attention to the output dynamics associated with the trade policy determined at point E^b in which the H -government gives an export subsidy $s_1^b > 0$ to firm 1 and the F -government imposes an export tax $s_2^b < 0$ on

firm 2. Since the equilibrium point E^b is stable, it is safe to assume that each firm receives the optimal value of the trade policy from the beginning of the dynamic process. Thus the output dynamic system reduces to

$$\begin{cases} (1 - \lambda)x(t + 1) + y(t) = (c_1 - s_1^b)(x(t + 1) + y(t))^{1+\lambda}, \\ x(t) + (1 - \lambda)y(t + 1) = (c_2 - s_2^b)(x(t) + y(t + 1))^{1+\lambda}. \end{cases} \quad (37)$$

In the numerical example presented in Figure 10, we take $c_1 = c_2 = 1$, $s_1^U = c_1$, $s_2^U = c_2$, $s_1^L = s_2^L = -0.854$ and $\lambda = 1.1$. The stationary values of the subsidies are $s_1^b \simeq 0.83$ and $s_2^b = -0.854 (= s_2^L)$. The ratio of the actual costs is

$$k = \frac{c_1 - s_1^e}{c_2 - s_2^e} \simeq 0.09153.$$

By (10) and (11), we can obtain the threshold values of the stability and feasibility,

$$\psi_2(\lambda)|_{\lambda=1.1} \simeq 0.121695 \text{ and } \phi_1(\lambda)|_{\lambda=1.1} \simeq 0.09036.$$

Therefore we have

$$\psi_2(\lambda)|_{\lambda=1.1} > k > \phi_1(\lambda)|_{\lambda=1.1}.$$

This inequality condition implies that the stationary point of the outputs is locally unstable but its trajectory can stay within the feasible (non-negativity) region. The output trajectory depicted in Figure 10 remains nonnegative for all $t \geq 0$ and aperiodically fluctuates around the stationary point C .

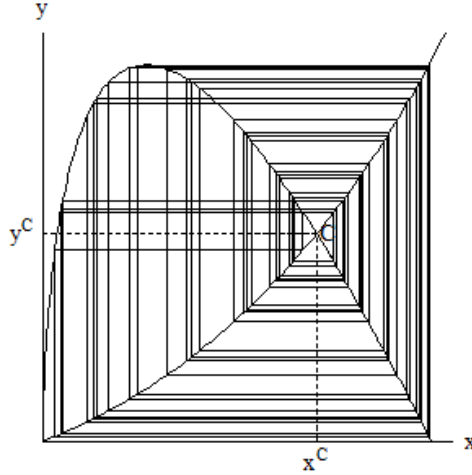


Figure 10. Birth of chaotic output dynamics

Depending on the values of the actual cost ratio, the output adjustment system (37) can generate a wide spectrum of dynamics ranging from stable dynamics to complex dynamics involving chaos. In Figure 11, a bifurcation diagram for the output is shown. Each point along the horizontal axis is a value

for the lower bound, s_2^L . This is a bifurcation parameter and has the effect of changing the actual cost ratio through relation,

$$k = \frac{c_1 - r_H(s_2^L)}{c_2 - s_2^L}.$$

Here the value of s_2^L is increased to -0.8 from -0.855 with 0.0025 increment. For each value of s_2^L the output equation is simulated for 1200 iterations. The first 1000 is discarded to eliminate transient changes. The remaining 200 data is plotted vertically against for s_2^L . As the absolute value of s_2^L increases, the stationary point is destabilized, bifurcates to a periodic cycle and finally fluctuates chaotically via a period-doubling cascade. Under the symmetric assumption $c_1 = c_2$, we can examine the output dynamics associated with point E^a in the same way. We summarize the results as follows:

Theorem 6 *Assume that the firms are symmetric, demand is inelastic and the governments takes the mixed policy, either E^a or E^b . The output dynamics exhibits complicated dynamics if its stationary point is locally unstable.*

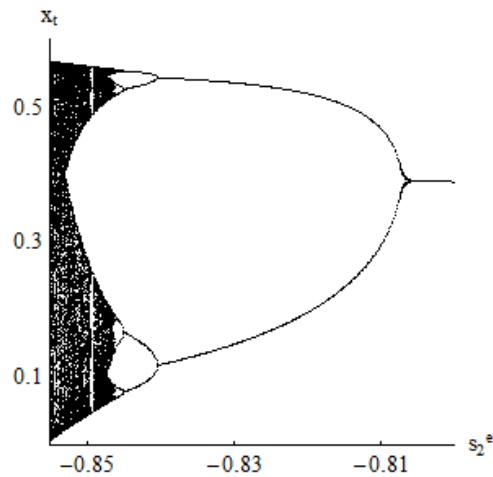


Figure 11. A bifurcation diagram for the output dynamic system (37).

4.1.2 Periodic Trade Policy

The shape of the basin of attraction is sensitive to the value of the adjustment coefficient α_i . Figure 12 illustrates two basins of attraction with two different values of α_i and indicates that the red regions become smaller with decreasing

value of α_i . We will reveal the mechanism which makes the red regions shrink.

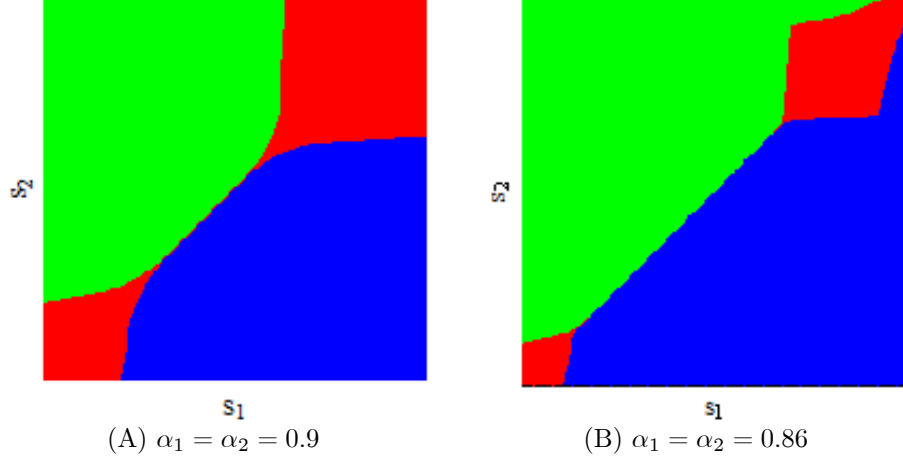


Figure 12. Basins of attraction with two different values of α_i

Any trajectory starting from a point in the red region converges to the period-2 cycle with its periodic points belonging to the diagonal. Hence one way to consider appearance or disappearance of a period-2 cycle is to restrict the dynamic system to the diagonal of the policy space by assuming that the two firms have the same adjustment coefficients, $\alpha_1 = \alpha_2 = \alpha$. This assumption together with the assumption of the identical production costs imply that the two firms behave identically if an initial point is selected on the diagonal. So their dynamic behavior can be described by the following one-dimensional map, which describes a representative firm:

$$s(t+1) = \varphi(s(t)) = (1-\alpha)s(t) + \alpha R(s(t))$$

with $R(s) = R_1(s) = R_2(s)$. The fixed point of $\varphi(s)$ is

$$s^e = \frac{(1-\lambda)\lambda}{2(3-2\lambda)}c$$

and the derivative of $\varphi(s)$ at the fixed point becomes

$$\left. \frac{d\varphi(s)}{ds} \right|_{s=s^e} = \frac{1}{2} \left(2(1-\alpha) + \frac{2\alpha(3\lambda-5)}{13-15\lambda+4\lambda^2} \right).$$

It is possible to show that the derivative is less than unity for $0 < \alpha \leq 1$ and $1 < \lambda \leq \lambda^*$ where $\lambda^* = (15 - \sqrt{17})/8$. It then follows that the stability condition for the fixed point is

$$-1 < \left. \frac{d\varphi(s)}{ds} \right|_{s=s^e}$$

or

$$0 > \frac{9-9\lambda+2\lambda^2}{13-15\lambda+4\lambda^2} (\alpha^S - \alpha)$$

with

$$\alpha^S = \frac{13 - 15\lambda + 4\lambda^2}{9 - 9\lambda + 2\lambda^2}.$$

Hence we arrive at the following stability conditions:

If $\alpha < \alpha^S$, then s^e is locally stable.

To examine the birth of a period-2 cycle, we define the second iterated function of $\varphi(s)$ by $\varphi(\varphi(s))$ and introduces a new function $\phi(s) = s - \varphi(\varphi(s))$. Figure 13 in which $\lambda = 1.1$ illustrates four graphs of $\phi(s)$ with $\alpha = 0.89$, $\alpha = 0.86$, $\alpha = 0.84$ and $\alpha = 0.8478$. The graphs with the first three values of α are depicted as solid curves and the one with $\alpha = 0.8478$ as the bold curve. The intersections with the horizontal line are the fixed points of $\varphi(\varphi(s))$. Since s^e is the fixed point of $\varphi(s)$, it is also a fixed point of $\varphi(\varphi(x))$ and all curves pass through this point. For $\alpha = 0.89$, the $\phi(s)$ curve has a negative slope at s^e because $\alpha^S \simeq 0.88156 < 0.89$ implies the instability of s^e . It has an *N*-shaped curve and its positive sloping parts cross the horizontal line to give rise to a periodic points of a stable period-2 cycle. For $\alpha = 0.86 < \alpha^S$, s^e is locally stable. The $\phi(s)$ curve crosses the horizontal line five times and thus generates two period-2 cycles, one inner unstable cycle and the other outer stable cycle. For $\alpha = 0.84$, s^e is locally stable but the $\phi(s)$ curve intersects the horizontal line only once at s^e . This implies that there exist no period-2 cycle. The threshold value of $\alpha^* \simeq 0.8478$ distinguishes the second case (i.e., emergence of two periodic cycles) from the third case (no periodic cycle). The $\phi(s)$ curve with α^* is depicted as bold and touches the positive part of the horizontal line from above and the negative part from below. In other word, the inner cycle coincides with the outer cycle. Therefore two distinct periodic cycles emerge for $\alpha > \alpha^*$ and no cycle emerges for $\alpha < \alpha^*$. For $\alpha < \alpha^*$ the red region disappears and the initial difference in the policy determines the equilibrium trade policy. Indeed any trajectory starting at a point with $s_1(0) > s_2(0)$ converges to the equilibrium point E^b while any trajectory starting a point with $s_1(0) < s_2(0)$

approaches the equilibrium point E^a .

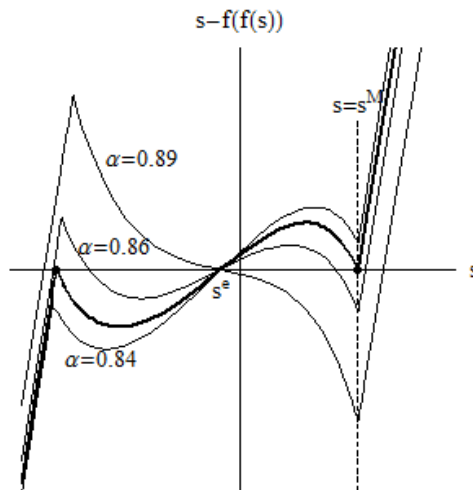


Figure 13. Emergence of a period-2 cycle

Theorem 7 *Suppose that the production costs are identical and the demand is inelastic. (i) The stationary point is locally unstable and one period-2 cycle exists for $1 \geq \alpha \geq \alpha^S$; (ii) The stationary point is locally stable and two period-2 cycles exist for $\alpha^S > \alpha \geq \alpha^*$; (iii) The stationary point is locally stable and no periodic cycles exist for $\alpha < \alpha^*$ where the critical value and the equilibrium value of the adjustment coefficient are*

$$\alpha^S \simeq 0.8816 \text{ and } \alpha^* \simeq 0.8478 \text{ if } \lambda = 1.1.$$

Since the output dynamics depends on the policy dynamics but not vice versa, we can be fairly certain that the output dynamics gives rise to a period-2 cycle which is synchronized with the period-2 cycle of trade policy.

4.2 Asymmetric Firms: $c_1 \neq c_2$

The cost difference does not change the qualitative properties of the dynamics when the demand is inelastic. The policy space is divided into three parts, each of which is a basin of attraction of a stationary state if the adjustment coefficients are close to unity. A period-2 cycle and its basin disappear if the coefficients become much smaller than unity. The cost difference and the value of the adjustment coefficient, however, quantitatively affects the shape of the basin. We fix $c_1 = 1$, $\lambda = 1.1$, $\alpha_1 = \alpha_2 = 0.95$ and take $c_2 = 1.05$ in Figure 14(A) and $c_2 = 0.95$ in Figure 14(B). We have already examined the effect on the determination of the optimal trade policy caused by the cost asymmetry in Figure 7 in which the same values of parameters are taken. It is still true that the higher-cost firm can enjoy the higher subsidy at point E^a in Figure 14(A) and at point E^b in Figure 14(B). On the other hand, the lower-cost firm receives the higher subsidy at point E^b in Figure 14(A) and at point E^a in

Figure 14(B). Comparing Figure 9(A) and Figure 14(A) reveals that the basin of E^a becomes smaller when the cost of firm 2 is larger and so does the basin of E^b when the cost of firm 1 is larger. Then it becomes probable that the lower-cost firm receives higher subsidy as the cost difference becomes larger. It is also numerically confirmed that the basin of the period-2 cycle disappears when the adjustment coefficient becomes smaller. Since the best replies of the firms become complicated with cost difference, it is not easy to reveal analytically the mechanism that causes disappearance of the period-2 cycle.

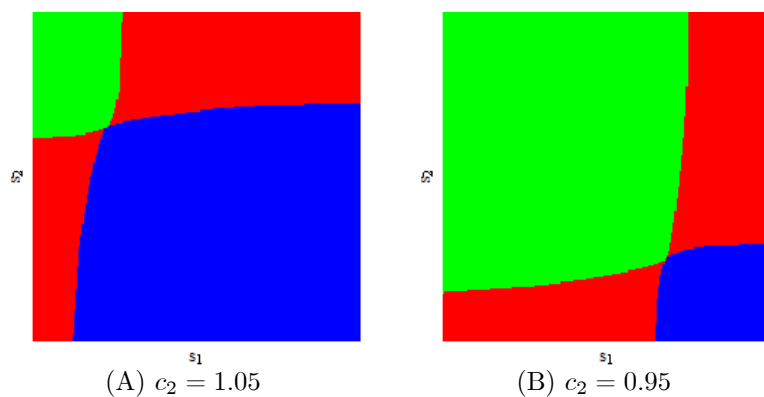


Figure14. Distorted basins of attraction with asymmetric firms

5 Concluding Remarks

In this paper we assume that the price function is hyperbolic and construct a three-country model with two active governments and two firms to consider dynamic behavior of the sequential subsidy game in which the governments determine their trade policy and the firms determine their optimal outputs. We first deal with the determination of the governments' optimal trade policy that depends on the actual cost ratio and strategic characteristics of the outputs. Our dynamic results are summarized as follows:

- (1) When the production costs are identical, a trade policy and the corresponding optimal output are stable if the demand is elastic while multistability (i.e., coexistence of multiple attractors) and complex dynamics of output occur if the demand is inelastic.
- (2) When the production costs are different, a stable trade policy can induce chaotic output fluctuations regardless of demand elasticity.
- (3) When the production costs are different, the trade policy can be chaotic and so does the output if the demand is elastic while multistability and chaotic output dynamics occur if the demand is inelastic.

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