Discussion Paper No.219

Mathematical Modeling of Financial Instability and Macroeconomic Stabilization Policies

Toichiro Asada
Faculty of Economics, Chuo University

March 2014

INSTITUTE OF ECONOMIC RESEARCH
Chuo University
Tokyo, Japan

Mathematical Modeling of Financial Instability and Macroeconomic Stabilization Policies

Toichiro Asada
Faculty of Economics, Chuo University
742-1 Higashinakano, Hachioji, Tokyo 192-0393, Japan
E-mail : asada@tamacc.chuo-u.ac.jp

March 2014

Abstract.

In this paper, we formulate a series of mathematical macrodynamic models that contribute to the theoretical analysis of financial instability and macroeconomic stabilization policies. Two-dimensional model of fixed prices without active macroeconomic stabilization policy, four-dimensional model of flexible prices with central bank's monetary stabilization policy, and six-dimensional model of flexible prices with monetary and fiscal policy mix are considered in order. In the final section, we provide an intuitive economic interpretation of the analytical results.

Key words : Financial instability, Minsky cycle, monetary stabilization policy, monetary and fiscal policy mix, inflation targeting, credibility

JEL classification : E12, E31, E32, E52, E62

1. Introduction

Minsky's(1975, 1982, 1986) 'financial instability hypothesis' implies that the financially dominated capitalist economy is inherently unstable. His hypothesis has been neglected for a long time by the mainstream economists, although it had a considerable influence among the heterodox economists such as Post Keynesians.[1] But, the situation dramatically changed soon after the worldwide financial crisis that was initiated by the so called 'subprime mortgage crisis' in 2007 in the United States. Since then, Minsky's hypothesis was 'rediscovered' by some mainstream economists such as Krugman.[2]

Minsky distinguishes three forms of investment financing, that is, 'hedge finance', 'speculative finance', and 'Ponzi finance'. He defines these three financing forms as follows.

"If realized and expected income cash flows are sufficient to meet the payment commitments on the outstading liabilities of a unit, then the unit will be hedge financing. However, the balance-sheet cash flows from a unit can be larger than the expected income receipt so that only way they can be met is by rolling over or even increasing debt; units that roll over debt are engaged in speculative finance and those that increase debt are engaged in Ponzi finance." (Minsky 1986, p. 203)

Minsky provides a description of the business cycle of the financially dominated capitalist economy that is based on the endogenous changes of these three financing forms, that is, Hedge finance→Speculative finance→Ponzi finance→Hedge finance and so on, which is called the 'Minsky cycle'.

By the way, it is important to note that Minsky did *not* think that such an inherent instability of the financially dominated capitalist economy is uncontrollable by the government and the central bank. In fact, he stressed that it is important to 'stabilize an unstable economy' by means of the proper macroeconomic stabilization policies by the government and the central bank.[3]

In this paper, we consider how to stabilize an unstable economy theoretically by using

---

[1] For the Post Keynesian-oriented theoretical literature on Minsky's financial instability hypothesis, see, for example, Asada(2001, 2004, 2012), Asada, Chiarella, Flaschel, Mouakil, Proaño and Semmler(2010), Keen(2000), Nasica(2000), Pally(1996), and Semmler(ed.)(1989).

[2] See, for example, Eggertsoon and Krugman(2012) and Krugman(2012).

[3] See Minsky(1986) and Asada, Chiarella, Flaschel, Mouakil, Proaño and Semmler(2010).

the analytical framework of 'high dimensional nonlinear Keynesian macrodynamic model' that was developed by Asada, Chiarella, Flaschel and Franke(2003, 2010) and Chiarella, Flaschel and Franke(2005).[4] In section 2, we formulate the basic Minskian two-dimensional fixed price model of financial instability without active macroeconomic stabilization policy. In section 3, we consider an extended flexible price four-dimensional model with central bank's monetary stabilization policy. In section 4, we study a further extended flexible price six-dimensional model of the macroeconomic stabilization policy by means of monetary and fiscal policy mix. Finally, in section 5, we provide an intuitive economic explanation of the analytical results. Some complicated mathematical proofs are relegated to the appendices.

2. Basic Model：Two-dimensional Model with Fixed Prices

The basic model that is the starting point of our analysis consists of the following system of equations.[5]

$$\dot{d} = \phi(g) - s_f(r - id) - (g + \pi)d \quad ; 0 < s_f < 1 \tag{1}$$

$$\dot{y} = \alpha(c + \phi(g) + v - y) \quad ; \quad \alpha > 0 \tag{2}$$

$$g = g(r, \rho - \pi^e, d) \quad ; \quad g_r = \partial g / \partial r > 0, \ g_{\rho - \pi^e} = \partial g / \partial(r - \pi^e) < 0,$$
$$g_d = \partial g / \partial d < 0 \tag{3}$$

$$c = (1 - s_1)\{y - r + (1 - s_f)r - \tau(y)\} + (1 - s_2)id + (1 - s_3)\rho b$$
$$; 0 < \tau_y = \tau'(y) < 1, \ 0 < s_1 < 1, \ 0 < s_2 \leqq 1, \ 0 < s_3 \leqq 1 \tag{4}$$

$$r = P / K = \beta Y / K = \beta y \quad ; 0 < \beta < 1 \tag{5}$$

$$i = \rho + \zeta(d) = i(\rho, d) \quad ; \quad \zeta(d) \geqq 0, \ i_d = \zeta'(d) > 0 \text{ for } d > 0 \tag{6}$$

$$m = l(y, \rho) \quad ; \quad l_y = \partial l / \partial y > 0, \ l_\rho = \partial l / \partial \rho < 0 \tag{7}$$

$$\pi = \pi^e = 0 \tag{8}$$

$$\rho = \text{constant} > 0 \tag{9}$$

$$v = \text{constant} > 0 \tag{10}$$

The meanings of the symbols are as follows. $D =$ stock of firms' nominal private debt. $p =$ price level. $K =$ real capital stock. $d = D/(pK) =$ private debt-capital ratio.

---

[4] The 'high dimensional' dynamic model means the dynamic model with many(at least three) endogenous variables.

[5] This model is essentially based on Asada's (2001) formulation.

$\pi = \dot{p}/p =$ rate of price inflation. $\pi^e =$ expected rate of price inflation. $g = \dot{K}/K =$ rate of capital accumulation. $\phi(g) =$ adjustment cost function of investment that has the properties $\phi'(g) \geqq 1$, $\phi''(g) > 0$, which was introduced by Uzawa(1969). $P =$ real profit. $r = P/K =$ rate of profit. $i =$ nominal rate of interest that is applied to firms' private debt. $\rho =$ nominal rate of interest of the government bond. $\rho - \pi^e =$ expected real rate of interest of the government bond. $s_f =$ firms' internal retention rate that is assumed to be constant. $Y =$ real output (real national income). $y = Y/K =$ output-capital ratio, which is a surrogate variable of the 'rate of capital utilization' and the 'rate of labor employment'. $G =$ real government expenditure. $v = G/K =$ government expenditure-capital ratio. $B =$ stock of nominal government bond. $b = B/(pK) =$ government bond-capital ratio. $\alpha =$ quantity adjustment speed of the disequilibrium in the goods market. $C =$ real private consumption expenditure. $c = C/K =$ private consumption expenditure-capital ratio. $T =$ real tax. $\tau = T/K =$ tax-capital ratio. $s_1 =$ average saving rate out of wage and profit income after tax that is assumed to be constant. $s_2 =$ average saving rate out of interest on private debt that is assumed to be constant. $s_3 =$ average propensity to save out of interest on public debt that is assumed to be constant. $\beta = P/Y =$ share of profit in national income that is assumed to be constant $(0 < \beta < 1)$. $M =$ nominal money stock. $m = M/(pK) =$ money stock-capital ratio. $L =$ real money demand. $l = L/K =$ money demand-capital ratio.

We can derive Eq. (1) as follows. The dynamic law of motion of the firms' private debt can be expressed by

$$\dot{D} = \phi(g)pK - s_f(rpK - iD). \tag{11}$$

On the other hand, by differentiating the definitional equation $d = D/(pK)$ by time, we have

$$\dot{d}/d = \dot{D}/D - \dot{p}/p - \dot{K}/K = \dot{D}/D - \pi - g. \tag{12}$$

Substituting Eq. (12) into Eq. (11), we obtain Eq. (1).

Eq. (2) describes the Keynesian quantity adjustment process of the disequilibrium in the goods market, which is called the dynamic multiplier process.[6]

Eq. (3) is the Keynesian type investment function that includes the Fisher's(1933)

[6] $E = \phi(g)K$ is the real investment expenditure including the adjustment cost, so that $E/K = \phi(g)$ is the real investment expenditure including the adjustment cost per capital stock. In this formulation, international trade is neglected for simplicity.

debt effect.[7]

Eq. (4) is the standard Keynesian type consumption function. In fact, it is assumed that

$$C = (1-s_1)\{W + (1-s_f)P - T\} + (1-s_2)iD + (1-s_3)\rho B, \tag{13}$$

$$Y = W + P \tag{14}$$

where W is the pre tax real wage income and P is the pre tax real profit. From these equations we have Eq. (4).[8]

Eq. (5) simply says that the share of profit in national income $\beta = P/Y$ is fixed, which is supposed to be determined by the 'degree of monopoly' in the sense of Kalecki(1971).

Eq. (6) captures the fact that i, the interest rate of the 'risky assets', will be higher than ρ, the interest rate of the 'safer asset', and the difference between them will reflect the degree of risk.

Eq. (7) is the equilibrium condition for the money market. The function $l(y,\rho)$ is the standard Keynesian real money demand function due to Keynes(1936).

Equations (8) – (10) imply that the price level is fixed and both of monetary and fiscal policies are inactive. These assumptions will be relaxed step by step in the subsequent sections.

We can rewrite the system of equations (1) – (10) as follows.

$$\dot{d} = \phi(g(\beta y,\rho,d)) - s_f\{\beta y - i(\rho,d)d\} - g(\beta y,\rho,d)d = f_1(d,y) \tag{15.1}$$

$$\dot{y} = \alpha[(1-s_1)\{(1-s_f\beta)y - \tau(y)\} + (1-s_2)i(\rho,d)d + (1-s_3)\rho b$$

$$+\phi(g(\beta y,\rho,d)) + v - y] = \alpha f_2(d,y,b) \tag{15.2}$$

$$m = l(y,\rho) \tag{15.3}$$

In this section, we assume that

---

[7] Asada(2001) derived this type of investment function from the firms' profit maximization behaviors by using both Kalecki's(1937) hypothesis of increasing risk of investment and Uzawa's(1969) hypothesis of increasing adjustment cost of investment, which is called 'Penrose effect'.

[8] In this formulation, it is assumed that the household is the creditor to both of firms and the government. Furthermore, it is assumed for simplicity that $\tau = T/K$ is independent of id and ρb but it solely depends on y. Incidentally, a possible formulation of the consumption function is

$$C = (1-s_1)\{W + (1-s_f)(P - iD) + iD - T\} + (1-s_3)\rho B.$$

In this particular case, we have $s_2 = 1-(1-s_1)s_f$ in Eq. (13).

$$s_3 = 1 \tag{16}$$

for simplicity of the analysis. In this case, equations (15.1) and (15.2) consist of the two-dimensional subsystem of dynamic equations with respect to d and y that is independent of Eq. (15.3). In such a case, Eq. (15.3) has the only role to determine the endogenous movement of the variable m, which does not feedback to other subsystem. In other words, this is a decomposable system.

We *assume* that this system has an equilibrium solution $(d^*, y^*) > (0,0)$ such that $\dot{d} = \dot{y} = 0$. The Jacobian matrix of this system *at the equilibrium point* becomes as follows.

$$J_1 = \begin{bmatrix} f_{11} & f_{12} \\ \alpha f_{21} & \alpha f_{22} \end{bmatrix} \tag{17}$$

where

$$f_{11} = \partial f_1 / \partial d = \{\underset{(+)}{\phi'(g)} - d\} \underset{(-)}{g_d} - g + s_f (\underset{(+)}{i_d} d + i), \tag{18}$$

$$f_{12} = \partial f_1 / \partial y = \beta\{(\underset{(+)}{\phi'(g)} - d) \underset{(+)}{g_r} - s_f\}, \tag{19}$$

$$f_{21} = \partial f_2 / \partial d = (1 - s_2)(\underset{(+)}{i_d} d + i) + \underset{(+)}{\phi'(g)} \underset{(-)}{g_d}, \tag{20}$$

$$f_{22} = \partial f_2 / \partial y = (1 - s_1)(1 - s_f \beta - \tau_y) + \beta \underset{(+)}{\phi'(g)} \underset{(+)}{g_r} - 1, \tag{21}$$

$$\begin{aligned} f_{11} f_{22} - f_{12} f_{21} = & -\underset{(+)}{\phi'(g)} \underset{(-)}{g_d} \{(1 - s_1) \underset{(+)}{\tau_y} + s_1 (1 - s_f \beta)\} + (1 - s_2)(\underset{(+)}{i_d} d + i)(\beta s_f - d \underset{(+)}{g_r}) \\ & + \beta \underset{(+)}{\phi'(g)} \underset{(+)}{g_r} \{-g + (s_f + s_2 - 1)(\underset{(+)}{i_d} d + i)\} \\ & + \{-g + s_f (\underset{(+)}{i_d} d + i)\} \{(1 - s_1)(1 - s_f \beta - \underset{(+)}{\tau_y}) - 1\}. \end{aligned} \tag{22}$$

Now, let us assume as follows.

Assumption 1.

$f_{11} < 0,\ f_{12} > 0,\ f_{21} < 0,\ f_{22} > 0,\ f_{11} f_{22} - f_{12} f_{21} > 0.$

These inequalities will in fact be satisfied if $\phi'(g)$ and $|g_d|$ are sufficiently large *at* *the equilibrium point*. Under **Assumption 1**, we have the following proposition.[9]

Proposition 1.

There exists a parameter value $\alpha_0 > 0$ that satisfies the following properties (1) – (3).

(1) The equilibrium point of the dynamic system (15) is *locally stable* for all $\alpha \in (0, \alpha_0)$.

(2) The equilibrium point of the dynamic system (15) is *locally totally unstable* for all $\alpha \in (\alpha_0, +\infty)$.

(3) There exist the *non-constant closed orbits* around the equilibrium point for some range of the parameter value α that is sufficiently close to α_0.

(Proof.)

The characteristic root of this system *at the equilibrium point* becomes

$$\Delta_1(\lambda) \equiv |\lambda I - J_1| = \lambda^2 + a_1\lambda + a_2 = 0, \tag{23}$$

where

$$a_1 = -(\lambda_1 + \lambda_2) = -traceJ_1 = -\underset{(-)}{f_{11}} - \alpha \underset{(+)}{f_{22}}, \tag{24}$$

$$a_2 = \lambda_1\lambda_2 = \det J_1 = \alpha(f_{11}f_{22} - f_{12}f_{21}) > 0, \tag{25}$$

and $\lambda_j (j = 1,2)$ are the characteristic roots of Eq. (23). Let us define

$$\alpha_0 = -\underset{(-)}{f_{11}} / \underset{(+)}{f_{22}} > 0. \tag{26}$$

(1) Suppose that $\alpha \in (0, \alpha_0)$. Then, it follows from equations (24) and (25) that we have $\lambda_1 + \lambda_2 < 0$ and $\lambda_1\lambda_2 > 0$. This means that the characteristic equation (23) has two roots with negative real parts, so that the equilibrium point becomes locally stable.

[9] In the models in this paper, the 'jump variables' are not allowed for unlike the mainstream 'New Keynesian' dynamic models that are represented by Woodford(2003), Galí(2009) and others, but it is assumed that all initial conditions of the endogenous variables are historically given. This means that we adopt the traditional notion of the local stability/instability that is popular in the 'Old Keynesian' dynamic models represented by Tobin(1994) as well as the 'Post Keynesian' models. That is to say, (1) the equilibrium point is considered to be *locally stable* if all characteristic roots have negative real parts, and (2) it is considered to be *locally unstable* if at least one characteristic root has positive real part, and (3) it is considered to be *locally totally unstable* if all characteristic roots have positive real parts. As for the critical assessment of 'New Keynesian' dynamic models, see, for example, Asada(2013), Asada, Chen, Chiarella and Flaschel(2006), Asada, Chiarella, Flaschel and Franke(2010), Chiarella, Flaschel and Semmler(2013), Flaschel, Franke and Proaño(2008) and Mankiw(2001).

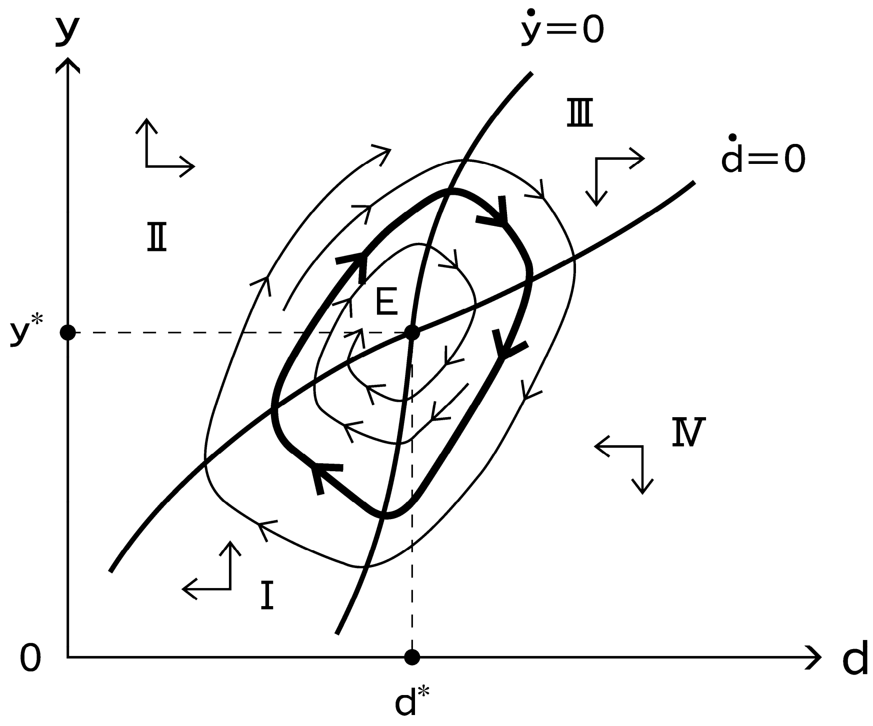

Figure 1. Closed orbit through 'subcritical' Hopf Bifurcation
($\alpha < \alpha_0$)

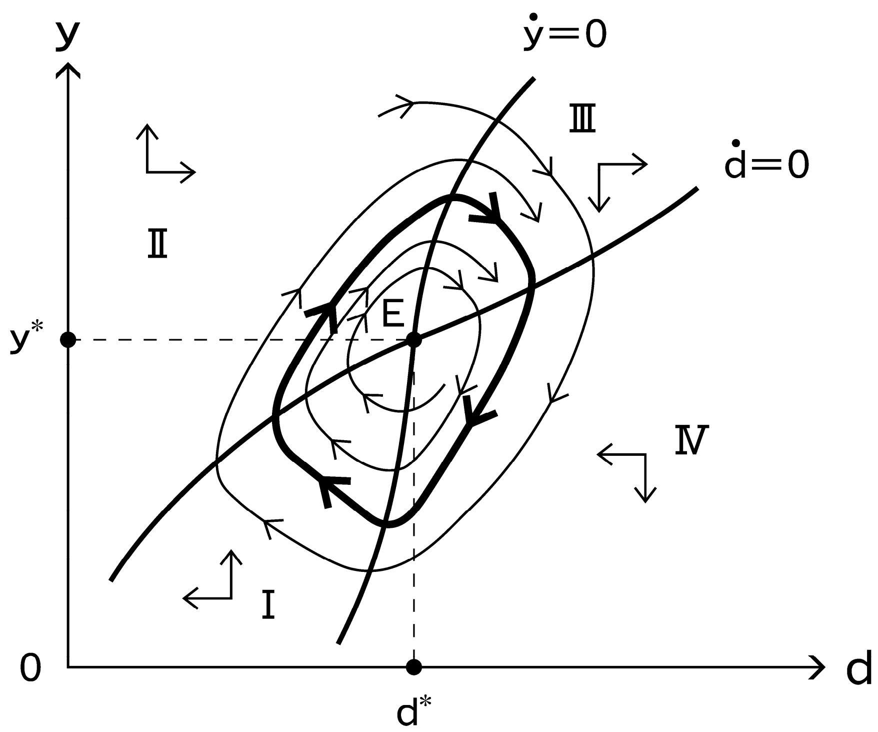

Figure 2. Closed orbit through 'supercritical' Hopf Bifurcation
($\alpha > \alpha_0$)

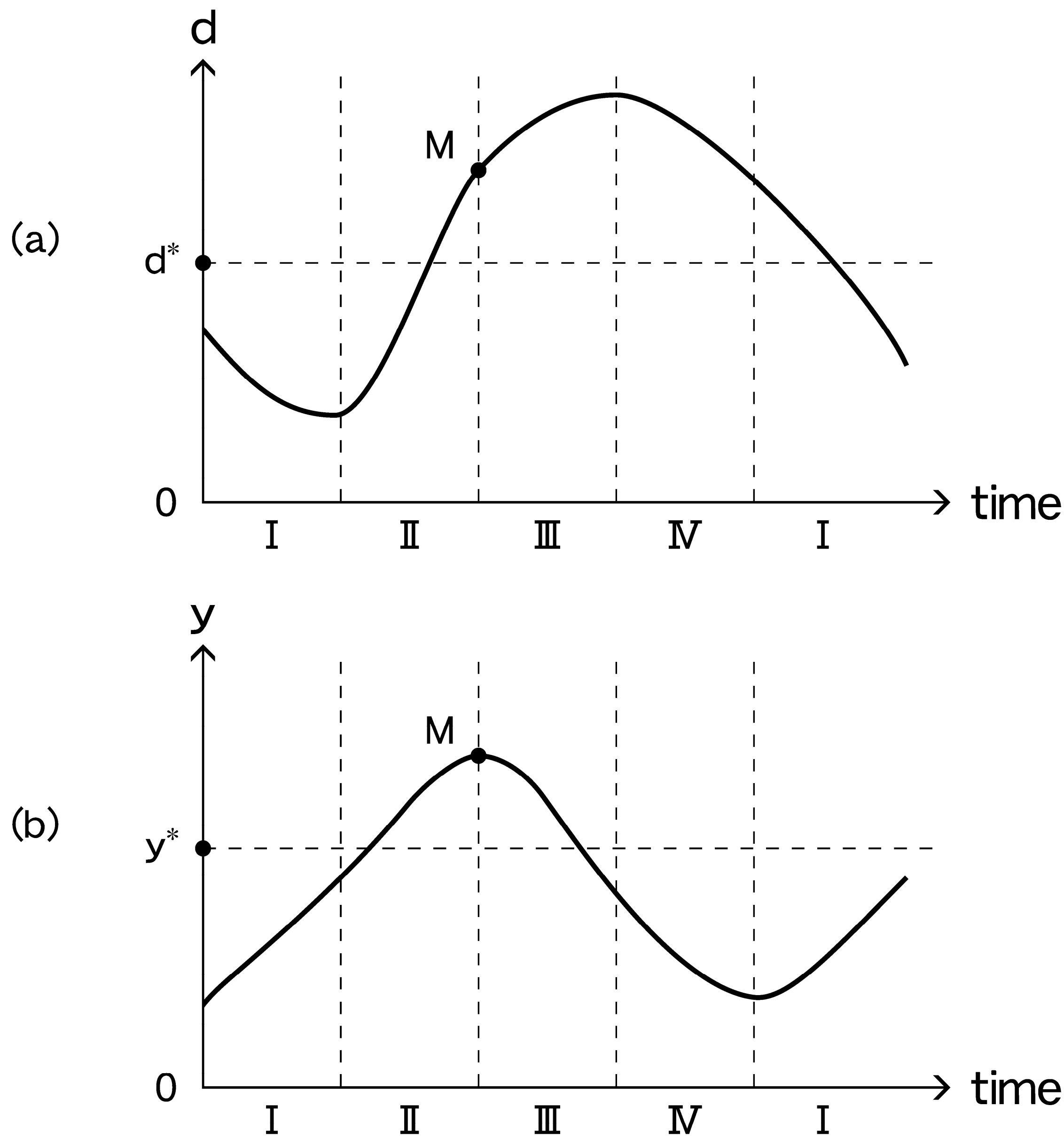

Figure 3. Minsky clycle

Ⅰ Recovery, Ⅱ Boom, Ⅲ Recession, Ⅳ Depression

(2) Suppose that $\alpha \in (\alpha_0, +\infty)$. Then, it follows from equations (24) and (25) that we have $\lambda_1 + \lambda_2 > 0$ and $\lambda_1 \lambda_2 > 0$. This means that the characteristic equation (23) has two roots with positive real parts, so that the equilibrium point becomes locally totally unstable.

(3) Suppose that $\alpha = \alpha_0$. Then, we have $\lambda_1 + \lambda_2 = 0$ and $\lambda_1 \lambda_2 > 0$, which means that the characteristic equation (23) has a pair of pure imaginary roots. Furthermore, we have $d(\mathrm{Re}\,\lambda)/d\alpha > 0$ at the point $\alpha = \alpha_0$, where $\mathrm{Re}\,\lambda$ is the real part of λ. This situation is enough to apply the Hopf Bifurcation theorem so that it ensures the existence of the non-constant closed orbits around the equilibrium point for some range of the parameter value α that is sufficiently close to α_0.[10] □

Proposition 1 (3) means that the endogenous fluctuations occur for some intermediate range of the parameter value α. We can consider that this is a mathematical expression of the 'Minsky cycle' that was proposed by Minsky(1975, 1982, 1986). Figures 1 – 3 are the graphical representations of such endogenous fluctuations.[11]

Figure 1 illustrates an example of the closed orbit that is produced through 'subcritical' Hopf Bifurcation. In this case, the closed orbit exists in the region $\alpha < \alpha_0$, and the closed orbit becomes unstable. Figure 2 illustrates an example of the closed orbit that is produced through 'supercritical' Hopf Bifurcation. In this case, the closed orbit exists in the region $\alpha > \alpha_0$, and the closed orbit becomes stable.

In general, both types of bifurcation can emerge. The case of Figure 1 corresponds to the 'corridor stability' that is described by Leijonfufvud(1973). In this case, the economic system is immune from the relatively small shock, but it is vulnerable if the shock is relatively large.

Figure 3 illustrates the time paths of two variables along the closed orbit, which represents the 'Minsky cycle'. The point M in this figure corresponds to the so called 'Minsky moment', which is a turning point between boom and recession. At this point, y begins to decrease, but d still continues to increase for a moment.

3. An Extension：Four-dimensional Model of Monetary Stabilization Policy with Flexible Prices

[10] See Gandolfo(2009) Chap. 24 for the exposition of the Hopf Bifurcation theorem.

[11] From equations (15) we have $y'(d)\big|_{\dot{d}=0} = -(f_{11}/f_{12}) > 0$, $y'(d)\big|_{\dot{y}=0} = -(f_{21}/f_{22}) > 0$, and $y'(d)\big|_{\dot{y}=0} - y'(d)\big|_{\dot{d}=0} = (f_{11}f_{22} - f_{12}f_{21})/f_{12}f_{22} > 0$ at the vicinity of the equilibrium point.

In the model of the previous section, it is assumed that the price level is fixed and the central bank's monetary policy is totally inactive. In this section, we relax these assumptions. We replace the equations (8) and (9) in the previous section with the following equations.

$$\pi = \varepsilon(y - \bar{y}) + \pi^e \;\; ; \;\; \varepsilon > 0, \;\; \bar{y} > 0 \tag{27}$$

$$\dot{\rho} = \begin{cases} \beta_1(\pi - \bar{\pi}) + \beta_2(y - \bar{y}) & if \quad \rho \succ 0 \\ \max[0, \beta_1(\pi - \bar{\pi}) + \beta_2(y - \bar{y})] & if \quad \rho = 0 \end{cases} \;\; ; \;\; \beta_1 > 0, \;\; \beta_2 > 0 \tag{28}$$

$$\dot{\pi}^e = \gamma[\xi(\bar{\pi} - \pi^e) + (1 - \xi)(\pi - \pi^e)] \;\; ; \;\; \gamma > 0, \; 0 \leqq \xi \leqq 1 \tag{29}$$

Eq. (27) is the quite standard 'expectation-argumented price Phillips curve'.

Eq. (28) formalizes an interest rate monetary policy rule by the central bank, which is a variant of the 'Taylor rule' type monetary policy that considers both of the rate of inflation and the level of real output, which is a surrogate variable of labor employment.[12] In this formulation, the zero bound of the nominal interest rate is explicitly considered. We can consider that this is a type of the flexible inflation targeting monetary policy rule, and $\bar{\pi}$ is the target rate of inflation that is set by the central bank.

Eq. (29) is a mixed type inflation expectation hypothesis. This is a mixture of the 'forward looking' and the 'backward looking'(adaptive) inflation expectations. In case of $\xi = 0,$ it is reduced to $\dot{\pi}^e = \gamma(\pi - \pi^e),$ which is a purely adaptive inflation expectation hypothesis. On the other hand, in case of $\xi = 1,$ it is reduced to $\dot{\pi}^e = \gamma(\bar{\pi} - \pi^e),$ which means that the publics' expected rate of inflation gravitates towards the target rate of inflation that is set and announced by the central bank. We can consider that the parameter value ξ is a measure of the 'degree of the credibility' of the central bank's inflation targeting, so that we call it the 'credibility parameter'.

The model in this section can be reduced to the following system of equations.[13]

$$\dot{d} = \phi(g(\beta y, \rho - \pi^e, d)) - s_f\{\beta y - i(\rho, d)d\}$$

$$- \{g(\beta y, \rho - \pi^e, d) + \varepsilon(y - \bar{y}) + \pi^e\}d = F_1(d, y, \pi^e, \rho) \tag{30.1}$$

$$\dot{y} = \alpha[(1 - s_1)\{(1 - s_f\beta)y - \tau(y)\} + (1 - s_2)i(\rho, d)d + (1 - s_3)\rho b$$

---

[12] For the original exposition of the 'Taylor rule', see Taylor(1993).

[13] Equations (30.2) and (30.3) imply that $\dot{y}$ is a *decreasing* function of $\rho - \pi^e,$ and $\dot{\pi}^e$ is an *increasing* function of $y.$ In other words, this model is immune from the notorious 'sign reversals' which are the peculiar characteristics of the 'New Keynesian' dynamic model (cf. Asada 2013, Asada, Chen, Chiarella and Flaschel 2006, Asada, Chiarella, Flaschel and Franke 2010, and Mankiw 2001).

$$+\phi(g(\beta y, \rho-\pi^e, d))+v-y]=\alpha F_2(d, y, \pi^e, \rho, b) \tag{30.2}$$

$$\dot{\pi}^e=\gamma[\xi(\bar{\pi}-\pi^e)+(1-\xi)\varepsilon(y-\bar{y})]=F_3(y, \pi^e) \tag{30.3}$$

$$\dot{\rho}=F_4(y, \pi^e)=\begin{cases} \beta_1(\pi^e-\bar{\pi})+(\beta_1\varepsilon+\beta_2)(y-\bar{y}) & if \quad \rho \succ 0 \\ \max[0, \beta_1(\pi^e-\bar{\pi})+(\beta_1\varepsilon+\beta_2)(y-\bar{y})] & if \quad \rho=0 \end{cases} \tag{30.4}$$

$$m=l(y, \rho) \tag{30.5}$$

Also in this section, we assume that $s_3=1$ for simplicity. In this case, the subsystem (30.1) – (30.4) becomes an independent four-dimensional system of dynamic equations with respect to d, y, π^e and ρ. In such a case, the role of Eq. (30.5) is only to determine the value of m endogenously.

We can express the equilibrium solution $(d^*, y^*, \pi^e{}^*, \rho^*, m^*)$ that satisfies the condition $\dot{d}=\dot{y}=\dot{\pi}^e=\dot{\rho}=0$ as follows if we neglect the nonnegative constraint of ρ.

$$F_1(d^*, \bar{y}, \bar{\pi}, z^*+\bar{\pi})=0 \tag{31.1}$$

$$F_2(d^* \bar{y}, \bar{\pi}, z^*+\bar{\pi})=0 \tag{31.2}$$

$$\pi^*=\pi^e{}^*=\bar{\pi} \tag{31.3}$$

$$y^*=\bar{y} \tag{31.4}$$

$$\rho^*=z^*+\bar{\pi} \tag{31.5}$$

$$m^*=l(\bar{y}, z^*+\bar{\pi}) \tag{31.6}$$

where z^* is the equilibrium real interest rate of the government bond.

We can determine the equilibrium values (d^*, z^*) from a system of the simultaneous equations (31.1) and (31.2). Incidentally, ρ^* becomes positive *if and only if* the inequality

$$\bar{\pi} > -z^* \tag{32}$$

is satisfied. We *assume* that this inequality is satisfied. In fact, we assume that $\bar{\pi}>0$ and $z^*>0$.

Next, let us study the local stability/instability of the equilibrium point. The Jacobian matrix of the system (30.1) – (30.4) *at the equilibrium point* becomes as follows.

$$J_2=\begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ \alpha F_{21} & \alpha F_{22} & \alpha F_{23} & \alpha F_{24} \\ 0 & \gamma\varepsilon(1-\xi) & -\gamma\xi & 0 \\ 0 & \beta_1\varepsilon+\beta_2 & \beta_1 & 0 \end{bmatrix} \tag{33}$$

We have $F_{11}=\partial F_1/\partial d=f_{11}-\bar{\pi}$, $F_{12}=\partial F_1/\partial y=f_{12}+\varepsilon d$, $F_{21}=\partial F_2/\partial d=f_{21}$, and $F_{22}=\partial F_2/\partial y=f_{22}$, where f_{11}, f_{12}, f_{21}, and f_{22} are defined by equations (18) –

(21) in the previous section.

Let us suppose that **Assumption 1** in the previous section is satisfied. Then, we obtain the following set of relationships.

$$F_{11} = \underset{(-)}{f_{11}} - \underset{(+)}{\overline{\pi}} < 0, \ \ F_{12} = \underset{(+)}{f_{12}} + \varepsilon d > 0, \ \ F_{21} = \underset{(-)}{f_{21}} < 0, \ \ F_{22} = \underset{(+)}{f_{22}} > 0 \tag{34}$$

$$F_{11}F_{22} - F_{12}F_{21} = \underbrace{(f_{11}f_{22} - f_{12}f_{21})}_{(+)} - \underset{(+)}{\overline{\pi}} \underset{(+)}{f_{22}} - \varepsilon d \underset{(-)}{f_{21}} \tag{35}$$

Other partial derivatives become as follows.

$$F_{13} = \partial F_1 / \partial \pi^e = -\{\underset{(+)}{\phi'(g)} - d\} \underset{(-)}{g_{\rho-\pi^e}} - d \tag{36}$$

$$F_{14} = \partial F_1 / \partial \rho = \{\underset{(+)}{\phi'(g)} - d\} \underset{(-)}{g_{\rho-\pi^e}} + s_f d \tag{37}$$

$$F_{23} = \partial F_2 / \partial \pi^e = -\underset{(+)}{\phi'(g)} \underset{(-)}{g_{\rho-\pi^e}} > 0 \tag{38}$$

$$F_{24} = \partial F_2 / \partial \rho = \underset{(+)}{\phi'(g)} \underset{(-)}{g_{\rho-\pi^e}} + (1-s_2)d + (1-s_3)b \tag{39}$$

$$\begin{aligned} F_{11}F_{23} - F_{13}F_{21} = {} & \underset{(+)}{\phi'(g)}[\underset{(-)}{g_{\rho-\pi^e}} \{(g+\overline{\pi}) - (1-s_f-s_2)(\underset{(+)}{i_d} d + i)\} + d \underset{(-)}{g_d}] \\ & + (1-s)(\underset{(+)}{i_d} d + i)(1 - \underset{(-)}{g_{\rho-\pi^e}}) \end{aligned} \tag{40}$$

$$\begin{aligned} F_{11}F_{24} - F_{14}F_{21} = {} & -\underset{(+)}{\phi'(g)} \underset{(-)}{g_{\rho-\pi^e}} \{(g+\overline{\pi}) + (1-s_f-s_2)(\underset{(+)}{i_d} d + i)\} \\ & + [\{\underset{(+)}{\phi'(g)} - d\} \underset{(-)}{g_d} - (g+\overline{\pi}) + s_f(\underset{(+)}{i_d} d + i)]\{(1-s_2)d + (1-s_3)b\} \\ & + (d \underset{(-)}{g_{\rho-\pi^e}} - s_f d)(1-s_2)(\underset{(+)}{i_d} d + i) \end{aligned} \tag{41}$$

We assume that the following assumption as well as **Assumption 1** in the previous section is satisfied.

Assumption 2.

$$F_{13} > 0, \ \ F_{14} < 0, \ \ F_{24} < 0, \ \ F_{11}F_{22} - F_{12}F_{21} > 0, F_{11}F_{23} - F_{13}F_{21} > 0, \ \ F_{11}F_{24} - F_{14}F_{21} > 0.$$

These inequalities will be satisfied if $\phi'(g)$, $\left| g_{\rho-\pi^e} \right|$, and i_d are sufficiently large *at* *the equilibrium point,* ε is sufficiently large and $1-s_f-s_2>0$.

The characteristic equation of the dynamic system (30.1) – (30.4) *at the equilibrium point* becomes

$$\Delta_2(\lambda)\equiv|\lambda I-J_2|=\lambda^4+b_1\lambda^3+b_2\lambda^2+b_3\lambda+b_4=0\,, \tag{42}$$

where

$$b_1=-traceJ_2=-\underset{(-)}{F_{11}}-\alpha\underset{(+)}{F_{22}}+\gamma\xi, \tag{43}$$

$b_2=$ sum of all principal second-order minors of J_2

$$=\alpha\begin{vmatrix}F_{11} & F_{12}\\ F_{21} & F_{22}\end{vmatrix}+\gamma\xi\begin{vmatrix}F_{11} & F_{13}\\ 0 & -1\end{vmatrix}+\begin{vmatrix}F_{11} & F_{14}\\ 0 & 0\end{vmatrix}+\alpha\gamma\begin{vmatrix}F_{22} & F_{23}\\ \varepsilon(1-\xi) & \xi\end{vmatrix}$$

$$+\alpha\begin{vmatrix}F_{22} & F_{24}\\ \beta_1\varepsilon+\beta_2 & 0\end{vmatrix}+\gamma\xi\beta_1\begin{vmatrix}-1 & 0\\ 1 & 0\end{vmatrix}$$

$$=\alpha(\underbrace{F_{11}F_{22}-F_{12}F_{21}}_{(+)})-\gamma\xi\underset{(-)}{F_{11}}+\alpha\gamma\{\xi\underset{(+)}{F_{22}}-\varepsilon(1-\xi)\underset{(+)}{F_{23}}\}-\alpha(\beta_1\varepsilon+\beta_2)\underset{(-)}{F_{24}}, \tag{44}$$

$b_3=-($ sum of all principal third-order minors of $J_2)$

$$=-\alpha\gamma\begin{vmatrix}F_{22} & F_{23} & F_{24}\\ \varepsilon(1-\xi) & -\xi & 0\\ \beta_1\varepsilon+\beta_2 & \beta_1 & 0\end{vmatrix}-\gamma\xi\beta_1\begin{vmatrix}F_{11} & F_{13} & F_{14}\\ 0 & -1 & 0\\ 0 & 1 & 0\end{vmatrix}-\alpha\begin{vmatrix}F_{11} & F_{12} & F_{14}\\ F_{21} & F_{22} & F_{24}\\ 0 & \beta_1\varepsilon+\beta_2 & 0\end{vmatrix}$$

$$-\alpha\gamma\begin{vmatrix}F_{11} & F_{12} & F_{13}\\ F_{21} & F_{22} & F_{23}\\ 0 & \varepsilon(1-\xi) & -\xi\end{vmatrix}$$

$$=\alpha[-\gamma\underset{(-)}{F_{24}}\{\varepsilon(1-\xi)\beta_1+\xi(\beta_1\varepsilon+\beta_2)\}+(\beta_1\varepsilon+\beta_2)(\underbrace{F_{11}F_{24}-F_{14}F_{21}}_{(+)})$$

$$+\gamma\varepsilon(1-\xi)(\underbrace{F_{11}F_{23}-F_{13}F_{21}}_{(+)})+\gamma\xi(\underbrace{F_{11}F_{12}-F_{12}F_{21}}_{(+)})]>0, \tag{45}$$

$$b_4 = \det J_2 = \alpha\gamma \begin{vmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ 0 & \varepsilon(1-\xi) & -\xi & 0 \\ 0 & \beta_1\varepsilon + \beta_2 & \beta_1 & 0 \end{vmatrix}$$

$$= \alpha\gamma\{(\beta_1\varepsilon + \beta_2)\xi + \beta_1\varepsilon(1-\xi)\}(\underbrace{F_{11}F_{24} - F_{14}F_{21}}_{(+)}) > 0. \tag{46}$$

It is well known that a set of *necessary* (but not sufficient) conditions for the local stability of the equilibrium point of the dynamic system (30) is given by the following set of inequalities (cf. Asada, Chiarella, Flaschel and Franke 2010, Mathematical appendix p. 416).

$$b_j > 0 \text{ for all } j \in \{1,2,3,4\} \tag{47}$$

The following 'instability proposition' is a direct corollary of this fact.

Proposition 2. (Instability Proposition)

Suppose that the parameter values α, β_1 and β_2 are fixed at any positive levels. Furthermore, suppose that (1) the 'credibility' parameter of the central bank's inflation targeting (ξ) is close to zero (including the case of $\xi = 0$), and (2) the adjustment speed of the inflation expectation (γ) is sufficiently large. Then, the equilibrium point of the dynamic system (30) becomes *locally unstable*.

(Proof.)

Suppose that $\xi = 0$. In this case, Eq. (44) becomes

$$b_2 = \alpha\{(\underbrace{F_{11}F_{22} - F_{12}F_{21}}_{(+)}) - \gamma\varepsilon \underset{(+)}{F_{23}} - (\beta_1\varepsilon + \beta_2)\underset{(-)}{F_{24}}\}. \tag{48}$$

Then, we have $b_2 < 0$ for all sufficiently large values of $\gamma > 0$, which violates one of the necessary conditions for local stability (47). It must be noted that we have $b_2 < 0$ for all sufficiently large values of $\gamma > 0$ even if $0 < \xi < 1$, as long as ξ is sufficiently close to zero, by continuity. □

On the other hand, we have the following 'stability proposition' in contrast to the above 'instability proposition'.

Proposition 3. (Stability Proposition)

Suppose that (1) the adjustment speed of the goods market disequilibrium (α) is sufficiently small, and (2) the 'credibility' parameter of the central bank's inflation targeting (ξ) is close to 1 (including the case of $\xi = 1$). Then, the equilibrium point of the dynamic equation (30) is *locally stable*.

(Proof.)

Suppose that $\xi = 1$. Then, the Jacobian matrix (33) becomes

$$J_2 = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ \alpha F_{21} & \alpha F_{22} & \alpha F_{23} & \alpha F_{24} \\ 0 & 0 & -\gamma & 0 \\ 0 & \beta_1\varepsilon + \beta_2 & \beta_1 & 0 \end{bmatrix}. \tag{49}$$

In this case the characteristic equation (42) becomes as follows.

$$\Delta_2(\lambda) \equiv |\lambda I - J_2| = |\lambda I - J_3|(\lambda + \gamma) = 0, \tag{50}$$

where

$$J_3 = \begin{bmatrix} F_{11} & F_{12} & F_{14} \\ \alpha F_{21} & \alpha F_{22} & \alpha F_{24} \\ 0 & \beta_1\varepsilon + \beta_2 & 0 \end{bmatrix} \tag{51}$$

and

$$|\lambda I - J_3| = \lambda^3 + w_1\lambda^2 + w_2\lambda + w_3 = 0, \tag{52}$$

$$w_1 = -traceJ_3 = -\underset{(-)}{F_{11}} - \alpha \underset{(+)}{F_{22}}, \tag{53}$$

$w_2 =$ sum of all principal second-order minors of J_3

$$= \alpha \begin{vmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{vmatrix} + \begin{vmatrix} F_{11} & F_{14} \\ 0 & 0 \end{vmatrix} + \alpha \begin{vmatrix} F_{22} & F_{24} \\ \beta_1\varepsilon + \beta_2 & 0 \end{vmatrix}$$

$$= \alpha\{\underbrace{(F_{11}F_{22} - F_{12}F_{21})}_{(+)} - (\beta_1\varepsilon + \beta_2)\underset{(-)}{F_{24}}\} > 0, \tag{54}$$

$$w_3 = -\det J_3 = -\alpha \begin{vmatrix} F_{11} & F_{12} & F_{14} \\ F_{21} & F_{22} & F_{24} \\ 0 & \beta_1\varepsilon + \beta_2 & 0 \end{vmatrix} = \alpha(\beta_1\varepsilon + \beta_2)\underbrace{(F_{11}F_{24} - F_{14}F_{21})}_{(+)} > 0, \tag{55}$$

$$w_1 w_2 - w_3 = \alpha\{(\underset{(-)}{F_{14}}\underset{(-)}{F_{21}} + \alpha \underset{(+)}{F_{22}}\underset{(-)}{F_{24}})(\beta_1\varepsilon + \beta_2)$$

$$+ (-\underset{(-)}{F_{11}} + \alpha \underset{(+)}{F_{22}})(\underbrace{F_{11}F_{22} - F_{12}F_{21}}_{(+)}). \tag{56}$$

The characteristic equation (50) has a negative real root $\lambda_4 = -\gamma < 0$, and other three roots are determined by Eq. (52). If α is sufficiently small, we have

$$w_j > 0 \text{ for all } j \in \{1,2,3\} \text{ and } w_1 w_2 - w_3 > 0, \tag{57}$$

which means that all of the Routh-Hurwitz conditions for stable roots of Eq. (52) are satisfied(cf. Gandolfo 2009, Chap. 16). In this case, all roots of the characteristic equation (50) have negative real parts. This conclusion in case of $\xi = 1$ is unchanged even if $0 < \xi < 1$, as long as ξ is sufficiently closed to 1, by continuity. □

Propositions 2 and 3 imply that the increase (the decrease) of the 'credibility' parameter of the central bank's inflation targeting (ξ) has a stabilizing effect (a destabilizing effect) of the macroeconomic system. Suppose that the equilibrium point of the dynamic system (30) is *locally unstable* in case of $\xi = 0$, and it becomes *locally stable* in case of $\xi = 1$. Then, there exists at least one 'bifurcation point' $\xi_0 \in (0,1)$ at which the switch between 'unstable' region and the 'stable' region occurs by continuity. It is clear that the real part of at least one characteristic root of Eq. (42) must become zero at the bifurcation point. On the other hand, it follows from equations (42) and (46) that

$$\Delta_2(0) = |-J_2| = \det J_2 = b_4 > 0, \tag{58}$$

which means that the characteristic equation (42) cannot have the real root such that $\lambda = 0$. This means that the characteristic equation (42) has at least one pair of *pure imaginary roots* at the bifurcation point $\xi = \xi_0$.

This means that the *endogenous cyclical fluctuations* occur at some range of the parameter value ξ that is sufficiently close to ξ_0.

4. A Further Extension：Six-dimensional Model of Monetary and Fiscal Stabilization Policy Mix with Flexible Prices

In the models of the previous sections, it was assumed that the government expenditure- capital ratio (v) is fixed. In this section, we relax this assumption, and study the effect of the monetary and fiscal stabilization policy mix. In the equations (30.1) – (30.4), v is no longer constant, and we add the following equations.

$$M/(pK) = m(\rho)H/K = l(y,\rho) = \varphi(\rho)y \;\; ; \;\; m_\rho = dm/d\rho > 0,$$

$$\varphi_\rho = d\varphi/d\rho < 0 \tag{59}$$

$$pT + \dot{B} + \dot{H} = pG + \rho B \tag{60}$$

$$\dot{v} = \beta_3[\theta(\bar{y} - y) + (1-\theta)(\bar{b} - b)] = F_5(y,b) \;\; ; \;\; \beta_3 > 0,\ 0 < \theta < 1 \tag{61}$$

where $M = mH =$ nominal money stock, $H =$ nominal high-powered money that is issued by the central bank, $m =$ money multiplier > 1, $\bar{b} =$ the target value of b that is set by the government. We assume that the private firms and the government are the debtors and the households are the creditors.

Eq. (59) is the LM equation that describes the equilibrium condition for the money market. The function $l(y,\rho) = \varphi(\rho)y$ is a particular form of the standard Keynesian real money demand function. We can rewrite this equation as

$$h = \psi(\rho)y \;\; ; \;\; h = H/(pK), \;\; \psi(\rho) = \varphi(\rho)/m(\rho), \;\; \psi'(\rho) = d\psi/d\rho < 0. \tag{62}$$

In our model which supposes that the central bank controls the nominal rate of interest (ρ), the high-powered money-capital ratio (h) becomes an endogenous variable that is determined by Eq. (62).

Eq. (60) is the budget constraint of the 'consolidated government' that includes the central bank. This equation means that the government expenditure including the interest payment of the government bond $(pG + \rho B)$ must be financed by (1) tax (pT), (2) bond financing $(\dot{B})$, or (3) money financing by the central bank $(\dot{H})$. [14]

Eq. (61) formalizes the government's fiscal policy rule. This equation means that the changes of the real government expenditure respond to both of the real national income (employment) and the level of the public debt. The parameter θ is the weight of the employment consideration rather than the public debt consideration in government's fiscal policy.

Differentiating the definitional equation $b = B/(pK)$ with respect to time and substituting Eq. (60) into it, we obtain[15]

$$\frac{\dot{b}}{b} = \frac{\dot{B}}{B} - \frac{\dot{p}}{p} - \frac{\dot{K}}{K} = \frac{pG + \rho B - pT - \dot{H}}{B} - \pi - g(\beta y, \rho - \pi^e, d). \tag{63}$$

---

[14] Also in the models of the previous sections, the definitional equation (60) must be satisfied, but this equation has no impact on the dynamics of the main variables in the models of the previous sections as long as $s_3 = 1$.

[15] Note that we have $\dot{K}/K = g(\beta y, \rho - \pi^e, d)$ from the investment function that is formulated in section 2.

We can rewrite this equation as

$$\dot{b} = v - \tau(y) - \frac{\dot{H}}{pK} + \{\rho - \pi - g(\beta y, \rho - \pi^e, d)\}b \;\; ; \;\; \tau = T/K = \tau(y). \tag{64}$$

This equation plays an important role in the dynamic of the public debt accumulation. If we neglect the impacts of the change of b on the changes of the variables such as v, y, $\dot{H}/(pK)$ etc., we have

$$\partial \dot{b} / \partial b = \rho - \pi - g. \tag{65}$$

Therefore, the inequality

real interest rate of government bond $= \rho - \pi < g =$ real rate of capital accumulation (66)

or equivalently,

nominal interest rate of government bond $\rho < g + \pi =$ nominal rate of capital accumulation (67)

is a *stabilizing factor* of the system, and the opposite inequality is a *destabilizing factor* of the system. The (partial) stabilizing condition (66) or (67) is called the 'Domar condition' after Domar(1957).[16]

Next, differentiating the definitional expression $h = H/(pK)$ with respect to time, we obtain the following expression.

$$\frac{\dot{H}}{pK} = (\pi + \frac{\dot{K}}{K})h + \dot{h} = \{\pi + g(\beta y, \rho - \pi^e, d)\}h + \dot{h} \tag{68}$$

On the other hand, differentiating Eq. (62) with respect to time and substituting equations (30.2) and (30.4) in section 3, we obtain

$$\dot{h} = \underset{(-)}{\psi'(\rho)} y\dot{\rho} + \psi(\rho)\dot{y} = \underset{(-)}{\psi'(\rho)} F_4(y, \pi^e) + \psi(\rho)\alpha F_2(d, y, \pi^e, \rho, v, b). \tag{69}$$

Substituting equations (27), (62), (68), and (69) into Eq. (64), we obtain the following equation that governs the dynamic of the variable b.[17]

$$\dot{b} = v - \tau(y) - \{\varepsilon(y - \bar{y}) + \pi^e + g(\beta y, \rho - \pi^e, d)\}\psi(\rho)y - \underset{(-)}{\psi'(\rho)} yF_4(y, \pi^e)$$
$$- \psi(\rho)\alpha F_2(d, y, \pi^e, \rho, v.b) + \{\rho - \varepsilon(y - \bar{y}) - \pi^e - g(\beta y, \rho - \pi^e, d)\}b$$

---

[16] There is a slight difference between the original 'Domar condition' and our 'Domar condition'. In Domar's(1957) original model, the dynamic stability of the ratio $B/(pY)$ rather than the ratio $b = B/(pK)$ is studied, so that in original Domar model, g is not $\dot{K}/K$ but it is $\dot{Y}/Y$.

[17] This means that the equilibrium condition for the money market (62) affects other parts of the system through Eq. (70) so that the dynamic system in this section is no longer the decomposable system.

$$= F_6(d, y, \pi^e, \rho, v, b) \tag{70}$$

Equations (30.1) – (30.4) with variable v and b together with equations (61) and (70) constitute a complete system of six-dimensional nonlinear differential equations.[18] We can summarize the system in this section as follows.[19]

$$\dot{d} = F_1(d, y, \pi^e, \rho) \tag{71.1}$$

$$\dot{y} = F_2(d, y, \pi^e, \rho, v, b) \tag{71.2}$$

$$\dot{\pi}^e = F_3(y, \pi^e) \tag{71.3}$$

$$\dot{\rho} = F_4(y, \pi^e) \tag{71.4}$$

$$\dot{v} = F_5(y, b) \tag{71.5}$$

$$\dot{b} = F_6(d, y, \pi^e, \rho, v, b) \tag{71.6}$$

The equilibrium solution of this system $(d^*, y^*, \pi^{e*}, \rho^*, v^*, b^*)$ that satisfies $\dot{d} = \dot{y} = \dot{\pi}^e = \dot{\rho} = \dot{v} = \dot{b} = 0$ can be expressed by the following system of equations.

$$F_1(d^*, \bar{y}, \bar{\pi}, \rho^*) = 0 \tag{72.1}$$

$$F_2(d^*, \bar{y}, \bar{\pi}, \rho^*, v^*, \bar{b}) = 0 \tag{72.2}$$

$$\pi^{e}* = \pi^* = \bar{\pi}, \quad y^* = \bar{y}, \quad b^* = \bar{b} \tag{72.3}$$

$$v^* = \tau(\bar{y}) + \{\bar{\pi} + g(\beta\bar{y}, \rho^* - \bar{\pi}, d^*)\}\psi(\rho^*)\bar{y}$$
$$+ \{g(\beta\bar{y}, \rho^* - \bar{\pi}, d^*) + \bar{\pi} - \rho^*\}\bar{b} = v^*(d^*, \rho^*, \bar{y}, \bar{\pi}, \bar{b}) \tag{72.4}$$

The system of simultaneous equations (72.1), (72.2) and (72.4) determines the equilibrium values (d^*, ρ^*, v^*). We *assume* that there exists the unique equilibrium point that satisfies

$$d^* > 0, \quad \rho^* > 0, \quad v^* > 0. \tag{73}$$

In addition to **Assumptions 1 and 2** in the previous sections, let us assume as follows.

Assumption 3.

$$0 < \rho^* - \bar{\pi} < g(\beta\bar{y}, \rho^* - \bar{\pi}, d^*)$$

---

[18] Unlike the previous sections, we do *not* necessarily assume that $s_3 = 1$ in this section.

[19] This six-dimensional system is a generalized version of the five-dimensional system that is formulated by Asada(2014), which does not consider the explicit dynamic of the variable d.

This assumption implies that the equilibrium real interest rate of the government bond is positive and the 'Domar condition' (66) is satisfied *at the equilibrium point.*

Next, let us consider the local stability/instability of the equilibrium point. We can express the Jacobian matrix of the dynamic system (71) *at the equilibrium point* as

$$J_4 = \begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} & 0 & 0 \\ \alpha F_{21} & \alpha F_{22} & \alpha F_{23} & \alpha F_{24} & \alpha & \alpha(1-s_3)\rho^* \\ 0 & \gamma\varepsilon(1-\xi) & -\gamma\xi & 0 & 0 & 0 \\ 0 & \beta_1\varepsilon+\beta_2 & \beta_1 & 0 & 0 & 0 \\ 0 & -\beta_3\theta & 0 & 0 & 0 & -\beta_3(1-\theta) \\ F_{61} & F_{62} & F_{63} & F_{64} & F_{65} & F_{66} \end{bmatrix} \tag{74}$$

where F_{ij} $(i, j = 1,2,3,4)$ are the same as those in the previous section, and other relevant partial derivatives at the equilibrium point become as follows.[20]

$$F_{61} = \partial F_6/\partial d = -\underset{(-)}{g_d}\{\psi(d^*)\bar{y} + \bar{b}\} - \psi(\rho^*)\alpha \underset{(-)}{F_{21}} > 0 \tag{75}$$

$$F_{64} = \partial F_6/\partial \rho = -\underset{(-)}{g_{\rho-\pi^e}}\{\psi(\rho^*)\bar{y} + \bar{b}\} - \{\bar{\pi} + g(\beta\bar{y}, \rho^* - \bar{\pi}, d^*)\}\underset{(-)}{\psi'(\rho^*)}\bar{y}$$

$$-\psi(\rho^*)\alpha \underset{(-)}{F_{24}} > 0 \tag{76}$$

$$F_{65} = \partial F_6/\partial v = 1 - \psi(\rho^*)\alpha \tag{77}$$

$$F_{66} = \partial F_6/\partial b = \rho^* - \bar{\pi} - g(\beta\bar{y}, \rho^* - \bar{\pi}, d^*) < 0 \tag{78}$$

$$F_{14}F_{61} - F_{11}F_{64} = -s_f d[\underset{(-)}{g_d}\{\psi(d^*)\bar{y} + \bar{b} + \underset{(+)}{\phi'(g)}\} + \psi(\rho^*)\alpha\{(1-s_2)(\underset{(+)}{i_d}\, d + i)]$$

$$+\{\underset{(+)}{\phi'(g)} - d\}[-\underset{(-)}{g_{\rho-\pi^e}}\psi(\rho^*)\{(1-s_2)(\underset{(+)}{i_d}\, d + i) + \underset{(+)}{\phi'(g)}\underset{(-)}{g_d}\} + \underset{(-)}{g_d}(g+\bar{\pi})\underset{(-)}{\psi'(\rho^*)}\bar{y}]$$

$$+\{-(g+\bar{\pi}) + s_f(\underset{(+)}{i_d}\, d + i)\}[\underset{(-)}{g_{\rho-\pi^e}}\{\psi(\rho^*)\bar{y} + \bar{b}\} + (g+\bar{\pi})\underset{(-)}{\psi'(\rho^*)}\bar{y}] \tag{79}$$

Now, we shall assume that the following inequality is satisfied.

Assumption 4.

$F_{14}F_{61} - F_{11}F_{64} > 0$

This inequality will be satisfied if $\phi'(g)$, $|g_{\rho-\pi^e}|$, $|g_d|$ and $|\psi'(\rho^*)|$ are sufficiently

[20] The values of F_{62} and F_{63} are irrelevant for our purpose.

large *at the equilibrium point*.

The characteristic equation of this system at the equilibrium point becomes

$$\Delta_4(\lambda) \equiv |\lambda I - J_4| = \lambda^6 + d_1\lambda^5 + d_2\lambda^4 + d_3\lambda^3 + d_4\lambda^2 + d_5\lambda + d_6 = 0, \tag{80}$$

$$d_1 = -traceJ_4, \tag{81}$$

$$d_j = (-1)^j (\text{ sum of all principal j'th order minors of } J_4) \quad (j = 2,3,4,5), \tag{82}$$

$$d_6 = \det J_4. \tag{83}$$

It is worth noting that the conditions

$$d_j > 0 \text{ for all } j \in \{1,2,\cdots,6\} \tag{84}$$

are the *necessary* (but not sufficient) conditions for the local stability of the equilibrium point of the dynamic system (71) (cf. Gandolfo 2009, Chap. 16).

Under assumptions 1 – 4, we can prove the following two propositions.[21]

Proposition 4. (Instability Proposition)

Suppose that the following conditions are satisfied.

(1) The credibility parameter of the central bank's inflation targeting (ξ) is close to zero.

(2) The adjustment speed of the inflation expectation (γ) is sufficiently large.

(3) The monetary policy parameters β_1 and β_2 are close to zero.

(4) The fiscal policy parameter that describes the weight of employment consideration (θ) is close to zero.

Then, the equilibrium point of the dynamic system (71) becomes *locally unstable.*

(Proof.) See **Appendix** A.

Proposition 5. (Stability Proposition)

Suppose that the following conditions are satisfied.

(1) The adjustment speed of the goods market disequilibrium (α) is sufficiently small.

(2) The credibility parameter of the central bank's inflation targeting (ξ) is close to 1 (including the case of $\xi = 1$).

(3) The monetary policy parameters β_1 and β_2 are nonnegative and at least one of them is positive.

[21] It is worth noting that **Assumptions 3 and 4** are *not* necessary for the proof of **Proposition 4**, but it is only used for the proof of **Proposition 5**.

(4) The fiscal policy parameter θ is less than 1, but it is close to 1.
(5) The average propensity to save out of the interest on the public debt (s_3) is close to 1 (including the case of $s_3 = 1$).

Then, the equilibrium point of the dynamic system (71) becomes *locally stable.*

(Proof.) See **Appendix B**.

Proposition 4 means that the 'Domar condition' (**Assumption 3**) is by no means the sufficient condition for the local stability of the equilibrium point of the full six-dimensional system in this section, but it is only a partial stability condition.

5. Concluding Remarks : Economic Interpretation of the Analytical Results

In this final section, we shall provide an intuitive economic interpretation of the analytical results which are presented in the previous section.

Proposition 4 means that the equilibrium point of the system (71) tends to become *dynamically unstable* if (1) the central bank's monetary policy is inactive and the central bank's inflation targeting is incredible, and (2) the real government expenditure responds sensitively to the amount of the outstanding public debt rather than the real national income (employment). This proposition characterizes an *inappropriate* fiscal and monetary policy mix. We can illustrate this destabilizing cumulative disequilibrium process by the following two coexisting positive feedback mechanisms $y \downarrow \Rightarrow y \downarrow$ and $b \uparrow \Rightarrow b \uparrow$.[22]

$$y \downarrow \Rightarrow \tau \downarrow \Rightarrow b \uparrow \Rightarrow v \downarrow \Rightarrow \text{(effective demand per capital stock)} \downarrow \Rightarrow y \downarrow \qquad (FM_1)$$

$$b \uparrow \Rightarrow v \downarrow \Rightarrow \{y \downarrow, \tau \downarrow, H/(pK) \downarrow\} \Rightarrow b \uparrow \qquad (FM_2)$$

In this depression process, the *decrease* of the government expenditure-capital ratio and the *increase* of the public debt-capital ratio coexist, and the actual and the expected rates of inflation continue to decline. In this process, the nominal interest rate of the government bond slowly declines and at last, it will reach to its lower bound. This theoretical scenario is quite consistent with the so called 'lost twenty years' of the

---

22 Suppose that the central bank's monetary policy is inactive so that both of the monetary policy parameters β_1 and β_2 are sufficiently small. In this case, the movement of the nominal interest rate of the government bond ρ becomes so sluggish that $h = H/(pK)$ moves to the same direction as that of the movement of y like (FM_2) (see Eq. (62) in the text). This means that the central bank continues to *reduce* the high-powered money-capital ratio in the process of depression, which has the pro-cyclical destabilizing effect.

Japanese economy that is characterized by the deflationary depression.[23]

On the other hand, **Proposition 5** means that the equilibrium point of the system (71) tends to be *dynamically stable* if (1) the central bank's inflation targeting is credible, and (2) the real government expenditure responds sensitively to the real national income(employment) rather than the amount of the outstanding public debt, under certain additional conditions. This proposition characterizes an *appropriate* fiscal and monetary policy mix.

We can schematically represent the stabilizing negative feedback mechanism of the government's fiscal policy $y\downarrow\Rightarrow y\uparrow$ that responds sensitively to the real national income(employment) rather than the amount of the outstanding public debt as follows.

$$y\downarrow\Rightarrow v\uparrow\Rightarrow \text{(effective demand per capital stock)}\uparrow\Rightarrow y\uparrow \qquad (FM_3)$$

The central bank's active monetary policy that accompanies the 'credible' inflation targeting will enhance this stabilizing negative feedback mechanism. We can consider that this is the rationale of new macroeconomic policy in Japan called 'Abenomics' that was initiated by Abe administration in 2013.[24]

Acknowledgments.

This research was financially supported by the Japan Society for the Promotion of Science (Grant-in Aid (C) 25380238) and the MEXT-Supported Program for the Strategic Research Foundation at Private Universities, 2013 – 2017.

Appendix A : Proof of Proposition 4.

Suppose that $\xi=\beta_1=\beta_2=\theta=0$. In this case, we have

$d_2=$ sum of all principal second-order minors of J_4

$$=-\gamma\alpha\varepsilon \underset{(+)}{F_{23}}+A, \qquad \text{(A1)}$$

where A is independent of the value of γ. This means that we have $d_2<0$ for all sufficiently large values of γ, which violates one of the necessary conditions for local stability (84). By continuity, this conclusion applies even if the parameters ξ, β_1, β_2, and θ are positive, as long as they are sufficiently small. □

Appendix B : Proof of Proposition 5.

[23] For the 'lost twenty years' of the Japanese economy, see Krugman(1998) and Asada (ed.) (2014).

[24] For the detailed exposition of 'Abenomics', see General Introduction of Asada(ed.)(2014).

Step 1.

Suppose that $\xi = s_3 = 1$. In this case, the characteristic equation (80) becomes

$$\Delta_4(\lambda) \equiv |\lambda I - J_4| = |\lambda I - J_5|(\lambda + \gamma) = 0, \tag{B1}$$

$$J_5 = \begin{bmatrix} F_{11} & F_{12} & F_{14} & 0 & 0 \\ \alpha F_{21} & \alpha F_{22} & \alpha F_{24} & \alpha & 0 \\ 0 & \beta_1\varepsilon + \beta_3 & 0 & 0 & 0 \\ 0 & -\beta_3\theta & 0 & 0 & -\beta_3(1-\theta) \\ F_{61} & F_{62} & F_{64} & F_{65} & F_{66} \end{bmatrix}. \tag{B2}$$

Equation (B1) has a negative real root $\lambda_6 = -\gamma$, and other five roots are determined by the equation

$$\Delta_5(\lambda) \equiv |\lambda I - J_5| = 0. \tag{B3}$$

Step 2.

Next, suppose that $\theta = 1$. In this case, Eq. (B3) is reduced to

$$\Delta_5(\lambda) = |\lambda I - J_6|(\lambda - F_{66}) = 0, \tag{B4}$$

$$J_6 = \begin{bmatrix} F_{11} & F_{12} & F_{14} & 0 \\ \alpha F_{21} & \alpha F_{22} & \alpha F_{24} & \alpha \\ 0 & \beta_1\varepsilon + \beta_2 & 0 & 0 \\ 0 & -\beta_3 & 0 & 0 \end{bmatrix}. \tag{B5}$$

Eq. (B4) has a negative real root $\lambda_5 = F_{66}$ and other four roots are determined by the following equation.

$$\Delta_6(\lambda) \equiv |\lambda I - J_6| = (\lambda^3 + z_1\lambda^2 + z_2\lambda + z_3)\lambda = 0, \tag{B6}$$

$$z_1 = -\underset{(-)}{F_{11}} - \alpha \underset{(+)}{F_{22}}, \tag{B7}$$

$$z_2 = \alpha\{\underbrace{(F_{11}F_{22} - F_{12}F_{21})}_{(+)} - \underset{(-)}{F_{24}}(\beta_1\varepsilon + \beta_2) + \beta_3\} > 0, \tag{B8}$$

$$z_3 = \alpha\{-\underset{(-)}{F_{11}}\beta_3 + (\beta_1\varepsilon + \beta_2)\underbrace{(F_{11}F_{24} - F_{14}F_{21})}_{(+)}\} > 0, \tag{B9}$$

$$z_1z_2 - z_3 = \alpha\{(-\underset{(-)}{F_{11}} - \alpha \underset{(+)}{F_{22}})\underbrace{(F_{11}F_{22} - F_{12}F_{21})}_{(+)} - \alpha\beta_3 \underset{(+)}{F_{22}}$$

$$+(\beta_1\varepsilon+\beta_2)(\underset{(-)}{F_{14}}\underset{(-)}{F_{21}}+\alpha\underset{(+)}{F_{22}}\underset{(-)}{F_{24}})\}. \tag{B10}$$

Eq. (B6) has a real root $\lambda_4 = 0,$ and other three roots are determined by the equation

$$\Delta_7(\lambda) \equiv \lambda^3 + z_1\lambda^2 + z_2\lambda + z_3 = 0 \tag{B11}$$

Step 3.

It is easy to see that all of the following Routh-Hurwitz conditions for stable roots of Eq. (B11) are satisfied if α is sufficiently small (cf. Gandolfo 2009, Chap. 16).

$$z_j > 0 \ \ (j = 1,2,3), \quad z_1z_2 - z_3 > 0 \tag{B12}$$

Hence, we have just proved the following result.

" Suppose that $\theta = 1$. Then, the characteristic equation (B3) has a real root $\lambda_4 = 0$ and other four roots of this equation have negative real parts under the conditions (1) and (3) of **Proposition 5**."

This means that Eq. (B3) has at least four roots with negative real parts under the conditions (1) and (3) of **Proposition 5** even if $0 < \theta < 1$, as long as θ is sufficiently close to 1 by continuity. On the other hand, in case of $0 < \theta < 1$, we have

$$\Delta_5(0) = -\prod_{j=1}^{5}\lambda_j = |-J_5| = -\det J_5$$
$$= \alpha(\beta_1\varepsilon+\beta_2)\beta_3(1-\theta)\{F_{65}(\underbrace{F_{11}F_{24}-F_{14}F_{21}}_{(+)}) + (\underbrace{F_{14}F_{61}-F_{11}F_{64}}_{(+)})\}, \tag{B13}$$

and F_{65} becomes positive if α is sufficiently small. Therefore, Eq. (B13) becomes positive so that we have $\prod_{j=1}^{5}\lambda_j < 0$ if $0 < \theta < 1$ and α is sufficiently small. This means that all roots of Eq. (B3) have negative real parts under the conditions (1), (3), and (4) of **Proposition 5**.

Step 4.

We have just proved the following result.

"Suppose that $\xi = s_3 = 1$. Then, all of six characteristic roots of Eq. (80) in the text have negative real parts under the conditions (1), (3), and (4) of **Proposition 5**."

By continuity, this conclusion applies even if $0 < \xi < 1$ and $0 < s_3 < 1$, as long as they are sufficiently close to 1. This proves **Proposition 5**. □

References

[1] Asada, T. (2001) : "Nonlinear Dynamics of Debt and Capital : A Post Keynesian Approach." In Aruka, Y. and Japan Association for Evolutionary Economics (eds.) *Evolutionary Controversies in Economics*, Springer, Tokyo, pp. 73 – 87.

[2] Asada, T. (2004) : "Price Flexibility and Instability in a Macrodynamic Model with a Debt Effect." *Journal of International Economic Studies* 32 – 2, pp. 19 – 38. (Hosei University, Tokyo)

[3] Asada, T. (2012) : "Modeling Financial Instability." *Intervention : European Journal of Economics and Economic Policies* 9 – 2, pp. 215 – 232.

[4] Asada, T. (2013) : "An Analytical Critique of 'New Keynesian' Dynamic Model." *Post Keynesian Review* 2 – 1, pp. 1 – 28. (http://ns.fujimori.cache.waseda.ac.jp/pkr.html)

[5] Asada, T. (ed.) (2014) : *The Development of Economics in Japan : From the Inter-war Period to the 2000s*. Routledge, London.

[6] Asada, T. (2014) : "Macrodynamics of Deflationary Depression : A Japanese Perspective." In Asada, T. (ed.) pp. 156 – 207.

[7] Asada, T., P. Chen, C. Chiarella and P. Flaschel (2006) : "Keynesian Dynamics and the Wage-Price Spiral : A Baseline Disequilibrium Model." *Journal of Macroeconomics* 28, pp. 90 – 130.

[8] Asada, T., C. Chiarella, P. Flaschel and R. Franke (2003) : *Open Economy Macrodynamics : An Integrated Disequilibrium Approach*. Springer, Berlin.

[9] Asada, T., C. Chiarella, P. Flaschel and R. Franke (2010) : *Monetary Macrodynamics*. Routledge, London.

[10] Asada, T., C. Chiarella, P. Flaschel, P. Mouakil, C. R. Proaño and W. Semmler (2010) : Stabilizing an Unstable Economy : On the Proper Choice of Policy Measures." *Economics : The Open-Access, Open Assessment E-Journal* 3 – 21, pp. 1 – 43.

[11] Chiarella, C., P. Flaschel and R. Franke (2005) : *Foundations for a Disequilibrium Theory of the Business Cycle : Qualitative Analysis and Quantitative Assessment*. Cambridge University Press, Cambridge.

[12] Chiarella, C., P. Flaschel and W. Semmler (2013) : "Keynes, Dynamic Stochastic General Equilibrium Model, and the Business Cycle." In Kuroki, R. (ed.) *Keynes and Modern Economics*, Routledge, London, pp. 85 – 116.

[13] Domar, E. (1957) : *Essays in the Theory of Economic Growth*. Oxford University Press, Oxford.

[14] Eggertsoon, G. B. and P. Krugman (2012) : "Debt, Deleveraging, and the Liquidity Trap : A Fisher-Minsky-Koo Approach." *Quarterly Journal of Economics* 127 – 3, pp. 1469 – 1513.

[15] Fisher, I.(1933) : "The Debt-deflation Theory of Great Depression." *Econometrica* 1,

pp. 337 – 357.

[16] Flaschel, P., R. Franke and C. R. Proaño (2008) : "On Equilibrium Determinacy in New Keynesian Models with Staggered Wage and Price Setting." *B. E. Journal of Economics* 8, Article 31(10 pages).

[17] Galí, J. (2008) : *Monetary Policy, Inflation, and the Business Cycle : A New Keynesian Framework.* Princeton University Press, Princeton.

[18] Gandolfo, G. (2009) : *Economic Dynamics, Fourth Edition.* Springer, Berlin.

[19] Kalecki, M. (1937) : "The Principle of Increasing Risk." *Economica* 4, pp. 440 – 447.

[20] Kalecki, M. (1971) : *Selected Essays on the Dynamics of the Capitalist Economy.* Cambridge University Press, Cambridge.

[21] Keen, S. (2000) : "The Nonlinear Economics of Debt Deflation." In Barnett, W. A., C. Chiarella, S. Keen, R. Marks and H. Scnabl (eds.) *Commerce, Complexity, and Evolution,* Cambridge University Press, Cambridge, pp. 83 – 110.

[22] Keynes, J. M. (1936) : *The General Theory of Employment, Interest and Money.* Macmillan, London.

[23] Krugman, P. (1998) : "It's Baaack : Japan's Slump and the Return of the Liquidity Trap." *Brookings Papers on Economic Activity* 2, pp. 137 – 205.

[24] Krugman, P. (2012) : *End This Depression Now!* W. W. Norton, New York.

[25] Leijonfufvud, A. (1973) : "Effective Demand Failures." *Swedish Journal of Economics* 75 – 1, pp. 27 – 48.

[26] Mankiw, G. (2001) : "The Inexorable and Mysterious Tradeoff between Inflation and Unemployment." *Economic Journal* 111, C45 – C 61.

[27] Minsky, H. P. (1975) : *John Maynard Keynes.* Columbia University Press, New York.

[28] Minsky, H. P. (1982) : *Can 'It' Happen Again? Essays on Instability and Finance.* M. E. Sharpe, Armonk, New York.

[29] Minsky, H. P. (1986) : *Stabilizing an Unstable Economy.* Yale University Press, New Haven.

[30] Nasica, E. (2000) : *Finance, Investment and Economic Fluctuations : An Analysis in the Tradition of Hyman Minsky.* Edward Elgar, Cheltenham, UK.

[31] Pally, T. (1996) : *Post-Keynesian Economics : Debt, Distribution and the Macroeconomy.* Macmillan, London.

[32] Semmler, W. (ed.) (1989) : *Financial Dynamics and Business Cycles : New Perspectives.* M. E. Sharpe, Armonk, New York.

[33] Taylor, J. B. (1993) : "Discretion versus Policy Rules in Practice." *Carnegie-Rochester Conference Series on Public Policy* 39, pp. 195 – 214.

[34] Tobin, J. (1994) : "Price Flexibility and Output Stability : An Old Keynesian View." In Semmler, W. (ed.) *Business Cycles : Theory and Empirical Methods,* Kluwer Academic Publishers, Boston, pp. 165 – 195.

[35] Uzawa, H. (1969) : "Time Preference and the Penrose Effect in a Two-Class Model of Economic Growth." *Journal of Political Econom*y 77, pp. 628 – 652.

[36] Woodford, M. (2003) : *Interest and Prices : Foundations of a Theory of Monetary Policy.* Princeton University Press, Princeton.