Note on Goodwin's 1951 Nonlinear Accelerator Model with an Investment Lag*

Akio Matsumoto
Department of Economics, Chuo University
742-1, Higashinakano, Hachioji
Tokyo, 211-0035, Japan

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Abstract

It has been well-known that nonlinearity, time delay and local instability are significant sources for a birth of cyclical dynamics since the pioneering work of Goodwin (1951). He has constructed a delay nonlinear business cycle model with the nonlninear acceleration principle which gives rise to cyclic oscillations when a stationary state is locally unstable. However very little is known about time delay effects caused by investment lags in Goodwin's cyclic dynamics, furthermore global dynamics in the locally stable case has not been considered yet. This study draws attentions to these unexplored aspects of Goodwin's business model. It is shown that increasing investment lag makes the length of business cycle longer and its amplitude larger. It is further demonstrated that multiple cycles coexist when the stationary state is locally stable. These results imply two issues: Goodwin's model is not only robust in cyclic oscillations regardless of local dynamic properties but also corridor stable when the stationary state is locally stable.

Key words: fixed time delay, continuously distributed time delay, corridor stability, coexistence of multiple limit cycles.

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1 Introduction

The contributions of Goodwin (1951) are reconsidered and further developed in this study. Goodwin introduced a nonlinear accelerator business cycle model with an investment lag, numerically specified it and graphically showed that it could generate a stable limit cycle when a stationary point is locally unstable. Since Goodwin's work, it is expected that instability, nonlinearity and delay could be significant sources for the birth of cyclic behavior. In view of the fact that it is difficult to analytically solve delay nonlinear models, it is a natural way to perform numerical studies or to convert the model to a tractable one by using approximation. Indeed, considerable effort has been devoted to investigate the nonlinear structure of the ordinary differential version of the unstable Goodwin's model. Recently, Sasakura (1996) gives an elegant proof of the stability and the uniqueness of Goodwin's cycle. More recently Lorenz and Nusse (2002), based on Lorenz (1987), reconstructs Goodwin's model as a forced oscillator system and demonstrates the emergence of chaos when nonlinearities become stronger. In the existing literature, however, there have been only limited analytical works on the delay differential version,[1] and, furthermore, very little has yet been revealed with respect to the circumstances under which the stationary point is locally asymptotically stable. The main purpose of this study is to provide an investigation of these unexplored aspects of Goodwin's business cycle model.

We add two new observations to the existing results. First, we reformate the model in terms of a nonlinear differential equation with the explicit treatment of time delay (i.e., fixed time delay and distributed time delay) and find out how the time delay affects the emergence and characteristics of a limit cycle. Second, we demonstrate that the nonlinear delay Goodwin's model has the corridor stability when a stationary point is locally stable. That is, the model is stable and the trajectory returns to the stationary state for smaller disturbances but is unstable and exhibits persistent fluctuations for larger disturbances.

In what follows, Section 2 overviews the basic structure of Goodwin's nonlinear accelerator model and introduces time delay to see its effect on cyclic dynamics. Section 3 shows a coexistence of a stable stationary point, an unstable limit cycle and a stable limit cycle. Section 4 concludes the paper.

[1] Yoshida and Asada (2007) investigates the impact of delayed government stabilization policy on the dynamic behavior of a Keynes-Goodwin model. Also see Bischi, Chiarella, Kopel and Szidarovszky (2007) for applications of the delay differential method to oligopoly models.

2 Goodwin's Business Cycle Model

This section is divided into three parts. In Section 2.1, we recapitulate the basic elements of Goodwin's model and perform simulations to see the birth of limit cycles when a stationary state is locally unstable and the investment function is nonlinear. We, then, adopt an explicit treatment of the investment lag into the model. In particular, we will examine fixed time delay in Section 2.2 and continuously distributed time delay in Section 2.3, and we will show see how such a delay affects the characteristics of cyclic dynamics.

2.1 Basic Model

Goodwin (1951) presents five different versions of the nonlinear accelerator model. The first version assumes a piecewise linear function with three levels of investment, which can be thought as the crudest or simplest version of the non-linear accelerator. This is a text-book model that can give a simple exhibition on how nonlinearities give rise to endogenous cycles without relying on structurally unstable parameters, exogenous shocks, etc. The second version replaces the piecewise linear investment function with a smooth nonlinear investment function. Although persistent cyclical oscillations of output are shown to exist, the second version includes a unfavorable phenomenon, namely, discontinuous investment jump , which is not realistic in the real economic world. "In order to come close to reality" (p.11, Goodwin (1951)), the investment lag is introduced in the third version. However, no analytical considerations are given to this third version. The existence of a business cycle is confirmed in the fourth version, which is a linear approximation of the third version with respect to the investment lag. Finally alternation of autonomous expenditure over time is taken into account in the fifth version.

To find out how nonlinearity works to generate endogenous cycles, we review the second version of Goodwin's model, which we call the *basic model*,

$$\begin{cases} \varepsilon\dot{y}(t) = \dot{k}(t) - (1-\alpha)y(t), \\ \\ \dot{k}(t) = \varphi(\dot{y}(t)). \end{cases} \tag{1}$$

Here k is capital stock, y national income, α the marginal propensity to consume, which is positive and less than unity, and the reciprocal of ε a positive adjustment coefficient. The dot over variables stands for time differentiation. The first equation of (1) defines an adjustment process of the national income. Accordingly, national income rises or falls if investment is larger or smaller than savings. The second equation, in which $\varphi(\dot{y}(t))$ denotes the induced investment, describes an accumulation process of capital stock based on the acceleration principle. According to the principle, investment depends on the rate of changes in the national income. A distinctive feature of Goodwin' model is to introduce a nonlinearity into the investment function in such a way that the investment is proportional to the change in the national income in the neighborhood of the equilibrium income but becomes inflexible (i.e., less elastic) for extremely larger or smaller values of the income. This *nonlinear* acceleration principle is crucial in obtaining endogenous cycles in Goodwin's model. We will next retain this nonlinear assumption and specify its explicit form. On the other hand, we depart from Goodwin's non-essential assumption of positive autonomous expenditure and will work with zero autonomous expenditure for the sake of simplicity. A direct consequence of this assumption is that an equilibrium solution or a stationary point of the basic model is $y(t) = \dot{y}(t) = 0$ for all t.

Inserting the second equation of (1) into the first one and movining the terms on the left hand side to the right gives a single dynamics equation for the national income y,

$$\varepsilon\dot{y}(t) - \varphi(\dot{y}(t)) + (1-\alpha)y(t) = 0. \tag{2}$$

This is a nonlinear differential equation. Although it is one-dimensional, its nonlinearity prevents deriving an explicit form of the solution. In spite of this simple form, it is possible to detect local dynamics by examining its linearized version in a neighborhood of the stationary point and global dynamics by performing numerical simulations.

The linear version of the income dynamic equation (2) is

$$\varepsilon\dot{y}(t) - \nu\dot{y}(t) + (1-\alpha)y(t) = 0, \tag{3}$$

where $\nu = \varphi'(0)$ is the slope of the investment function at the stationary point. This is a first-order ordinary differential equation. Applying separation of variables gives a complete solution,

$$y(t) = y_0 e^{\lambda t} \text{ with } \lambda = \frac{1-\alpha}{\nu - \varepsilon}, \tag{4}$$

where y_0 is an initial condition. The stationary point is stable or unstable according to whether the eigenvalue λ is negative or positive. Since $1-\alpha$ is the positive marginal propensity to save, the sign of the eigenvalue depends on whether the numerator is positive or negative. Thus the stationary point is locally stable if $\nu < \varepsilon$ and unstable if $\nu > \varepsilon$.

We turn our attentions to global dynamics under the assumption of local instability, namely, where $\nu > \varepsilon$. We first specify the investment function as well as the values of the coefficients of (2) and then perform simulations to see what dynamics of y can be generated by the nonlinear equation (2). Although Goodwin (1951) assumed piecewise linear investment function, we, for the sake of analytical convenience, adopt a smooth nonlinear investment function of the form of an arctangent,

$$\varphi(\dot{y}(t)) = \delta \left\{ \tan^{-1}(\dot{y}(t) - a) - \tan^{-1}(-a) \right\}, \quad \delta > 0 \text{ and } a > 0. \tag{5}$$

This function has endogenous "ceiling" and "floor" and is asymmetric when the parameter, a, is non-zero. In what follows, we set $a = 1$ and $\delta = \frac{12}{\pi}$ for which the investment function (5) passes through the origin, and its ceiling is three time higher than its floor as it was the case in Goodwin's model.

In numerical examples below, the initial point is set at $y(0) = 10$, and we set of $\varepsilon = 0.5$ as the adjustment coefficient and $\alpha = 0.6$ as the marginal propensity to consume as in Goodwin's model. The numerical results are given in Figure 1 in which an endogenous cycle is illustrated on the right and the corresponding time path of output on the left. Along the upside-down N-shaped locus on the right, the initial point, denoted by I_0, is displaced slightly upward to point A so that the output is increasing to y_H, the highest level. Investment immediately switches discontinuously from positive to negative. Consequently, the orbit jumps from point A to point B. With negative $\dot{y}(t)$ at point B, the national income gradually declines from point B to point C so that the output is decreasing to y_L, the lowest level. Once point C is reached, investment switches again discontinuously from negative to positive. In other words, the orbit jumps again to point D from point C, from which the national income glides toward point A, and then the process repeats itself. Thus we have a closed orbit constituting a self-sustaining cycle. The points A and C are critical points at which one of the variables makes a discontinuous jump. The same dynamics are depicted as a function of time t on the left in Figure 1. The kinked points of the time-trajectory correspond to the discontinuous jumps of the cyclic behavior. This is a simple exhibition of generating an endogenous cycle of output. We can summarize these numerical results as follows.

Result 1 *Given $\nu > \varepsilon$ (i.e., locally unstable stationary point), a slow-rapid limit cycle is brought about in the basic model due to the nonlinear investment accelerator (5).*

Insert Figure 1 Here.

2.2 Delay Model with Fixed Time Lags

Due to the fact that in real economy, plans and their realizations need time to take effects, Goodwin (1951) introduces the investment lag, θ, between decisions to invest and the corresponding outlays. Inserting θ into the investment function of the basic model yields the third version,

$$\varepsilon\dot{y}(t) - \varphi(\dot{y}(t-\theta)) + (1-\alpha)y(t) = 0. \tag{6}$$

This is a *neutral delayed nonlinear differential equation*, which we call the *fixed delay model.* Goodwin does not analyze dynamics generated by this fixed delay model. Furthermore, to the best of our knowledge, no analytical solutions of the delayed model are available yet. However, it is possible, again, to investigate dynamics of the delayed model by using linearization for local dynamics and numerical simulations for global dynamics. Since a cyclic oscillation has been shown to exist in the basic model, our main concern is to see how the presence of the investment lag affects characteristics of such a slow-rapid cycle. To this end, we analytically investigate the stability of the cycle generated in the linearized model and numerically detect what effects are caused by the lag on cyclical dynamics.

The fixed delay model is autonomous and its special solution is constant (i.e., $y(t) = 0$) so that its linearized version takes the form of a *linear neutral autonomous delay* differential equation,

$$\varepsilon\dot{y}(t) - v\dot{y}(t-\theta) + (1-\alpha)y(t) = 0. \tag{7}$$

It is well known that if the characteristic polynomial of a linear neutral equation has roots only with negative real parts, the stationary point is locally asymptotically stable. The normal procedure for solving this equation is to try an exponential form of the solution. Substituting $y(t) = y_0 e^{\lambda t}$ into (7) and rearranging terms, we obtain the corresponding characteristic equation:

$$\varepsilon\lambda - v\lambda e^{-\lambda\theta} + (1-\alpha) = 0.$$

To check stability, we determine conditions under which all roots of this characteristic equation lie in the left or right of the complex plane. Dividing both sides of the characteristic equation by ε and introducing the new variables $A = \frac{1-\alpha}{\varepsilon}$ and $B = -\frac{v}{\varepsilon}$, we rewrite the characteristic equation as

$$\lambda + A + B\lambda e^{-\lambda\theta} = 0. \tag{8}$$

Kuang (1993) derives explicit conditions for stability/instability of the n-th order linear real scalar neutral differential difference equation with a single delay. Since (8) is a special case of the n-th order equation, applying the result of Kuang (1993, Theorem 1.2) implies that the real parts of the solutions of equation (8) are positive for all θ if $|B| > 1$. Hence we have the following result.

Theorem 1 *If $\nu > \varepsilon$, then the stationary point of (7) is unstable for all $\theta > 0$.*

If $v < \varepsilon$ (i.e., $|B| < 1$), (8) has at most finitely many eigenvalues with positive real part. The roots of the characteristic equation are functions of the delay. As the lengths of the delay change, the roots may change their signs from positive to negative or vise versa so that the stability of the solution may also change. Such phenomena are often referred to as *stability switches*. We will next show that such stability switchings cannot take place in the fixed delayed model.

For the following discussion we assume that $v < \varepsilon$. The case $v = \varepsilon$ will be treated later as a critical case. It can be checked that $\lambda = 0$ is not a solution of (8) because substituting $\lambda = 0$ yields $A = 0$ that contradicts $A > 0$. In the case of $v < \varepsilon$, Kuang (1993, Theorem 1.4) shows that if the stability switches at $\theta = \bar{\theta}$, then (8) must have a pair of pure conjugate imaginary roots with $\theta = \bar{\theta}$. Thus to find the critical value of $\bar{\theta}$, we assume that $\lambda = i\omega$, with $\omega > 0$ is a root of (8) for $\theta = \bar{\theta}$, $\bar{\theta} \geq 0$. Substituting $\lambda = i\omega$ into (8), we have

$$A + B\omega \sin \omega\theta = 0,$$

and

$$\omega + B\omega \cos \omega\theta = 0.$$

Moving A and ω to the right hand side and adding the squares of the resultant equations, we obtain

$$A^2 + (1 - B^2)\omega^2 = 0.$$

Since $A > 0$ and $1 - B^2 > 0$ as $|B| < 1$ is assumed, there is no ω that satisfies the above equation. In other words, there are no roots of (8) crossing the imaginary axis when θ increases. Therefore, there are no stability switches for any θ.

In case $\varepsilon = \nu$ in which $|B| = 1$, the characteristic equation becomes

$$\lambda(1 - e^{-\lambda\theta}) + A = 0. \tag{9}$$

It is clear that $\lambda = 0$ is not a solution of (9) since $A > 0$. Thus we can assume that a root of (9) have non-negative real part, $\lambda = u + iv$ with $u \geq 0$ for some $\theta > 0$. From (9), we have

$$(u + A)^2 + v^2 = e^{-2u\theta}(u^2 + v^2) \leq (u^2 + v^2),$$

where the last inequality is due to $e^{-2u\theta} \leq 1$ for $u \geq 0$ and $\theta > 0$. Hence

$$2uA + A^2 \leq 0,$$

where the direction of inequality contradicts the assumption that $u \geq 0$ and $A > 0$ in case $\varepsilon = \nu$. Hence it is impossible that the characteristic equation has roots with nonnegative real parts. Therefore, all roots of (9) must have negative real parts for all $\theta > 0$. Summarizing the above discussions gives the following theorem.

Theorem 2 *In case of $v \leq \varepsilon$, the the stationary point of (7) is asymptotically stable for all $\theta > 0$.*

Result 1 and Theorems 1 and 2 show that the stability condition of the fixed delay model is the same as the one of the basic model. Thus it can be seen that if the basic model is stable (resp. unstable), then the fixed delay model is also stable (resp. unstable). In other words, introducing fixed time lag does not change the stability condition of the basic model. It is, however, numerically confirmed that the fixed production lag has distinctive effects on the global dynamics as shown in Figure 2 in which we illustrate four different limit cycles for four different values of θ. It can be observed that the limit cycles change their shapes from parallelorgram-type cycle to vertically elongated parallelogram with rounded corners as the investment lag increases from 0.125 to unity by doubling θ, that is, the width of the cycles becomes smaller and the height becomes larger. Since the peak and the bottom of the output-cycle are reached at the point where the time derivative is zero, the amplitude of the cycle is equal to the distance between the two points at which the limit cycles cross the vertical axis. It is also observed that the amplitude of the cycles increases and the rate of output change becomes smaller as the production lag increases. These numerical results are summarized in:

Result 2 *The fixed time delay gets rid of discontinuous jumps and increasing investment lag makes the length of cycles longer and its amplitude larger in the case of $\nu > \varepsilon$.*

Insert Figure 2 about here.

2.3 Delay Model with Continuously Distributed Lags

Continuously distributed time delay is an alternative approach to deal with a time lag in investment. If the expected change of national income is denoted by $\dot{y}^e(t)$ at time t and is based on the entire history of the actual changes of national income from zero to t, the dynamic system can be written as the system of integro-difference equations,

$$\begin{aligned} &\varepsilon\dot{y}(t) - \varphi(\dot{y}^e(t)) + (1-\alpha)y(t) = 0, \\ &\dot{y}^e(t) = \int_0^t w(t-s,\theta,m)\dot{y}(s)ds, \end{aligned} \tag{10}$$

where the weighting function is

$$w(t-s,\theta,m) = \begin{cases} \dfrac{1}{\theta}e^{-\frac{t-s}{\theta}} & \text{if } m = 0, \\ \dfrac{1}{m!}\left(\dfrac{m}{\theta}\right)^{m+1}(t-s)^m e^{-\frac{m(t-s)}{\theta}} & \text{if } m \geq 1. \end{cases}$$

Here m is a nonnegative integer and θ is a positive real parameter, which is associated with the length of the delay. We call this dynamic system the *distributed delay model*.

To examine local dynamics of the above system in the neighborhood of the stationary point, we consider the linearized version,

$$\varepsilon\dot{y}(t) - \nu\int_0^t w(t-s,\theta,m)\dot{y}(s)ds + (1-\alpha)y(t) = 0.$$

Looking for the solution in the usual exponential form

$$y(t) = y_0e^{\lambda t} \text{ and } \dot{y}(t) = \lambda y_0 e^{\lambda t},$$

we substitute $y(t)$ and $\dot{y}(t)$ into the linearized version to obtain

$$\varepsilon\lambda - \nu\lambda\int_0^t w(t-s,\theta,m)e^{-\lambda(t-s)}ds + (1-\alpha) = 0.$$

As shown in Bischi et al, (2007), introducing the new variable $z = t-s$ simplifies the integral as

$$\int_0^t w(t-s,\theta,m)e^{-\lambda(t-s)}ds = \int_0^t w(z,\theta,m)e^{-\lambda z}dz.$$

By letting $t \to \infty$ and assuming that $\mathrm{Re}(\lambda) + \frac{m}{\theta} > 0$, we have

$$\int_0^\infty \frac{1}{\theta} e^{-\frac{z}{\theta}} e^{-\lambda z} dz = (1 + \lambda\theta)^{-1} \text{ if } m = 0,$$

and

$$\int_0^\infty \frac{1}{m!} \left(\frac{m}{\theta}\right)^{m+1} z^m e^{-\frac{mz}{\theta}} e^{-\lambda z} dz = (1 + \frac{\lambda\theta}{m})^{-(m+1)} \text{ if } m > 1.$$

That is,

$$\int_0^\infty w(z, \theta, m) e^{-\lambda z} ds = \left(1 + \frac{\lambda\theta}{q}\right)^{-(m+1)}$$

with

$$q = \begin{cases} 1 & \text{if } m = 0, \\ m & \text{if } \ m \geq 1. \end{cases}$$

Then the characteristic equation becomes

$$(\varepsilon\lambda + (1 - \alpha)) \left(1 + \frac{\lambda\theta}{q}\right)^{m+1} - \nu\lambda = 0. \tag{11}$$

If there are no time delays, $\theta = 0$, then the above equation is reduced to the same characteristic equation as the one we have already derived above. We will next show some simple cases in which analytical results can be obtained.

Since the case of $m = 0$ will be rigorously discussed in the next section, we examine stability in cases with $m \geq 1$. We expand the characteristic equation (11) by using the binomial theorem to obtain,

$$a_0\lambda^{m+2} + a_1\lambda^{m+1} + ... + a_{m+1}\lambda + a_{m+2} = 0, \tag{12}$$

where the coefficients a_i are defined as

$$\begin{aligned} a_0 &= \varepsilon\theta^{m+1} > 0, \\ a_k &= \left\{ \begin{pmatrix} m+1 \\ k+1 \end{pmatrix} m\varepsilon + \begin{pmatrix} m+1 \\ k \end{pmatrix} (1-\alpha)\theta \right\} m^k \theta^{m-k} > 0 \text{ for } k = 1, 2, ...m, \\ a_{m+1} &= m^m\{m\varepsilon + (m+1)(1-\alpha)\theta - m\nu\} \gtreqless 0, \\ a_{m+2} &= m^{m+1}(1-\alpha) > 0. \end{aligned}$$

According to the Routh-Hurwitz criterion, the necessary and sufficient conditions that all roots of the characteristic equation (12) have negative real parts are the following:

(1) the coefficients are positive, $a_k > 0$ for $k = 1, 2, ...2m + 1$,

(2) the principle minors of the Routh-Hurwitz determinant are positive,

$$D_2^m = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} > 0,\ D_3^m = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_0 \\ a_5 & a_4 & a_3 \end{vmatrix} > 0,\ D_4^m = \begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ a_5 & a_4 & a_3 & a_2 \\ a_7 & a_6 & a_5 & a_4 \end{vmatrix} > 0, ...$$

Case 1. $m = 1$

Substituting $m = 1$ into (12) yields

$$a_0\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{13}$$

where

$$\begin{aligned} a_0 &= \varepsilon\theta^3 > 0, \\ a_1 &= (2\varepsilon + (1-\alpha)\theta)\theta > 0, \\ a_2 &= \varepsilon + (1-\alpha)2\theta - \nu \gtreqless 0, \\ a_3 &= 1 - \alpha > 0. \end{aligned}$$

It can be seen that the sign of a_2 is not determined. In addition to $a_2 > 0$, the Routh-Hurwitz criterion requires that the following second- and third-order Routh-Hurwitz determinants are positive,

$$D_2^{m=1} = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} > 0 \text{ and } D_3^{m=1} = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & 0 & a_3 \end{vmatrix} > 0.$$

Since $D_3^{m=1} = a_3 D_2^{m=1}$ and $a_3 = 1 - \alpha > 0$, we have

$$\operatorname{sign}\left(D_3^{m=1}\right) = \operatorname{sign}\left(D_2^{m=1}\right),$$

where

$$D_2^{m=1} = \theta\left\{2(\varepsilon + (1-\alpha)\theta)^2 - (2\varepsilon + (1-\alpha)\theta)\nu\right\}. \tag{14}$$

Consider an $a_2 = 0$ curve and a $D_2^{m=1} = 0$ curve in the positive quadrant of the (θ, ν)-plane. First we assert that both curves are upward sloping and intersect only once for $\theta = 0$. Moreover we can assert that $a_2 < 0$ to the left of the $a_2 = 0$ curve and $a_2 > 0$ to the right and that $D_2^{m=1} < 0$ to the left of the $D_2^{m=1} = 0$ curve and $D_2^{m=1} > 0$ to the right. Substituting $a_2 = 0$ into $D_2^{m=1}$ gives $D_2^{m=1} = -(1-\alpha)\varepsilon\theta^2 < 0$. This implies that the $a_2 = 0$ curve is located in the region where $D_2^{m=1} < 0$. Thus, $a_2 > 0$ and $D_3^{m=1} > 0$ in the region where $D_2^{m=1} > 0$. Therefore the $D_2^{m=1} = 0$ curve is the partition line that divides the (θ, ν)-plane into two regions: one region below the line in which the stationary state is stable as the Routh-Hurwitz criterion is satisfied and the other region above the line in which the stationary state is unstable. Solving $D_2^{m=1} = 0$ for ν yields the explicit expression of the partition line,

$$\nu = \frac{2(\varepsilon + (1-\alpha)\theta)^2}{2\varepsilon + (1-\alpha)\theta}. \tag{15}$$

We now return to equation (13) to show the existence of a limit cycle even with continuously distributed delay by applying the Hopf bifurcation theorem. According to the theorem, we can establish the existence if the cubic characteristic equation has a pair of pure imaginary roots and the real part of these roots vary with a bifurcation parameter. We select ν as the bifurcation parameter and then calculate its value at the point for which loss of stability just occurs. Substituting (15) into (13), we can obtain a factorized expression of the characteristic equation along the partition line,

$$(2\varepsilon + (1-\alpha)\theta + \varepsilon\theta\lambda)(1 - \alpha + (2\varepsilon\theta + (1-\alpha)\theta^2)\lambda^2) = 0,$$

which can be explicitly solved for λ. One of the characteristic roots is real and negative and the other two are pure imaginary:

$$\begin{aligned}
\lambda_1 &= -\frac{2\varepsilon + (1-\alpha)\theta}{\varepsilon\theta} < 0, \\
\lambda_{2,3} &= \pm i\sqrt{\frac{1-\alpha}{2\varepsilon\theta + (1-\alpha)\theta^2}} = \pm i\omega.
\end{aligned}$$

In order to apply the Hopf bifurcation theorem, we need to check whether the real part of the conjugate complex roots change its sign as the bifurcation parameter passes through its critical value. Suppose that λ depends on ν, $\lambda(\nu)$, and then implicit-differentiation of (13) shows that

$$3\varepsilon\theta^3\lambda^2 + \left(2\varepsilon\theta + (1-\alpha)\theta^2\right)\lambda + \varepsilon + (1-\alpha)2\theta - \nu\frac{d\lambda}{d\nu} = \lambda.$$

Thus

$$\begin{aligned}
sign\left\{\frac{d(\operatorname{Re}\lambda)}{d\nu}\right\}_{\lambda=i\omega} &= sign\left\{\operatorname{Re}\left(\frac{d\lambda}{d\nu}\right)^{-1}\right\}_{\lambda=i\omega} \\
&= sign\left\{(2\varepsilon\theta + (1-\alpha)\theta^2)\right\}
\end{aligned}$$

where we used the facts that the terms with λ are imaginary and the constant terms are real. Therefore we have

$$\left.\frac{d(\operatorname{Re}\lambda)}{d\nu}\right|_{\lambda=i\omega} > 0.$$

This implies that the roots cross the imaginary axis at $i\omega$ from left to right as ν increases. Therefore the Hopf bifurcation theorem applies, and thus a birth of limit cycles is assured around the stationary point. The left part of Figure 3 illustrates a limit cycle in a 3D space when the stationary state is unstable, and the right side shown an orbit approaching the stationary state when stable.[2]

Insert Figure 3 about here.

Case 2. $m = 2$

The characteristic equation is quartic in λ and its coefficients are all positive except a_3 whose sign is not determined,

$$a_3 = 4(2\varepsilon + 3(1-\alpha)\theta) - 2\nu).$$

The Routh-Hurwitz determinants can be defined in the same way as before,

$$D_2^{m=2} = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix}, \; D_3^{m=2} = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ 0 & a_4 & a_3 \end{vmatrix} \text{ and } D_4^{m=2} = \begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_4 & a_3 & a_2 \\ 0 & 0 & 0 & a_4 \end{vmatrix},$$

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[2] $z(t)$ and $w(t)$ are defined as follows:

$$z(t) = \int_0^t \frac{1}{\theta^2}(t-s)\,e^{-\frac{t-s}{\theta}}\dot{y}(s)ds$$

and

$$w(t) = \int_0^t \frac{1}{\theta}e^{-\frac{t-s}{\theta}}\dot{y}(s)ds.$$

ν is selected as

$$\frac{2(\varepsilon + (1-\alpha)\theta)^2}{2\varepsilon + (1-\alpha)\theta} + 0.05 \text{ in the unstable case,}$$

and

$$\frac{2(\varepsilon + (1-\alpha)\theta)^2}{2\varepsilon + (1-\alpha)\theta} - 0.05 \text{ in the stable case.}$$

where

$$D_2^{m=2} = 2\theta^3\left\{32\varepsilon^2 + 3(1-\alpha)^2\theta^2 + \varepsilon(18(1-\alpha)+4v)\right\} > 0,$$

$$D_3^{m=2} = 8\theta^3\left\{8(2\varepsilon+(1-\alpha)\theta)^3 - 2(28\varepsilon^2 + 12(1-\alpha)\varepsilon\theta) + 3(1-\alpha)^2\theta^2)\nu - 8\varepsilon\nu^2\right\},$$

$$D_4^{m=2} = 8(1-\alpha)D_3^{m=2}.$$

By the same way as given in the case of $m = 1$, we assert that the $a_3 = 0$ curve and the $D_3^{m=2} = 0$ curve are upward sloping, intersect once for $\theta = 0$ and the former curve is located above the latter curve. Substituting $a_3 = 0$ into $D_3^{m=2}$ gives

$$D_3^{m=2} = -64(1-\alpha)^2\theta^4(6\varepsilon + (1-\alpha)\theta)^2 < 0.$$

The direction of this inequality implies that the $a_3 = 0$ curve is located in the region in which $D_3^{m=2} < 0$. Thus we have $a_3 > 0$ and $D_4^{m=2} > 0$ in the region in which $D_3^{m=2} > 0$. Hence the $D_3^{m=2} = 0$ locus is the partition line that divides the (θ, ν)-plane into two regions: one region below the line in which the stationary state is stable and the other region above the line in which the stationary state is unstable. From $D_2^{m=1} = 0$ we obtain the explicit expression of the partition line,

$$8(2\varepsilon+(1-\alpha)\theta)^3 - 2(28\varepsilon^2 + 12(1-\alpha)\varepsilon\theta) + 3(1-\alpha)^2\theta^2)\nu - 8\varepsilon\nu^2 = 0. \quad (16)$$

It can be confirmed that the $D_2^{m=1} = 0$ curve is steeper than the $D_3^{m=2} = 0$ curve, and both curves have the same vertical intercept for $\theta = 0$. This means that the distributed delay model with $m = 1$ has a larger stable region than the model with $m = 2$. By the same procedure as in the case of $m = 1$ above, we can show the birth of a limit cycle in the case of $m = 2$ as well.

After we repeat the above procedure for all values of m, it is then possible to show that $D_{m+1} > 0$ implies $a_{m+1} > 0$ and $D_i > 0$ for all i. Hence the $D_{m+1}^m = 0$ locus is the partition line that divides the (θ, ν)-plane into two regions, the stable region and the unstable region. The five partition lines with m from 1 to 5 are depicted in Figure 4. It can be seen that all lines cross the vertical axis for $\nu = \varepsilon$ and their slopes become smaller as m increases. Notice that the dotted horizontal line is the partition line in the case of fixed time delay. This implies that the stable region becomes smaller as the value of m increases and converges to the region defined with the fixed time delay when m tends to infinity.

Insert Figure 4 about here.

3 Coexistence of Multiple Cycles

In this section, we demonstrate a coexistence of multiple cycles when the stationary point is locally stable. This finding implies the robustness of cyclic properties of Goodwin's model regardless of whether the stationary point is locally stable or unstable. In what follows, we select the slope of the investment function evaluated at the stationary point as the bifurcation parameter and investigate the possibility of the supercritical bifurcation in Section 3.1. After proving the existence of an unstable limit cycle, we construct an invariant set in the state space and apply the Poincaré-Bendixson theorem to find a stable limit cycle that encloses the unstable limit cycle in Section 3.2. The coexistence of multiple cycles has been already shown for a multiplier-accelerator model in Puu (1986), for Kaldor's business cycle model in Grasman and Wentzel (1994) and for a Metzlerian inventory cycle model in Matsumoto (1996) using an approach that will be further explored.

3.1 Distributed Delay Model with $m = 0$

We consider the distributed delay model with $m = 0$ in which the weighting function becomes exponentially declining and thus gives the most weight to the most recent income change. The dynamic system of the Volterra-type integro-differential equation is

$$\begin{aligned} &\varepsilon \dot{y}(t) - \varphi(\dot{y}^e(t)) + (1-\alpha)y(t) = 0, \\ &\dot{y}^e(t) = \int_0^t \frac{1}{\theta} e^{-\frac{t-s}{\theta}} \dot{y}(s)ds. \end{aligned} \tag{17}$$

As shown below, we have rewritten the continuously distributed delay equation as a system of ordinary differential equations. By doing so, we can use all tools known from the stability theory of ordinary differential equations to analyze the asymptotic behavior of the stationary point.

The time-differentiation of the second equation of (17) and introducing $z = \dot{y}^e$ reduce it to

$$\dot{z}(t) = \frac{1}{\theta}\left(\dot{y}(t) - z(t)\right). \tag{18}$$

Solving the first equation for $\dot{y}$, replacing $\dot{y}^e$ with z, replacing $\dot{y}$ in (18) with the new expression of $\dot{y}$ and then adding the new dynamic equation of z will transform the integro-differential equation to the following 2D system of ordinary differential equations,

$$\begin{aligned}\dot{y}(t) &= -\frac{1-\alpha}{\varepsilon}y(t) + \frac{1}{\varepsilon}\varphi(z(t)),\\ \dot{z}(t) &= \frac{1}{\theta}\left(-\frac{1-\alpha}{\varepsilon}y(t) + \frac{1}{\varepsilon}\varphi(z(t)) - z(t)\right).\end{aligned} \tag{19}$$

It can be seen that the stationary state of (19) is $y = z = 0$. By linearizing the system at the stationary state, we get the Jacobian matrix,

$$J = \begin{pmatrix} -\dfrac{1-\alpha}{\varepsilon} & \dfrac{\nu}{\varepsilon} \\ -\dfrac{1-\alpha}{\varepsilon\theta} & \dfrac{1}{\theta}\left(\dfrac{\nu}{\varepsilon} - 1\right) \end{pmatrix}.$$

The corresponding characteristic equation is quadratic in λ,[3]

$$\lambda^2 + \frac{\varepsilon + (1-\alpha)\theta - \nu}{\varepsilon\theta}\lambda + \frac{1-\alpha}{\varepsilon\theta} = 0.$$

Setting $k = \nu - [\varepsilon + (1-\alpha)\theta]$ gives,

$$\lambda_{1,2} = \frac{1}{2}\left\{\frac{k}{\varepsilon\theta} \pm \sqrt{\left(\frac{k}{\varepsilon\theta}\right)^2 - \frac{4(1-\alpha)}{\varepsilon\theta}}\right\}.$$

The product of the eigenvalues is positive,

$$\lambda_1\lambda_2 = \frac{1-\alpha}{\varepsilon\theta} > 0$$

due to the assumptions imposed on parameters, $0 < \alpha < 1$ and $(\varepsilon, \theta) > 0$. These parametric restrictions ensure that the stationary point is not a saddle point. It also follows that the sum of the eigenvalues is either positive or negative according to whether k is negative or positive,

$$\lambda_1 + \lambda_2 = \frac{k}{\varepsilon\theta} \gtreqless 0 \Leftrightarrow k \gtreqless 0.$$

[3]Notice that this characteristic equation is identical with the characteristic equation that can be derived from Goodwin's approximated model (i.e., equation (5f) of Goodwin (1951)). This means that both equations generates exactly the same dynamics in the neighborhood of $\theta = 0$. See Szidarovszky and Matsumoto (2007) for the similarities and dissimilarities between the two dynamic equations.

To confirm the dependency of the stability on the parameters θ and ν, we define the parameter region by $\Omega = \{(\theta, \nu) \,|\theta > 0 \text{ and } \nu > 0\}$, considering the values of the other two parameters α and ε given. Since the stability of the stationary point depends of the sign of k so that a $k = 0$ curve becomes the partition line that divides Ω into two regions, one stable and the other unstable. Solving $k = 0$ gives the partition line

$$\nu = [\varepsilon + (1 - \alpha)\theta], \tag{20}$$

which is positive-sloping in Ω. Thus the stationary state is stable in the region below this line in which $k < 0$ and unstable in the region above. It also depends on the value of the discriminant of the characteristic equation whether the local dynamics is oscillatory or monotonic. The curve along which the discriminant is zero is determined by

$$\nu = [\varepsilon + (1 - \alpha)\theta] \pm 2\sqrt{(1 - \alpha)\varepsilon\theta}, \tag{21}$$

which distinguishes the parameter region for real roots from that for complex roots.

These two curves divide the parameter region Ω as shown in Figure 5. For combinations of θ and ν in a region marked as either [MS] (in the bottom left and bottom right corners) or [MU], the stationary state is monotonic stable and unstable while for those in [OS] or [OU], the stationary state is oscillatory stable and unstable.[4] In Figure 5, we plot θ on the horizontal axis and v on the vertical axis. For certain combinations of the parameters, θ and ν, in either the light-gray region or the dark-gray region, the characteristic roots are complex and thus the system produces oscillations. Moreover, the oscillations are explosive for the combinations in the light-gray region and damped for those in the dark-gray region. On the other hand, for the combinations of θ and ν in the white regions, the characteristic roots are real and thus the system produces monotonic dynamics that is convergent or divergent according to whether the combination is in the lower-white region or the upper-white region. Figure 5 also implies that the distributed delay model may produce qualitatively the same dynamics as ν increases regardless of the value of the production lag. Thus, for a given θ, the linearized model first generates monotonic stable dynamics, then oscillatory stable dynamics, oscillatory unstable dynamics and finally monotonic unstable dynamics as ν increases from zero.

[4] The partition line crosses the vertical axis for $\nu = \varepsilon$ and is positive sloping while the zero-discriminant locus is tangent to the vertical line for $\nu = \varepsilon$ and to the horizontal line for $\theta = \frac{\varepsilon}{1-\alpha}$. Thus the qualitatively similar divisions are obtained regardless of the specified values of parameters, although we specify $\alpha = 0.6$ and $\varepsilon = 0.5$ in Figure 5.

Insert Figure 5 about here.

3.2 Hopf Cycle

By applying the Hopf bifurcation theorem, we investigate whether there is a limit cycle in the stable distributed delay model with $m = 0$. According to the theorem, Hopf bifurcation occurs if the complex conjugate roots as functions of the bifurcation parameters cross the imaginary axis. Obviously, the characteristic roots are complex conjugate with zero real part if $k = 0$. As there are no other roots in the two-dimensional system, a limit cycle exists if the eigenvalues cross the imaginary axis with non-zero speed at the bifurcation point. Though there may exist several possibilities to parametrize the distributed delay model, it seems interesting to choose the slope of the investment function evaluated at the stationary point as the bifurcation parameter.

Since we consider the case of local stability (i.e., $k < 0$ or $\nu < \varepsilon + (1 - \alpha)\theta$) in this section, it can be easily seen that there exists a value ν_0 for which

$$\nu_0 - [\varepsilon + (1 - \alpha)\theta] = 0,$$

implying that the complex conjugate roots cross the imaginary axis. For $\nu > \nu_0$, respectively $\nu < \nu_0$, the real part becomes positive, respectively negative. Hence, v_0 is indeed a bifurcation value of the distributed delay model. Due to the Hopf bifurcation theorem, this establishes the existence of closed orbits in a neighborhood of the stationary point $(0, 0)$ at $\nu = \nu_0$.[5]

The Hopf theorem, however, has no indication about the nature of the limit cycle. There are two possibilities, one is that orbits spiral outward from the stationary point toward a stable limit cycle, called the *supercritical bifurcation*, and the other is that all orbits starting inside the cycle spiral in toward the stationary point and becomes explosive outside the cycle, called the *subcritical bifurcation*. To make the distinction between the sub- and super-critical Hopf bifurcation, we calculate the stability index.

The distributed delay model can be written as

$$\begin{pmatrix} \dot{y} \\ \dot{z} \end{pmatrix} = J \begin{pmatrix} y \\ z \end{pmatrix} + \begin{pmatrix} g^1(z) \\ g^2(z) \end{pmatrix}$$

[5] See Lorenz (1993) for the Hopf bifurcation theorem, the stability index and the following coordinate transformations.

where J is the Jacobian matrix defined above, and $g^i(z)$ for $i = 1, 2$ are nonlinear terms that can be derived as

$$\begin{aligned} g^1(z) &= \frac{1}{\theta}\left(\varphi(z) - \nu_0 z\right), \\ g^2(z) &= \frac{1}{\varepsilon\theta}\left(\varphi(z) - \nu_0 z\right). \end{aligned}$$

In order to transform the Jacobian matrix of the distributed delay model into the normal form, we introduce the coordinate transformation

$$\begin{pmatrix} y \\ z \end{pmatrix} = D \begin{pmatrix} u \\ v \end{pmatrix} \text{ with } D = \begin{pmatrix} 0 & 1 \\ d_{21} & d_{22} \end{pmatrix},$$

where

$$d_{21} = \frac{\varepsilon}{\nu_0\theta}\sqrt{\frac{\nu_0 - \varepsilon}{\varepsilon}} \text{ and } d_{22} = \frac{\nu_0 - \varepsilon}{\nu_0\theta}.$$

Since matrix D transforms the coordinate system (y, z) into a new coordinate system (u, v), the distributed delay model becomes

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & -\sqrt{\dfrac{1-\alpha}{\varepsilon\theta}} \\ \sqrt{\dfrac{1-\alpha}{\varepsilon\theta}} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \mathbf{a}g(u,v) \\ \mathbf{b}g(u,v) \end{pmatrix}$$

where

$$g(u, v) = \varphi(d_{21}u + d_{22}v) - \nu_0(d_{21}u + d_{22}v).$$

$$\mathbf{a} = \frac{1}{\sqrt{\varepsilon(\nu_0 - \varepsilon)}} \text{ and } \mathbf{b} = \frac{1}{\varepsilon}.$$

It is well-known that the stability of the emerging cycle depends on up to the third-order derivatives of the nonlinear function $g(u, v)$. The stability index is

$$\begin{aligned} I &= \frac{1}{16}\left[\mathbf{a}\left(g_{uuu} + g_{uvv}\right) + \mathbf{b}\left(g_{uuv} + g_{vvv}\right)\right] \\ &\quad + \frac{1}{16}\sqrt{\frac{\varepsilon\theta}{1-\alpha}}\left[\left(\mathbf{a}^2 - \mathbf{b}^2\right) g_{uv}\left(g_{uu} + g_{vv}\right) + \mathbf{ab}\left(\left(g_{uu}\right)^2 - \left(g_{vv}\right)^2\right)\right] \end{aligned}$$

where the partial derivatives are

$$g_{uu} = \varphi''\left(d_{21}\right)^2, \; g_{vv} = \varphi''\left(d_{22}\right)^2, \; g_{uv} = \varphi'' d_{21} d_{22},$$

$$g_{uuu} = \varphi'''(d_{21})^3, \; g_{vvv} = \varphi'''(d_{22})^3, \; g_{uvv} = \varphi''' d_{21}(d_{22})^2, \; g_{uuv} = \varphi'''\left(d_{21}\right)^2 d_{22}.$$

Then arranging terms gives

$$I = \frac{\varphi'''(0)\nu_0(\nu_0 - \varepsilon)}{16\varepsilon\theta^3} > 0.$$

This inequality indicates that the emerging cycle, enclosing the stable stationary point, is repelling (i.e., unstable). A subcritical Hopf bifurcation occurs for $\nu < \nu_0$ in the distributed delay model with $m = 0$. We can summarize our results as follows.

Theorem 3 *Given the stability of the stationary point, there is an unstable limit cycle that encloses a stable equilibrium in the distributed delay model with $m = 0$.*

3.3 Stable and Unstable Limit Cycles

Assuming that $\nu < \nu_0$, we investigate a number of a limit cycle in this subsection. As shown in Theorem 3, an unstable limit cycle exists for $v < v_0$ due to the Hopf theorem. We will also show that a stable limit cycle exists and it encloses the unstable Hopf cycle. We will finally construct an invariant set in such a way that once an orbit enters the set, it cannot escape from it at any future time and then will apply the Poincaré-Bendixon theorem to examine whether a stable cycle can arise in the set. A typical application of the theorem is in the case in which there is a single unstable equilibrium in some invariant set. Remove the neighborhood of the stationary point in which all orbits move away from it. Then the theorem ensures that the remaining set must contain a limit cycle.[6] Since the stationary point is now assumed to be stable in the distributed delay model, we need another approach for searching for such an invariant set.

In Figure 6 below, we may find a point A located so high on the vertical axis that an orbit starting there comes back to cross again the vertical axis at a lower point E after crossing not only the horizontal axis at points B and D but also the vertical axis at point C. Consider a region bounded by $ABCDEA$ and an open region bounded by the unstable limit cycle that is shown to exist in Theorem 3. Then the invariant set to be considered can be obtained by deleting the latter region from the former region. The result is shown in Figure 6 in which the shaded region represents this invariant set. The boundary of the inner white region corresponds to the unstable limit cycle that surrounds the stable stationary point. The shaded region thus has no stationary point, and any orbit starting inside the region stays within this

[6]See Chaper 2.2 of Lorenze (1993) for example.

region. Formally, the Poincaré-Bendixson theorem guarantees the existence of one stable limit cycle in the shaded region as illustrated as the bold cycle in Figure 6. Combining the discussion above with Theorem 3 yields the following:

Theorem 4 *Given the stability of the stationary point, a stable limit cycle coexists with an unstable limit cycle that encloses a stable equilibrium point for the distributed delay model with $m = 0$.*

Insert Figure 6 Here

In Figure 7, we present a bifurcation diagram in which the amplitude of the cycle is on the vertical axis and the bifurcation parameter ν on the horizontal axis. ν_u is the critical value at which the distributed delay model loses its stability, and a limit cycle is born. For $\nu > \nu_u$, the model is destabilized for small perturbations so that any orbit moves away from the stationary point. Nonlinearity of the model prevents it from diverging globally but leads to a unique stable limit cycle. This is essentially the same cycle as the one that Goodwin (1951) demonstrates by applying the Lienard method. On the other hand, for $\nu < \nu_u$, it is locally stabilized but generates an unstable cycle as well as a stable cycle for ν in the interval $[\nu_s, \nu_u]$. It can be seen that as ν decreases from ν_u, the amplitude of the inner unstable cycle increases and that of the outer stable cycle decreases. v_s is the other critical value for which the two cycles coincide. For a lower $\nu < \nu_s$, limit cycles no longer exit since the invariant set vanishes.

The coexistence of multiple cycles and a stable stationary point reminds us to "corridor stability," the notion which was introduced by Leijonhufvud (1973). It implies that a dynamic system is stable for small perturbations but unstable for large perturbations. The interior of the white elliptic region in Figure 7 is the corridor in which the stationary point is stable. Thus, if perturbations around the stationary point are small enough not to take the trajectory out of the corridor, the restoring effect of the model forces the trajectory to return to the stationary point. To the contrary, if perturbations are large enough to take orbits outside the corridor, the lasting effect of the model leads to persistent cyclic oscillations. The foregoing numerical analysis indicates that the nonlinearity of the investment function and the time delay can be the sources of corridor stability in the distributed delay model with $m = 0$.

Insert Figure 7 Here

4 Concluding Remarks

We reconsidered Goodwin's 1951 nonlinear accelerator model of business cycle and demonstrated two new features, the effects caused by investment lag on the characteristic of Goodwin's cycle when the stationary state was locally unstable and the coexistence of multiple business cycles when locally stable. Concerning the first feature, it is numerically confirmed that increasing investment lag makes the length of the business cycle longer and its amplitude larger. Moreover, it is analytically confirmed that the fixed investment lag has the stronger destabilizing effect in comparison to the continuously distributed investment lag.

Concerning the second feature, we showed, by combining the result obtained from the Hopf bifurcation theorem with the one due to Poincare-Bendixon theorem, that two limit cycles can coexist with the stable stationary state: one cycle is unstable and surrounds the stationary state, and the other is stable and encloses the unstable limit cycle. This finding indicates the corridor stability of Goodwin's model in which a damping force dominates and makes trajectories approach the stationary state for small disturbances but an anti-damping force dominates and makes trajectories converge to the outer stable limit cycle for larger disturbances. The results imply global stability of Goodwin's model regardless of the local dynamic properties.

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Caption of Figures

Figure 1. Limit cycle without time lag.

Figure 2. Limit cycles with time lags.

Figure 3. Limt cycle and stable trajectory in 3D system.

Figure4. Partion lines.

Figure 5. Stable and unstable regions.

Figure 6. Coexistence of multiple limit cycles.

Figure 7. Bifurcation diaram.

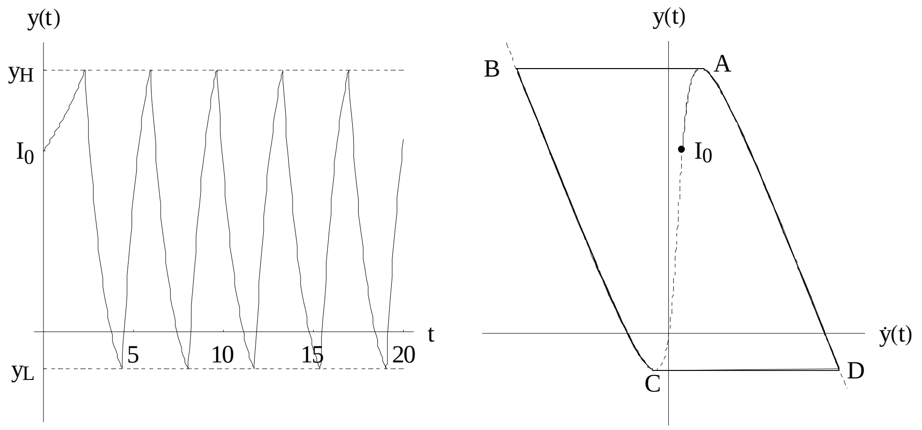

Figure 1:

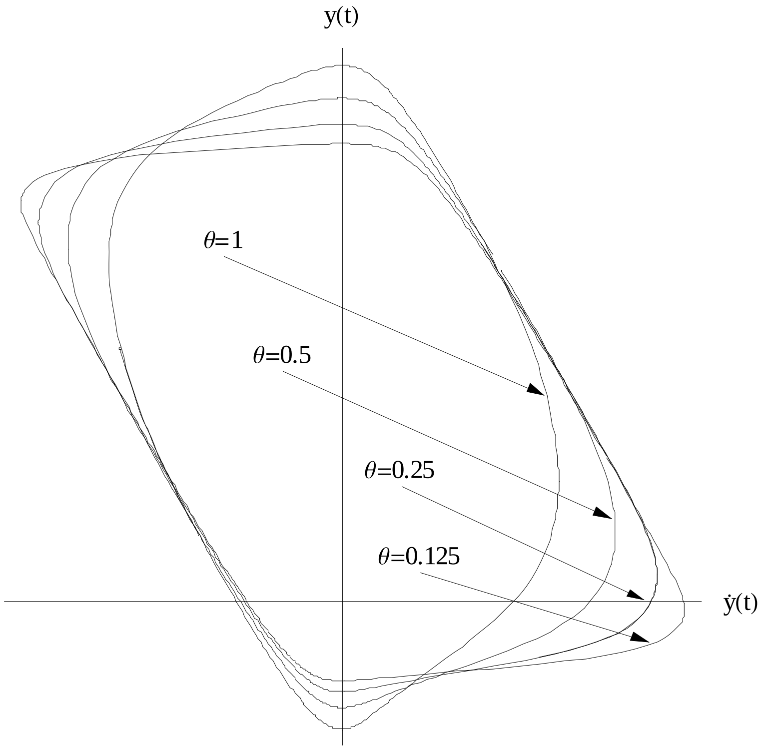

Figure 2:

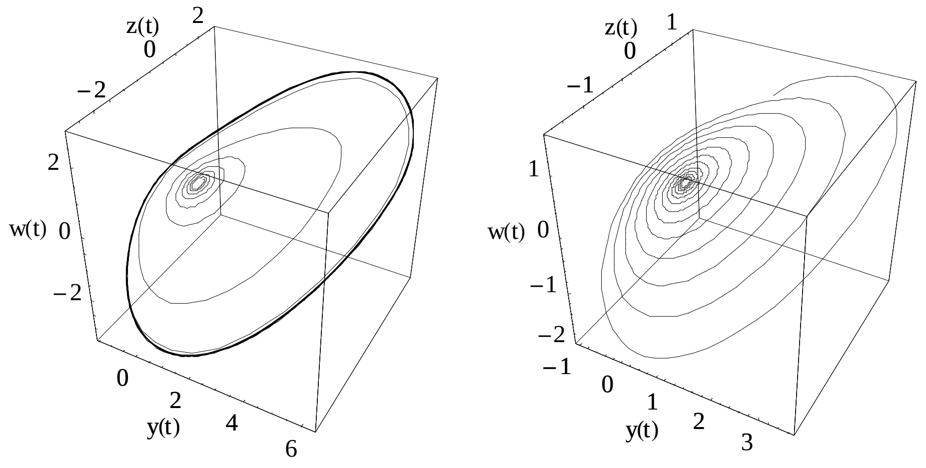

Figure 3:

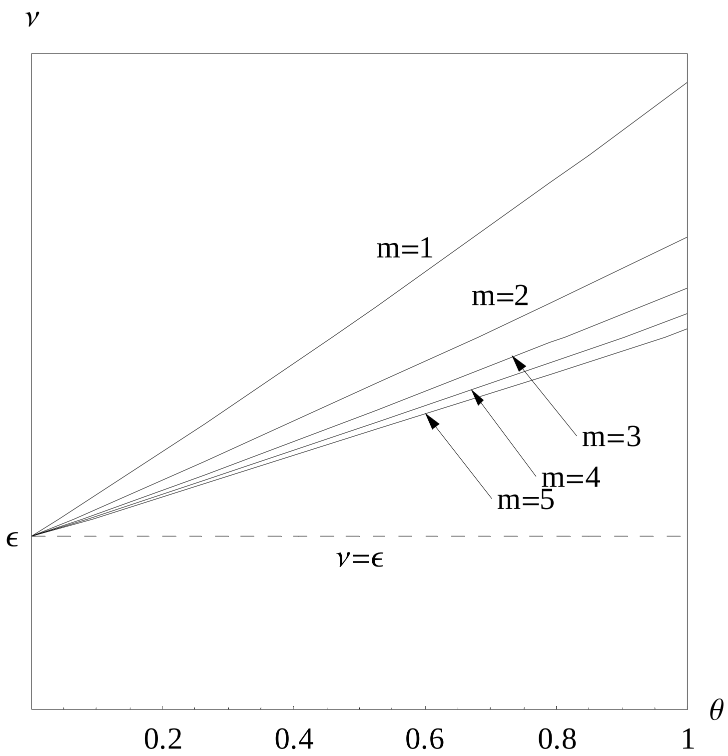

Figure 4:

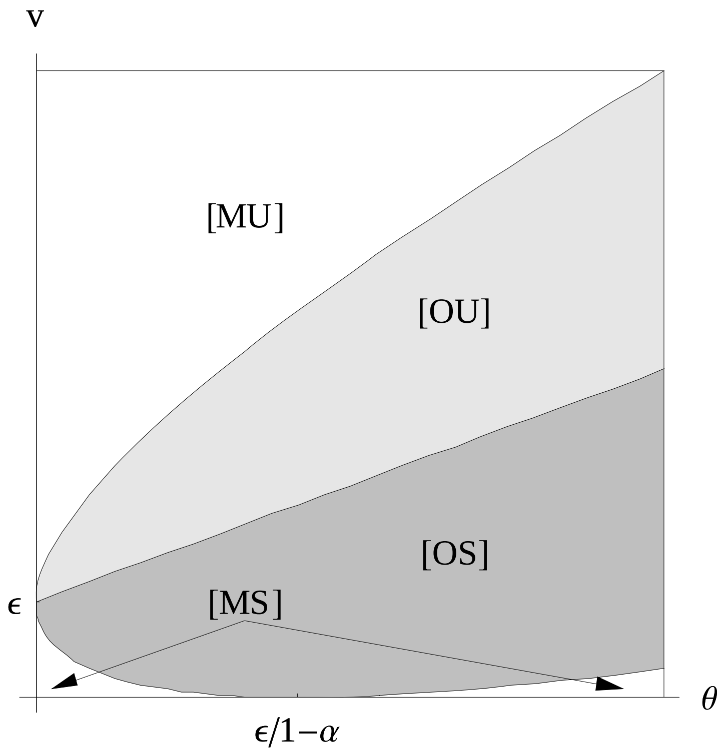

Figure 5:

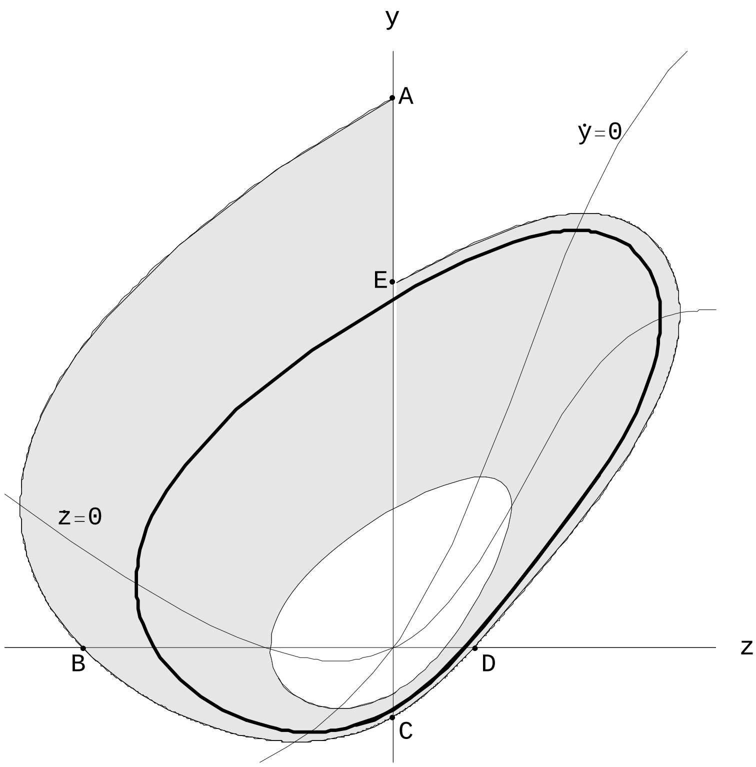

Figure 6:

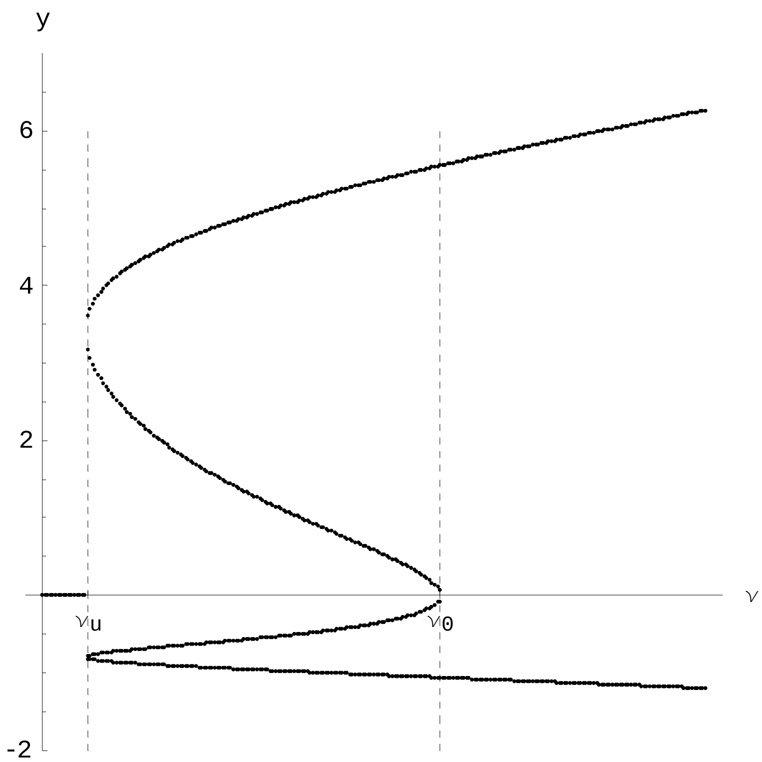

Figure 7: