A Further Note on Price and Quantity Competition in Differerntiated Oligopolies*

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Abstract

This study complements the results developed by Häckner (2000) and Hus and Wang (2005). It constructs a n-firm oligopoly model with product differentiation and compares optimal prices, profits and welfare obtained under Cournot competition with those under Bertrand competition. Three main results are demonstrated: (1) higher-qualified firms charge higher price under Bertrand competition than under Cournot competition when the goods are complements; (2) it depends on the ratio of the market average quality to the individual quality whether Cournot profit is higher than Bertrand profit or not; (3) social welfare (the sum of consumer surplus and profits) can be higher under Cournot competition than under Bertrand competition in the case of higher-qualified firms.

1 Introduction

In Bertrand and Cournot markets using the duopoly framework, Singh and Vives (1984) show, among others, the followings clear-cut results:

(i) prices are higher and welfare is lower under Cournot competition than under Bertrand competition regardless of whether the goods are substitutes or complements;

(ii) profits are higher under Cournot competition than under Bertrand competition if the goods are substitutes;

(iii) profits are higher under Bertrand competition than under Cournot competition if the goods are complements.

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In n-firm Bertrand and Cournot markets, Häckner (2000) shows that some of Singh and Vives' results are sensitive to the duopoly framework. Although (iii) is robust in the n-firm framework, the first half of (i) and (ii) can be reversed in the n-firm framework with $n > 2$. Namely, prices can be higher under Bertrand competition than under Cournot competition when the goods are complements. There are also cases in which Cournot profit can be higher than Bertrand profits when the goods are complements. Recently, using the same n-firm oligopoly model, Hsu and Wang (2005) demonstrate that the second half of (i) always holds: consumer surplus and total surplus are higher under Bertrand competition than under Cournot competition, regardless of whether goods are substitutes or complements.

In this study, we extend the results developed by Häckner (2000) and Hsu and Wang (2005). In particular, adopting the n-firm oligopoly model and confining our attention to the case of the stronger complementability, we show three main results: (1) higher-qualified firms charge a higher price under Bertrand competition than under Cournot competition when the goods are complements; (2) it depends on the ratio of the market average quality to the individual quality whether Cournot profit is higher than Bertrand profit or not; (3) social welfare (the sum of consumer surplus and profits) can be higher under Cournot competition than under Bertrand competition with higher-qualified firms.

The rest of the paper is organized as follows. In Section 2 we present an n-firm linear model. In Section 3, prices, profits and social welfare are compared under Cournot and Bertrand competitions. Concluding remarks are given in Section 4.

2 n-Firm Oligopoly Model

We will assume consumer's utility maximization in Section 2.1 to obtain a special demand function. In Section 2.2, the firm's profit maximization will be considered under quantity (Cournot) competition, and in Section 2.3 we will derive the optimal prices, outputs and profits under price (Bertrand) competition.

2.1 Consumers

As in Singh and Vives (1985), there is a continuum of consumers of the same type. Following Häckner (2000), the utility function of the representative consumer is simplified as

$$U(\mathbf{q}, I) = \sum_{i=1}^{n} \alpha_i q_i - \frac{1}{2}\left(\sum_{i=1}^{n} q_i^2 + 2\gamma \sum_{i \neq j}^{n} q_i q_j\right) - I \tag{1}$$

where $\mathbf{q} = (q_i)$ is the quantity vector, α_i measures the quality of good i and $\gamma \in [-1, 1]$ measures the substitutability between the goods: $\gamma > 0$, $\gamma < 0$ or $\gamma = 0$ imply that the goods are substitutes, complements or independent. Moreover, the goods are perfect substitutes if $\gamma = 1$ and perfect complements if $\gamma = -1$. In this study, we confine our analysis to the case in which the goods are imperfect substitutes or complements and are not independent, by assuming that $|\gamma| < 1$ and $\gamma \neq 0$.

The inverse demand function of good k is reduced from the first-order condition of the optimal consumption of good k,

$$p_k = \alpha_k - q_k - \gamma \sum_{i \neq j}^{n} q_j \tag{2}$$

where $n \geq 2$ is assumed. That is, the price vector is a linear function of the output vector:

$$\mathbf{p} = \boldsymbol{\alpha} - \mathbf{Bq} \tag{3}$$

where $\mathbf{p} =(p_i)$, $\boldsymbol{\alpha} = (\alpha_i)$ and $\mathbf{B} = (B_{ij})$ with $B_{ii} = 1$ and $B_{ij} = \gamma$ for $i \neq j$. Assuming that $\mathbf{B}$ is invertible[1] and then solving (3) for $\mathbf{q}$ yield the direct demand

$$\mathbf{q} = \mathbf{B}^{-1}(\boldsymbol{\alpha} - \mathbf{p}) \tag{4}$$

where the diagonal and the off-diagonal elements of $\mathbf{B}^{-1}$ are

$$\frac{1 + (n-2)\gamma}{(1-\gamma)(1+(n-1)\gamma)} \text{ and } -\frac{\gamma}{(1-\gamma)(1+(n-1)\gamma)},$$

respectively. Hence the direct demand of good k, the k^{th}-component of $\mathbf{q}$ in (4), is given by

$$q_k = \frac{(1+(n-2)\gamma)(\alpha_k - p_k) - \gamma \sum_{i \neq k}^{n} (\alpha_i - p_i)}{(1-\gamma)(1+(n-1)\gamma)}. \tag{5}$$

2.2 Quantity-adjusting firms

In Cournot competition, firm k chooses a quantity of good k to maximize its profit $\pi_k = (p_k - c_k)q_k$ subject to (2), taking the other firms' quantities given. The constant marginal cost c_k is assumed to be positive. To avoid negative optimal production, the net quality $\alpha_i - c_k$ is also assumed to be positive.

Assumption 1. $c_k > 0$ and $\alpha_k - c_k > 0$ for all k.

Solving the first-order condition of the profit maximizing problem yields the best response,

$$q_k = \frac{1}{2}\left(\alpha_k - c_k - \gamma \sum_{i \neq k}^{n} q_i\right). \tag{6}$$

It is easily checked that the second-order condition is satisfied. The Cournot equilibrium output and price for firm k are given by

$$q_k^C = \frac{(2+(n-1)\gamma)(\alpha_i - c_k) - \gamma \sum_{i=1}^{n} (\alpha_i - c_i)}{(2-\gamma)(2+(n-1)\gamma)} \tag{7}$$

and

$$p_k^C = \frac{(2+(n-1)\gamma)(\alpha_i + (1-\gamma)c_k) - \gamma \sum_{i=1}^{n} (\alpha_i - c_i)}{(2-\gamma)(2+(n-1)\gamma)}. \tag{8}$$

[1] The n by n matrix B is invertible if $\det B = (1-\gamma)^{n-1}(1+(n-1)\gamma) \neq 0$.

Subtracting (7) from (8) yields $p_k^C - c_k = q_k^C$ that is substituted into the profit function to obtain the Cournot profit,

$$\pi_k^C = \left(q_k^C\right)^2. \tag{9}$$

2.3 Price-adjusting firms

In Bertrand competition, firm k chooses the price of good k to maximize the profit $\pi_k = (p_k - c_k)q_k$ subject to (5), taking the other firms' prices given. Solving the first-order condition yields the best response,

$$p_k = \frac{(1+(n-1)\gamma)(\alpha_k + c_k) - \gamma \sum_{i \neq k}^{n} (\alpha_i - p_i)}{2 + 2(n-3)\gamma}. \tag{10}$$

The second-order condition for the optimum solution (henceforth referred to as SOC) is

$$\frac{\partial^2 \pi_k}{\partial p_k^2} = -\frac{1+(n-2)\gamma}{(1-\gamma)(1+(n-1)\gamma)} < 0, \tag{11}$$

where the direction of inequality depends on the parameter configuration. If $1 > \gamma > 0$ and $n \geq 2$, this SOC is satisfied. On the other hand, if $-1 < \gamma < 0$ and $n \geq 2$, then we need additional conditions to satisfy the SOC. Since

$$1 + (n-1)\gamma < 1 + (n-2)\gamma,$$

the conditions are $1 + (n-1)\gamma > 0$ or $1 + (n-2)\gamma < 0$. We call the first condition SOC_1 and the second condition SOC_2. In Figure 1 where n is taken to be 20, SOC_1 is satisfied in the darker-gray region while SOC_2 is satisfied in the lighter-gray region.[2] The upper boundary of the darker-gray region is the $1 + (n-1)\gamma = 0$ curve and the lower boundary of the lighter-gray region is the $1 + (n-2)\gamma = 0$ curve. Häckner (2000) as well as Hsu and Wang (2005) assumes SOC_1 and give a detailed analysis. Therefore in this paper we assume that SOC_2 holds. As can be seen, the darker-gray region shrinks and the lighter-gray region expands as n increases. A natural question arising is whether the results obtained by Häckner (2000) and by Hsu and Wang (2005) still hold under SOC_2.

Assumption 2 $1 + (n-2)\gamma < 0$ when the goods are complements.

Insert Figure 1 about here

The Bertrand equilibrium price and output for firm k are given by

$$p_k^B = \frac{(2+(n-3)\gamma)[(1+(n-1)\gamma)(\alpha_k + c_k) - \gamma c_k] - \gamma(1+(n-2)\gamma)\sum_{i=1}^{n}(\alpha_i - c_i)}{(2+(2n-3)\gamma)(2+(n-3)\gamma)} \tag{12}$$

and

$$q_k^B = \frac{1+(n-2)\gamma}{(1-\gamma)(1+(n-1)\gamma)}(p_k^B - c_k) \tag{13}$$

[2] We refere to the dashed and dotted cuves in the lighter-gray region in the next section.

where

$$p_k^B - c_k = \frac{(2+(n-3)\gamma)(1+(n-1)\gamma)(\alpha_k - c_k) - \gamma(1+(n-2)\gamma)\sum_{i=1}^{n}(\alpha_i - c_i)}{(2+(2n-3)\gamma)(2+(n-3)\gamma)}. \tag{14}$$

Due to (13), the Bertrand profit of firm k can be simplified as

$$\pi_k^B = \frac{(1-\gamma)(1+(n-1)\gamma)}{1+(n-2)\gamma}(q_k^B)^2. \tag{15}$$

3 Comparison of the Optimal Behavior

In this section we will compare the optimal behavior under Cournot competition with that under Bertrand competition and examine whether the results obtained by Häckner (2000) and Hsu and Wang (2005) still hold under SOC_2.

3.1 Price comparison

We first seek the non-negativity conditions for the equilibrium outputs and then examine the price difference with non-negative outputs.

From (7), the Cournot output q_k^C is non-negative if

$$\frac{\gamma}{2+(n-1)\gamma}\left(z^C(\gamma) - \frac{\bar{\beta}}{\beta_k}\right) \geq 0 \tag{16}$$

where

$$z^C(\gamma) = \frac{2+(n-1)\gamma}{n\gamma}, \tag{17}$$

$$\beta_k = \alpha_k - c_k$$

and

$$\bar{\beta} = \frac{1}{n}\sum_{i=1}^{n}(\alpha_i - c_i).$$

Here β_k can be referred to as the net quality offered by firm k and $\bar{\beta}$ the average net quality offered by the n firms. Both are positive due to Assumption 1. We call firm k *higher-qualified* if $\beta_k > \bar{\beta}$ and *lower-qualified* if $\beta_k < \bar{\beta}$. When the goods are substitutes (i.e., $0 < \gamma < 1$), the first factor of (16) is positive. It is then clear that the Cournot output in this case is non-negative if

$$z^C(\gamma) \geq \frac{\bar{\beta}}{\beta_k}. \tag{18}$$

Notice that this inequality is always true if firm k is higher-qualified (that is, $\bar{\beta} < \beta_k$) since $z^C(\gamma) > 1$ for $0 < \gamma < 1$.

We turn to the case in which the goods are complements (i.e., $-1 < \gamma < 0$). (17) indicates that the sign of $z^C(\gamma)$ depends on the sign of $2 + (n-1)\gamma$. The $2 + (n-1)\gamma = 0$ curve corresponds to the dotted (lower) hyperbola in the lighter-gray region of Figure 1. It divides the lighter-gray region into two parts: $2 + (n-1)\gamma < 0$ above the curve and $2 + (n-1)\gamma > 0$ below. The non-negativity condition for the Cournot output differs according to from which region the parameter (γ, n) is chosen. If (γ, n) is such that $2 + (n-1)\gamma < 0$, then $z^C(\gamma)$ is positive and so is the first factor of (16). It follows that the Cournot output is non-negative if (18) holds. On the other hand, if (γ, n) is such that $2 + (n-1)\gamma > 0$, then $z^C(\gamma)$ is negative and so is the first factor of (16). Hence $q_k^C > 0$ always if $2 + (n-1)\gamma > 0$.

Given n, $z^C(\gamma)$ strictly decreases in γ and takes its maximum at $\gamma = -1$. If we define γ^C by the solution of equation

$$z^C(\gamma^C) = \frac{\bar{\beta}}{\beta_k}$$

and introduce the notation[3]

$$\gamma_2 = -\frac{2}{n-1} \text{ and } \gamma_3 = -\frac{1}{n-2},$$

then the non-negativity condition in the case of $-1 < \gamma < 0$ can be summarized as follows:

$$\text{(i) if } \frac{\bar{\beta}}{\beta_k} \geq z^C(-1), \text{ then } q_k^C \geq 0 \text{ for } \gamma_2 < \gamma < \gamma_3, \tag{19}$$

and

$$\text{(ii) otherwise, } q_k^C > 0 \text{ for } -1 < \gamma < \gamma^C \text{ or } \gamma_2 < \gamma < \gamma_3. \tag{20}$$

Note that $\gamma^C \leq -1$ under the inequality condition (19). In order to have $\gamma^C > -1$ in (20), we need to assume that $n > 3$.

Let us turn next our attention to the non-negativity conditions of q_k^B. From (13) and (14), q_k^B is non-negative if

$$\frac{\gamma(1+(n-2)\gamma)}{(2+(n-3)\gamma)(2+(2n-3)\gamma)} \left(z^B(\gamma) - \frac{\bar{\beta}}{\beta_k} \right) \geq 0 \tag{21}$$

where

$$z^B(\gamma) = \frac{(2+(n-3)\gamma)(1+(n-1)\gamma)}{n\gamma(1+(n-2)\gamma)}. \tag{22}$$

It is clear again that since the first factor of (21) is positive for $\gamma > 0$, the Bertrand output is non-negative if

$$z^B(\gamma) \geq \frac{\bar{\beta}}{\beta_k}. \tag{23}$$

Notice again that this inequality is always satisfied if firm k is higher-qualified (that is, $\bar{\beta} < \beta_k$) since $z^B(\gamma) > 1$ for $\gamma \in (0, 1)$.

Assume next that $\gamma < 0$. Although $1 + (n-1)\gamma < 0$ and $2 + (2n-3)\gamma < 0$ under Assumption 2, the sign of $2+(n-3)\gamma$ is ambiguous. The $2+(n-3)\gamma = 0$ curve corresponds to the dashed (upper) hyperbola in Figure 1 and divides the lighter-gray region into two parts: $2 + (n-3)\gamma < 0$ above the curve and $2 + (n-3)\gamma > 0$ below. As in the case of Cournot competition, the non-negativity conditions differ according to from which region the parameters (γ, n) are selected. If (γ, n) is such that $2 + (n-3)\gamma < 0$, then $z^B(\gamma)$ is positive and so is the first factor of (16). Therefore the Bertrand output is non-negative if

[3] γ_2 and γ_3 are the solutions of equations $2+(n-1)\gamma = 0$ and $1+(n-2)\gamma = 0$, respectively.

(23) holds. On the other hand, if (γ, n) is such that $2 + (n-3)\gamma > 0$, then $z^B(\gamma)$ is negative and so is the first factor of (16). Hence $q_k^B > 0$ always if $2 + (n-3)\gamma > 0$.

Given n, $z^B(\gamma)$ strictly decreases in γ and takes its maximum at $\gamma = -1$. If we define γ^B by the solution of equation

$$z^B(\gamma^B) = \frac{\bar{\beta}}{\beta_k}$$

and introduce the notation

$$\gamma_1 = -\frac{2}{n-3},$$

then the non-negativity condition of q_k^B in the case of $-1 < \gamma < 0$ can be summarized as follows:

$$\text{(i) if } \frac{\bar{\beta}}{\beta_k} \geq z^B(-1), \text{ then } q_k^B > 0 \text{ for } \gamma_1 < \gamma < \gamma_3 \tag{24}$$

and

$$\text{(ii) otherwise, } q_k^B > 0 \text{ for } -1 < \gamma < \gamma^B \text{ or } \gamma_1 < \gamma < \gamma_3. \tag{25}$$

Note again that $\gamma^B \leq -1$ under the inequality condition in (24). In order to have $\gamma^B > -1$ in (20), we need to assume that $n > 5$.[4]

Having the non-negativity conditions for the equilibrium outputs, we can now examine the optimal price difference. Since $q_k^C \geq 0$ and $q_k^B \geq 0$ imply $p_k^C \geq 0$ and $p_k^B \geq 0$, respectively, the non-negative conditions for the outputs guarantee the non-negativity of the equilibrium prices. Assuming $n > 2$ and subtracting (12) from (8) yield the following relation:

$$p_k^C - p_k^B = \frac{n(n-1)(n-2)\gamma^4\beta_k}{(2-\gamma)(2+(2n-3)\gamma)} \frac{(z_k^*(\gamma) - \bar{\beta}/\beta_k)}{(2+(n-1)\gamma)(2+(n-3)\gamma)} \tag{26}$$

where

$$z_k^*(\gamma) = \frac{(2+(n-1)\gamma)(2+(n-3)\gamma)}{n(n-2)\gamma^2}. \tag{27}$$

In the case of $\gamma > 0$, it is easy to see that

$$sign\left[p_k^C - p_k^B\right] = sign\left[z_k^*(\gamma) - \frac{\bar{\beta}}{\beta_k}\right]. \tag{28}$$

Relations (17), (22) and (27) imply that

$$z^B(\gamma) < z^C(\gamma) < z_k^*(\gamma) \text{ for } 0 < \gamma < 1.$$

(18) and (23) imply that the non-negativity condition of the equilibrium outputs as well as that of the optimal prices is

$$\min[z^B(\gamma), z^C(\gamma)] \geq \frac{\bar{\beta}}{\beta_k}.$$

[4] $z^B(\gamma^B) = \bar{\beta}/\beta_k$ has two solutions. The smaller root does not statisfy SOC_2 and therefore it is eliminated from consideration.

Therefore, under the non-negative Cournot and Bertrand outputs, we have

$$z_k^*(\gamma) > \frac{\bar{\beta}}{\beta_k}. \tag{29}$$

With this inequality, (28) indicates that the Cournot price is always higher than the Bertrand price when the goods are substitutes (that is, $0 < \gamma < 1$) and the optimal outputs are non-negative. This result is the same as the one shown in Proposition (i) of Häckner (2000).

In the case of $\gamma < 0$, we should notice that the first factor of (26) is negative as $(2 + (2n - 3)\gamma) < 0$ under Assumption 2. We also should look at the division of the lighter-gray region in Figure 1 in more detail. The region is divided into three parts: the upper region above the $2 + (n - 3)\gamma = 0$ curve, the lower region below the $2 + (n - 1)\gamma = 0$ curve and the middle region between these two curves. Furthermore,

$$1 > z_k^*(\gamma_3) > z^C(-1) > z^B(-1) > z_k^*(-1).$$

Depending on the value of $\bar{\beta}/\beta_k$, the price difference is determined by different conditions. First, suppose that

$$\frac{\bar{\beta}}{\beta_k} > z_k^*(\gamma_3). \tag{30}$$

Under this condition, (19) and (24) imply that the optimal outputs are non-negative in the lower region

$$q_k^B \geq 0 \text{ and } q_k^C \geq 0 \text{ for } \gamma_2 < \gamma < \gamma_3.$$

Furthermore in the lower region, we have $(2 + (n - 3)\gamma)(2 + (n - 1)\gamma) > 0$ and

$$\frac{\bar{\beta}}{\beta_k} > z_k^*(\gamma) \text{ for } -1 < \gamma < \gamma_3.$$

Hence (26) implies that the Cournot price is higher than the Bertrand price in the lower region,

$$p_k^C > p_k^B \text{ for } \gamma_2 < \gamma < \gamma_3. \tag{31}$$

See Figure 2A in which the lower region is shaded in darker-gray and $\bar{\beta}/\beta_k$ is taken to be 2. Since $z_k^*(\gamma_3) < 1$, (30) is fulfilled when firm k is lower-qualified (that is, $\beta_k < \bar{\beta}$). Hence (31) implies that lower-qualified firms charge higher prices under Cournot competition than under Bertrand competition. On the other hand, Häckner (2000, Proposition 1(ii)) shows that lower-qualified firms charge higher prices under Bertrand competition than under Cournot competition, which apparently differs from our result just obtained.

Next suppose that

$$z_k^*(-1) > \frac{\bar{\beta}}{\beta_k}. \tag{32}$$

It can be easily verified that

$$-1 < \gamma_S^* < \gamma^B < \gamma_1 < \gamma^C < \gamma_2 < \gamma_L^* < \gamma_3 \tag{33}$$

where γ_S^* and γ_L^* are the smaller and larger roots of $z_k^*(\gamma) = \bar{\beta}/\beta_k$. Relations (20) and (25) imply the following non-negativity conditions for the optimal outputs,

$$q_k^C \geq 0 \text{ and } q_k^B \geq 0 \text{ for } -1 < \gamma < \gamma^B \text{ or } \gamma_2 < \gamma < \gamma_3. \tag{34}$$

Since $(2+(n-3)\gamma)(2+(n-1)\gamma) > 0$ in both of the upper and lower regions, we have

$$sign\left[p_k^C - p_k^B\right] = sign\left[\frac{\bar{\beta}}{\beta_k} - z_k^*(\gamma)\right].$$

With (33) and (34), (**??**) implies that

$$p_k^C > p_k^B \text{ for } \gamma_S^* < \gamma < \gamma^B \text{ or } \gamma_2 < \gamma < \gamma_L^* \text{ as } \frac{\bar{\beta}}{\beta_k} > z_k^*(\gamma) \tag{35}$$

and

$$p_k^C < p_k^B \text{ for } -1 < \gamma < \gamma_S^* \text{ or } \gamma_L^* < \gamma < \gamma_3 \text{ as } \frac{\bar{\beta}}{\beta_k} < z_k^*(\gamma). \tag{36}$$

See Figure 2B in which $\bar{\beta}/\beta_k$ is taken to be 1/2. Notice that (35) holds in the lighter-gray region and (36) holds in the darker-gray region. Either q_k^B or q_k^C or both are negative in the white region between the curves denoted by r^B and γ_2. Since $z^C(-1) \leq 1$, (32) is fulfilled when firm k is higher-qualified (that is, $\beta_k > \bar{\beta}$). (35) implies that a higher-qualified firm charges higher price under Cournot competition than under Bertrand competition if the complementability are medium in the sense that $\gamma_S^* < \gamma < \gamma^B$ or $\gamma_2 < \gamma < \gamma_L^*$. On the other hand, (36) implies that a higher-qualified firm changes its price strategy and charges a higher price under Bertrand competition than under Cournot competition if the complementability is extremely large or small in the sense that $-1 < \gamma < \gamma_S^*$ or $\gamma_L^* < \gamma < \gamma_3$. The last two results are not discussed in Häckner (2000) since this kind of division does not appear under condition SOC_1. Depending on the actual value of $\bar{\beta}/\beta_k$, we can identify several other cases. The result that Bertrand price can be higher than Cournot price depending on the complementability of the goods is also obtained in those cases.[5]

Insert Figure 2 about here.

We can summarize our results as follows:

Proposition 1 *Assume that $n > 2$ and $\gamma > 0$ or $1 + (n-2)\gamma < 0$ if $\gamma < 0$. (i) when the goods are substitutes, prices are higher under Cournot competition than under Bertrand competition. (ii) when the goods are complements, lower-qualified firms charge higher prices under Cournot competition than under Bertrand competition. (iii) when the goods are complement and their complementability is medium, higher-qualified firms charge higher prices under Cournot competition than under Bertrand competition. (iv) when the goods are complement and their complementability is extremely large or small, higher-qualified firms may charge higher prices under Bertrand competition than under Cournot competition.*

[5] In particular, we have three more cases: $z_k^*(\gamma_3) > \bar{\beta}/\beta_k > z^C(-1)$, $z^C(-1) > \bar{\beta}/\beta_k > z^B(-1)$ and $z^B(-1) > \bar{\beta}/\beta_k > z_k^*(-1)$. In the same way as demonstrated above, we can reveal the conditions under which the Bertrand price can be higher than the Cournot price.

Notice the sharp difference between the result of Häckner (2000) that lower-qualified firms charge higher prices under Bertrand competition and our result that higher-qualified firms charge higher price. This is not a contradiction, since these results were obtained under different conditions (SOC_1 and SOC_2).

3.2 Profit comparison

Following the method of Häckner (2000), we compare Cournot and Bertrand profits to see which strategy makes more profits. Denote the ratio of the Cournot profit over the Bertrand profit of firm k by $G(z_k) = \pi_k^C/\pi_k^B$ where

$$z_k = \frac{\bar{\beta}}{\beta_k}.$$

From (7), (9), (13) and (15), we have

$$G(z_k) = K\left(\frac{z^C - z_k}{z^B - z_k}\right)^2$$

with

$$K = \frac{1+(n-2)\gamma}{(1-\gamma)(1+(n-1)\gamma)}\left(\frac{(1-\gamma)(1+(n-1)\gamma)(2+(n-3)\gamma)(2+(2n-3)\gamma)}{(2-\gamma)(2+(n-1)\gamma)(1+(n-2))^2}\right)^2$$

It can be easily verified that $K > 0$,

$$G'(z_k) = 2K\frac{(z_k - z^C)(z^C - z^B)}{(z_k - z^B)^3} \tag{37}$$

and

$$G(0) = \frac{(1-\gamma)(2+(2n-3)\gamma)^2}{(2-\gamma)^2(1+(n-2)\gamma)(1+(n-1)\gamma)}.$$

The difference of the denominator and the numerator of $G(0)$ reveals that

$$G(0) \lesseqgtr 1 \text{ according to } 2+(n-2)\gamma \gtreqless 0.$$

In the case of $\gamma > 0$, the domain of $G(z_k)$ is $(0, z^B(\gamma))$ as $z_k \le \min[z^B(\gamma), z^C(\gamma)]$ for nonnegative outputs and $z^B < z^C$. $G'(z_k) > 0$ over this domain and $G(z_k)$ approaches infinity as z_k approaches z^B from below. Furthermore, $2+(n-2)\gamma > 0$ if $\gamma > 0$, which implies $G(0) < 1$. Let us define the threshold value $\bar{z}_k$ as $G(\bar{z}_k) = 1$. Hence we have the following:

$$\pi_k^C < \pi_k^B \text{ for } 0 < z_k < \bar{z}_k, \tag{38}$$

and

$$\pi_k^C > \pi_k^B \text{ for } \bar{z}_k < z_k < z^B(\gamma). \tag{39}$$

This result is qualitatively the same as Häckner's result (see his Proposition 2(ii)) if the condition $0 < z_k < \bar{z}_k$ is interpreted as a large quality difference and $\bar{z}_k < z_k < z^B(\gamma)$ as a small difference. However there is a difference. In his result, higher-qualified firms earn higher profits under Bertrand competition. On the other hand, in our results in which $z^B(\gamma) > 1$ for $\gamma > 0$, a firm having the value z_k closer to $z^B(\gamma)$ in (39) is lower-qualified but earns higher profit under Cournot competition. Furthermore, a firm having the value z_k closer to zero in (38) is higher-qualified but earns higher profit under Bertrand competition.

In the case of $\gamma < 0$, $G(z_k)$ approaches infinity as z_k approaches z^B. Relation (37) implies that

$$G'(z_k) > 0 \text{ for } z_k < z^B(\gamma) \text{ or } z_k > z^C(\gamma)$$

and

$$G'(z_k) < 0 \text{ for } z^B(\gamma) < z_k < z^C(\gamma).$$

Solving $G(z_k) = 1$ for z yields two roots, a smaller root z_k^S and a larger root z_k^L, which then implies $G(z_k) < 1$ for $z_k > z_k^L$. Furthermore the $2 + (n-2)\gamma = 0$ curve divides the SOC$_2$ region into two parts: $2 + (n-2)\gamma < 0$ above the curve implies $G(0) > 1$, and $2 + (n-2)\gamma > 0$ below implies $G(0) < 1$.[6] Hence we have the following:

$$\text{If } 2 + (n-2)\gamma > 0, \text{ then } \begin{cases} \pi_k^C > \pi_k^B \text{ for } 0 < z_k < z_k^L \ (z_k \neq z^B), \\ \pi_k^B > \pi_k^C \text{ for } z_k > z_k^L. \end{cases} \tag{40}$$

and

$$\text{if } 2 + (n-2)\gamma < 0, \text{ then } \begin{cases} \pi_k^C > \pi_k^B \text{ for } z_k^S < z_k < z_k^L \ (z_k \neq z^B), \\ \pi_k^B > \pi_k^C \text{ for } 0 < z_k < z_k^S \text{ or } z_k^L < z_k. \end{cases} \tag{41}$$

These results in the case of complementary goods are different than Hacker's result (see his Proposition 2(i)). The source of the difference can be found in the difference in the assumptions on the second order condition. If $1 + (n-1)\gamma > 0$ as assumed in Häckner (2000), then $z^B - z^C > 0$ that makes $G(z_k)$ decreasing in z_k. Since $G(0) < 1$, the profit ratio is always less than unity as shown by Häckner (2000). Our results are summarized as follows:

Proposition 2 *(i) When the goods are complements and $2+(n-2)\gamma > 0$, then Cournot profits are higher than Bertrand profits for $0 < z_k < z_k^L$ and lower for $z_k > z_k^L$; (ii) when the goods are complements and $2+(n-2)\gamma < 0$, then Cournot profits are lower than Bertrand profits for $0 < z_k < z_k^S$ or $z_k > z_k^L$ and higher for $z_k^S < z_k < z_k^L$ where the threshold values, z_k^S and z_k^L, are the smaller root and the larger root of $G(z_k) = 1$. (iii) when the goods are substitutes, Cournot profits are lower than Bertrand profits for $0 < z_k < \bar{z}_k$ and higher for $\bar{z}_k < z_k < z^B$ where the threshold values $\bar{z}_k$ is defined by $G(\bar{z}_k) = 1$.*

3.3 Welfare comparison

Let us denote consumer surplus and total surplus by CS and TS. In the case of $\gamma > 0$ in which $1 - \gamma > 0$ and $1 + (n-1)\gamma > 0$, we can directly apply the formula developed by Hsu and Wang (2005) to obtain

$$CS^C < CS^B \text{ and } TS^C < TS^B$$

[6] $n > 4$ is necessary to make this distinction.

where "C" or "B" indicates that the corresponding surplus is evaluated at Cournot equilibrium or at Bertrand equilibrium. Their result, welfare is higher under Bertrand competition than under Cournot competition, still holds in our framework when the goods are substitutes. Although we have $1+(n-1)\gamma<0$ under Assumption 2 in the case of $\gamma<0$, their formula is still useful to find condition under which the opposite result (that is, welfare can be higher under Cournot competition than under Bertrand competition) can be produced.

The formula given by Hsu and Wang (2005) is as follows:

$$\begin{aligned} & sign[CS^C - CS^B] \\ & = sign\left[\tfrac{(3n-5)\gamma^2-2(2n-5)\gamma-4}{(1-\gamma)A^2} - \tfrac{(2n^2-7n+5)\gamma^2+2(3n-5)\gamma+4}{(1+(n-1)\gamma)B^2}\right] \end{aligned} \tag{42}$$

with

$$\sigma_\beta^2 = \frac{\sum_{i=1}^n(\beta_i-\bar\beta)}{n},$$

$$A = \frac{(2-\gamma)(2+(2n-3)\gamma)}{\sigma_\beta}$$

and

$$B = \frac{(2+(n-1)\gamma)(2+(n-3)\gamma)}{\bar\beta}.$$

The sign of the consumer surplus difference depends on the signs of the numerators in the brackets of the right hand side of (42). To simplify the notation, we denote the numerators by

$$f(\gamma) = (3n-5)\gamma^2-2(2n-5)\gamma-4 \text{ and } g(\gamma) = (2n^2-7n+5)\gamma^2+2(3n-5)\gamma+4.$$

We can focus on the welfare obtained in the upper-left region of Figure 1 in which $2+(n-3)\gamma<0$. Then $f(\gamma)>0$ and $g(\gamma)>0$ for $\gamma\in[-1,\gamma_1]$. With $1+(n-1)\gamma<0$, (42) implies that

$$CS^C > CS^B.$$

Total surplus is the sum of consumer surplus and firms profits,

$$TS^C = CS^C + \sum_{i=1}^n \pi_i^C \text{ and } TS^B = CS^B + \sum_{i=1}^n \pi_i^B.$$

It has been shown in Proposition2(ii) that π_k^C can be higher than π_k^B for all k under $2+(n-3)\gamma<0$ if firm k is higher-qualified in the sense that $z_k\in(z_k^S,z_k^L)$. Hence we have

$$TS^C > TS^B.$$

Proposition 3 *(i) When the goods are substitutes, consumer surplus and total surplus are higher under Bertrand competition than under Cournot competition. (ii) when the goods are complements and* $2+(n-3)\gamma<0$*, consumer surplus and total surplus can be higher under Cournot competition than under Bertrand competition if firms are higher-qualified..*

Following the discussion of Hsu and Wang (2005), we can offer an intuitive reasoning for this result. Output shares of firm k under Bertrand and Cournot competitions are

$$s_k^B = \frac{q_k^B}{\sum_{i=1}^n q_i^B} = \frac{1}{n} + \delta^B \frac{\beta_k - \bar{\beta}}{\bar{\beta}}$$

and

$$s_k^C = \frac{q_k^C}{\sum_{i=1}^n q_i^C} = \frac{1}{n} + \delta^C \frac{\beta_k - \bar{\beta}}{\bar{\beta}},$$

furthermore their difference is

$$s_k^B - s_k^C = \delta^{CB}$$

where quantities δ^B, δ^C and δ^{CB} are defined as

$$\delta^B = \frac{(1+(n-2)\gamma)(2+(n-3)\gamma)}{n(1-\gamma)(2+(2n-3)\gamma)},\ \delta^C = \frac{2+(n-1)\gamma}{n(2-\gamma)} \text{ and } \delta^{CB} = \frac{(n-1)\gamma^3}{(1-\gamma)(2-\gamma)(2+(2n-3)\gamma)}.$$

Notice that δ^B and δ^C are negative and δ^{CB} is positive if $2+(n-3)\gamma < 0$. In either equilibrium, the output shares are ranked with quality as the lowest-quality firm selling the most and the highest-quality firm selling the least amount. In our framework, lower-quality firms have significant effect on the consumer welfare. As shown in Proposition 1(ii), the low-quantity firms charge lower prices under Bertrand competition than under Cournot competition. Therefore their changing strategies from quantity-adjusting to price-adjusting have the significant enough effect to make welfare larger under Cournot competition than under Bertrand competition if condition SOC_2 holds.

4 Concluding Remarks

We construct an n-firm oligopoly model and compare its equilibrium prices, profits and welfare under Cournot and Bertrand competitions. The main feature of this study is to assume $1 + (n-2)\gamma < 0$ (that is, SOC_2) to fulfill the second-order condition for the profit maximization problems of the price-adjusting firms, while Häckner (2000) and Hsu and Wang (2005) have a different assumption, $1+(n-1)\gamma > 0$ (that is, SOC_1). This difference in the assumptions is a source of the sharp differences between our results and their results. Concerning price comparisons, it follows from Proposition 1(iv) that when the goods are complements, higher-qualified firms may charge higher Bertrand price than Cournot price under SOC_2. On the contrary, Häckner (2000) shows that lower-qualified firms may have higher Bertrand price. Concerning profit comparisons Proposition 2 reveals that it depends on the value of the ratio of the average net quality to the individual net quality (i.e., $\bar{\beta}/\beta_k$) whether Cournot profits can be higher than Bertrand profits or not. On the other hand, Häckner (2000) shows that higher-qualified firms may earn higher Bertrand profits. And finally, as social welfare is concerned, Proposition 3(ii) indicates that higher-qualified firms behavior leads to higher consumer surplus and higher profits under Cournot competition while Hsu and Wang (2005) demonstrate that both of consumer surplus and total surplus are higher under Bertrand competition.

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