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Monetary policy in the unique growth cycle of post Keynesian systems

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Abstract

This paper deals with the following two issues in a post Keynesian system: the existence and *uniqueness* of a growth cycle and the effectiveness of a counter-cyclical monetary policy for stabilization. It is found that the unique growth cycle, represented by the unique limit cycle, can be observed if the rate of interest is set to a constant level while that the long-run equilibrium can be globally asymptotically stable if an appropriate counter-cyclical monetary policy is conducted.

Keywords: Economic stability; Growth cycle; Keynesian economics; Monetary policy

JEL classification: C62; E12; E32; E52

1 Introduction

In economic theory, persistent cyclical fluctuations such as business cycles and growth cycles are usually described by periodic orbits including limit cycles,[1] and the main theme of theory of business cycles is, in a lot of cases, to establish the presence of a periodic orbit in dynamic models. Since the contributions of Kaldor (1940), Hicks (1950) and Goodwin (1951), in economic theory, the concept of "nonlinearity" has been recognized as a key element in "detecting" a periodic orbit.[2] Indeed, mathematical theory of nonlinear dynamical systems, such as the Poincaré-Bendixson theorem and the Hopf bifurcation theorem,[3] has intensively been employed to verify the existence of a

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[1]As is often the case in economics, limit cycles are often confused with periodic orbits, but mathematically, they are different (though slightly) concepts. Strictly speaking, a limit cycle is a periodic orbit such that there is another solution path of the dynamical system under consideration converges to it as $t \to \infty$ or as $t \to -\infty$. (cf. Coddington and Levinson 1955, pp. 391-392; Hirsch and Smale 1974, p. 250). In two-dimensional autonomous differential equations, periodic orbits are *generically* limit cycles due to the concept of structural stability (cf. Peixoto 1962), but there is a case in which a periodic orbit is not a limit cycle. In Goodwin's (1967) model of growth cycles, which can be reduced to a Lotka-Volterra system, all solution paths except for the equilibrium are periodic orbits, but none of them is a limit cycle (cf. Velupillai 1979; Flaschel 1984). In particular, if the number of periodic orbits is finite, they are all limit cycles, either stable or unstable.

[2]See Yasui (1953), Ichimura (1955) or Morishima (1959) for early development of "nonlinear" economic theory. For the contributions of these Japanese economists, see, for instance, Velupillai (2008) or Asada (2014).

[3]For the Poincaré-Bendixson theorem and the Hopf bifurcation theorem, see, for example, Coddington and Levinson (1955, chap. 16) and Marsden and McCracken (1976), respectively.

periodic orbit including a limit cycle.[4]

Although the mechanism of persistent business cycles can be explained by the existence of a periodic orbit, the existence does not necessarily imply the uniqueness. Unless the uniqueness of a periodic orbit is obtained, the characteristics of (persistent) business cycles such as periods and amplitudes of fluctuations can dramatically be changed after large external shocks occur because the macroeconomic system under consideration can converge to a periodic orbit different from the original one. With the uniqueness of a periodic orbit, on the other hand, the macroeconomic system converges to the unique (original) business cycle even after huge shocks.[5] It is thus much easier to predict future economic situations when a periodic orbit is unique than when it is not. In this respect, the uniqueness of a periodic orbit has a practical merit, especially in terms of predictions.

Unfortunately, however, the uniqueness of a periodic orbit has not much been examined in economic theory due to technical difficulty. Indeed, there have been only a few papers which address the uniqueness. Ichimura (1955), Kosobud and O'Neill (1972), Lorenz (1986, 1993) and Galeotti and Gori (1989) examined the possibility that a periodic orbit is unique in Kaldor's (1940) model of business cycles, but they failed to find a realistic sufficient condition for the uniqueness.[6] Sasakura (1996), on the other hand, established the uniqueness in Goodwin's (1951) model of business cycles without any stringent assumption, especially on the saving function, but the investment function in Goodwin's (1951) model is itself an unrealistic one based on the "acceleration principle" of investment.[7] Thus, the uniqueness of a periodic orbit has not, until recently, been established theoretically in realistic situations.

Recently, in Murakami (2018a, 2018b), I tackled the issue of the unique periodic orbit and verified the uniqueness, as well as existence, of a limit cycle, which can be interpreted as persistent growth cycles in terms of economics, in post Keynesian systems with realistic features. In these works, I made use of a linear Keynesian consumption or saving function which is empirically plausible and of two investment functions, one of which is consistent with Keynes' (1936) marginal efficiency theory of investment (cf. Murakami 2018a) and the other of which is consistent with the profit principle and the "utilization principle" of investment (cf. Murakami 2018b).[8] In these studies, I drew the conclusion that the unique persistent business (or growth) cycle is observed, without policy interventions, in the (post) Keynesian system, which is characterized by fast quantity (or utilization) adjustment or by frequent revisions of the long-term expectation. In them, however, I took no account of the role of policy implementations.

In this paper, I begin with reviewing the conclusion on the unique growth cycle drawn in Murakami (2018b) and

[4]For classical applications of the Poincaré-Bendixson theorem to economic theory, see, for instance, Rose (1967), Stiglitz (1967) or Chang and Smyth (1971), while, for those of the Hope bifurcation theorem, see, for instance, Torre (1977), Benhabib and Nishimura (1979) or Benhabib and Miyao (1981).

[5]Precisely speaking, the unique business cycle has to be (periodically) stable for obtaining this conclusion.

[6]Specifically, all of them had to assume that aggregate saving is a sigmoid (nonlinear) function of aggregate income (and of aggregate capital stock), but as Lorenz (1993) admitted, this assumption cannot be supported from an empirical viewpoint.

[7]The acceleration principle, adopted in Harrod (1936), Samuelson (1939), Hicks (1950) and Goodwin (1951), is often confused with the "profit principle" of investment, which states that investment is dependent on the level or rate of profit and was established by Kalecki (1939) and Kaldor (1940). As argued by Kaldor (1940, 1951), however, these two principles are different and the profit principle is more plausible from theoretical and empirical viewpoints than the acceleration principle is. Furthermore, as clarified by Chenery (1952), the acceleration principle implicitly assumes that capital stock is operated at full (or nearly full) capacity, but this assumption is neither realistic nor reasonable. Also, if variations in capacity utilization are taken into account and the so-called flexible accelerator is introduced, the accelerator principle is reduced to a simple version of the profit principle.

[8]The utilization principle states that investment is affected by the rate of utilization and was postulated by Steindl (1952, 1979) and Rowthorn (1981).

then discuss the possibility that the monetary authority can, with policy interventions, stabilize the macroeconomic system exposed to persistent economic fluctuations along the unique growth cycle. This paper is organized as follows. In Section 2, I set up a post Keynesian model based on Murakami (2018b). In Section 3, I analyze the characteristics of the post Keynesian model, separately, in the cases in which the rate of interest is kept constant and in which it is changed by the monetary authority's feedback (counter-cyclical) policy. Specifically, I confirm the conclusion given by Murakami (2018b) that, when the rate of interest is constant, a periodic orbit or a limit cycle, which can be viewed as persistent growth cycles, uniquely exists, if the speed of quantity adjustment is high enough (Section 3.1) and then demonstrate that the unique limit cycle (or growth cycle) is eliminated and the long-run equilibrium gains the "global" asymptotic stability, if the monetary authority conducts a proper (or "moderate") counter-cyclical interest rate policy (Section 3.2). In Section 4, I summarize my analysis and conclude this paper. In Appendix, I provide the mathematical theorem to be utilized in my analysis.

2 The model

In this section, I first formalize the consumption and investment functions from Keynesian perspectives and then set up a dynamical system composed of differential equations of the output-capital ratio, which can be identified with the rate of utilization,[9] and of per capita stock of capital.

2.1 Consumption

To begin, I describe aggregate consumption in the following way:

$$C = c_0 N + c(Y - T), \tag{1}$$

where c is a positive constant less than unity and c_0 is a nonnegative constant. In (1), Y, C, T and N stand for aggregate income or output, aggregate consumption expenditure, government income tax and the size of population, respectively; c and c_0 represent the marginal propensity to consume and the base level of individual consumption, respectively. Equation (1) implies that aggregate consumption C is affected positively by aggregate disposable income $Y - T$ and that aggregate base consumption is proportionate to population N.[10] The consumption function (1) can be adapted to the context of economic growth (or of economic decline) because aggregate base consumption $c_0 N$ varies at the same rate as the (natural) rate of economic growth.

[9] By assuming that the level of potential output is proportional to that of capital stock, I may identify the output-capital ratio with the rate of utilization because the latter is proportional to the former.

[10] The consumption function (1) is conceptually identical with those given by Murakami (2018a, 2018b). This consumption function can be related to the saving functions presented in recent studies on the Kaleckian system (cf. Allain 2015; Lavoie 2016).

2.2 Investment

Based on the Keynesian theory of investment, I assume that the rate of gross capital formation (i.e., the ratio of aggregate gross investment to aggregate capital stock) is expressed by a function of the rate of utilization (or the ratio of aggregate output to aggregate capital stock) and the rate of interest in the following way:

$$\frac{I}{K} = f(u, r). \tag{2}$$

In (2), I, K, u and r stand for aggregate gross investment, aggregate capital stock, the rate of utilization (or the output-capital ratio) Y/K and the rate of interest, respectively; f is the (gross) capital formation function. Equation (2) states that the rate of gross capital formation I/K is related to the rate of utilization u and to that of interest r, and it is consistent with the Keynesian theory of investment.[11]

Following the same argument as in Murakami (2018b), I give a specific form to the capital formation function f. For this purpose, I introduce the hypothetical postulate that all firms can choose only two types of investment behavior: "optimistic" and "pessimistic" plans. A firm is assumed to set its rate of gross capital formation to f_o (resp. f_p) if it chooses the "optimistic" (resp. "pessimistic") plan, where f_o and f_p are nonnegative constants with $f_p < f_o$.[12] Under this hypothesis, I can describe the rate of gross capital formation, in the aggregate sense, as a consequence of the distribution of heterogeneous firms.[13] Denoting the share of "optimistic" firms by p, I can calculate the (aggregate) rate of gross capital formation f as follows:

$$f = pf_o + (1-p)f_p. \tag{3}$$

I turn to the relationship of the rate of utilization $u = Y/K$ and the rate of interest r to the share of "optimistic" firms p. Since u is positively related to the rate of gross profit (on capital),[14] I may say that the larger u is, the more profitable (fixed) investment is. I can thus suppose that the share of "optimistic" firms p is affected positively by u and negatively by r. In the same way as in logistic regression analysis, I may relate p to u and r in the following form:

$$\ln\Big(\frac{p}{1-p}\Big) = \eta_u u - \eta_r r - \eta_0, \tag{4}$$

where η_u and η_r are positive constants and η_0 is a real constant.

[11] Equation (2) reflects Keynes' (1936, chap. 11) theory of investment and the profit and utilization principles of investment (cf. Kalecki 1939, 1971; Kaldor 1940; Steindl 1952, 1979; Rowthorn 1981). Note that when aggregate capital share is constant, the profit and utilization principles are identical with each other. For a microeconomic foundation of the profit principle, see Murakami (2016).

[12] It is also assumed that $f_p < \delta + \nu < f_o$, where δ and ν are defined later.

[13] Based upon the concept of statistical physics, Aoki and Yoshikawa (2007) and Yoshikawa (2015) suggested that aggregate data, as a consequence of macroeconomic or collective behavior of heterogeneous agents, should be interpreted in terms of "distributions." My hypothetical treatment is consistent with their suggestions.

[14] Denoting aggregate capital share by $\pi \in (0,1)$, I can give the rate of (gross) profit by πu. In this paper, I suppose that aggregate capital share π is constant because it has been known to be roughly constant in the long run (cf. Kaldor 1961; Jones 2016). Under this postulate, the rate of utilization u can be viewed as an index of profitability.

By substituting (4) in (3), I can provide the following form of f:

$$\frac{I}{K} = f(u, r) = \frac{f_o(\eta_u u - \eta_r r - \eta_0) + f_p}{1 + \exp(\eta_u u - \eta_r r - \eta_0)}. \tag{5}$$

Since p is allowed to vary in $[0, 1]$, any number in $[f_p, f_o]$ can be realized as the resultant aggregate rate of gross capital formation f. If f_o and f_p are taken as the upper and lower limits of each firm's rate of gross capital formation, respectively, our "binary-choice" hypothesis can be seen to describe any feasible outcome of aggregate capital formation.[15] Note that this capital formation function, illustrated in figure 1, is consistent with Kaldor's (1940) sigmoid investment function.

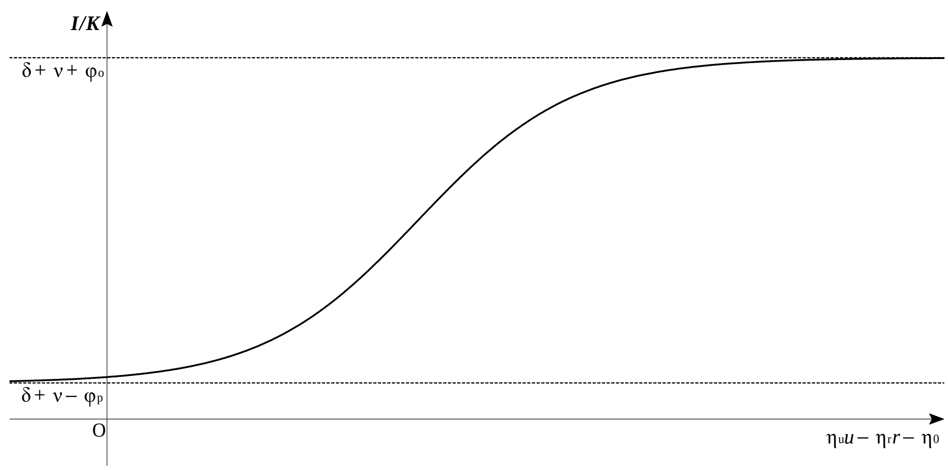

Figure 1: Capital formation function

2.3 Government expenditure and income tax

For simplicity, I assume that government expenditure G is proportionate to the size of population N as follows:

$$G = gN, \tag{6}$$

where g is a positive constant. In (6), it is assumed that the government conducts no counter-cyclical fiscal policy. In this paper, I simply suppose that government expenditure G changes with the size of population N.

I also assume that the government collects its income tax T by the following simple rule:

$$T = \tau Y - \tau_0 N, \tag{7}$$

where τ is a positive constant less than unity and τ_0 is a nonnegative constant. In (7), τ and τ_0 represent the marginal rate of tax and the base level of individual tax, respectively.

[15]For similar specifications of the investment function, see, for instance, Skott (1989, chap. 6), Franke (2014) or Murakami (2018b).

Therefore, I can, substituting (7) in (1), obtain the following consumption function:

$$C = (c_0 + c\tau_0)N + c(1-\tau)Y. \tag{8}$$

2.4 Quantity or utilization adjustment

Following the Keynesian or Kaleckian tradition, I introduce the following formulation for utilization adjustment, one kind of quantity adjustment:

$$\dot{u} = \alpha\Big(\frac{C+I+G}{K} - \frac{Y}{K}\Big), \tag{9}$$

where α is a positive constant. In (9), α represents the speed of quantity (utilization) adjustment. Equation (9) means that the rate of utilization u is adjusted in response to that difference between aggregate demand per unit of capital stock $(C+I+G)/K$ and $u = Y/K$, which is equal to aggregate excess demand or supply per unit of capital stock.[16]

Putting (5), (6) and (8) in (9), I can obtain the following equation:

$$\dot{u} = \alpha\Big[\frac{f_o \exp(\eta_u u - \eta_r r - \eta_0) + f_p}{1 + \exp(\eta_u u - \eta_r r - \eta_0)} - \sigma u + \frac{a}{k}\Big]. \tag{10}$$

where $\sigma = 1 - c(1-\tau) > 0$ and $a = c_0 + g + c\tau_0 > 0$. In (10), k stands for per capita stock of capital K/N.

2.5 Capital formation

The following equation describes aggregate capital formation process:

$$\frac{\dot{K}}{K} = \frac{I}{K} - \delta, \tag{11}$$

where δ is a positive constant. In (11), δ represents the rate of capital depreciation.

Substituting (5) in (11), I can obtain the following expression:

$$\frac{\dot{K}}{K} = \frac{f_o \exp(\eta_u u - \eta_r r - \eta_0) + f_p}{1 + \exp(\eta_u u - \eta_r r - \eta_0)} - \delta. \tag{12}$$

[16]Since firms determine their level of production on the basis of their (current) level of capacity, it is reasonable to think that the rate of utilization, proportionate to the output-capital ratio, is an adjusting variable in firms' daily production process. Indeed, the formalization in (9) has been adopted in a lot of studies of the post Keynesian or Kaleckian system (cf. Chiarella and Flaschel 2000; Sasaki 2014; Murakami and Asada 2018).

2.6 Population changes

To allow for economic growth or decline due to demographic changes, I assume that the size of population N varies at a constant rate as follows:

$$\frac{\dot{N}}{N} = \nu, \tag{13}$$

where ν is a real constant. For the constant rate of population change ν, it is assumed that $f_p < \delta + \nu < f_o$.

By combining (12) and (13), I can obtain the following differential equation of per capita capital stock $k = K/N$:

$$\frac{\dot{k}}{k} = \frac{\varphi_o \exp(\eta_u u - \eta_r r - \eta_0) - \varphi_p}{1 + \exp(\eta_u u - \eta_r r - \eta_0)}, \tag{14}$$

where $\varphi_o = f_o - (\delta + \nu) > 0$ and $\varphi_p = \delta + \nu - f_p > 0$.

2.7 Full system: System (K)

The dynamical system to be analyzed is given as follows:

$$\begin{aligned} \dot{u} &= \alpha\Big[\frac{f_o \exp(\eta_u u - \eta_r r - \eta_0) + f_p}{1 + \exp(\eta_u u - \eta_r r - \eta_0)} - \sigma u + \frac{a}{k}\Big] \\ &= \alpha\Big[\delta + \nu + \frac{\varphi_o \exp(\eta_u u - \eta_r r - \eta_0) - \varphi_p}{1 + \exp(\eta_u u - \eta_r r - \eta_0)} - \sigma u + \frac{a}{k}\Big], \end{aligned} \tag{10}$$

$$\frac{\dot{k}}{k} = \frac{\varphi_o \exp(\eta_u u - \eta_r r - \eta_0) - \varphi_p}{1 + \exp(\eta_u u - \eta_r r - \eta_0)}. \tag{14}$$

The system of (10) and (14) is denoted by "System (K)" (to signify "Keynesian") in what follows.

3 Analysis

To figure out the effect of counter-cyclical monetary policy, I examine first the case in which the rate of interest r is constant and then the one in which r is adjusted in response to the gap between the rate of utilization and that corresponding to full utilization, denoted by $u - u_f$.

3.1 The constant rate of interest: System (K*)

To begin, I consider the case in which the rate of interest r is set to a constant by the monetary authority. In this case, System (K) can be written in the following form:

$$\dot{u} = \alpha\Big[\delta + \nu + \frac{\varphi_o \exp(\eta_u u - \eta_r r^* - \eta_0) - \varphi_p}{1 + \exp(\eta_u u - \eta_r r^* - \eta_0)} - \sigma u + \frac{a}{k}\Big], \tag{15}$$

$$\frac{\dot{k}}{k} = \frac{\varphi_o \exp(\eta_u u - \eta_r r^* - \eta_0) - \varphi_p}{1 + \exp(\eta_u u - \eta_r r^* - \eta_0)}, \tag{16}$$

where r^* is a nonnegative constant set by the monetary authority. In what follows, the system of (15) and (16), or System (K) with $r = r^*$, is denoted by "System (K*)."

I define an equilibrium point of System (K*) as a point $(u, k) \in \mathbb{R}^2_{++}$ at which $\dot{u} = \dot{k} = 0$. Then, an equilibrium point of System (K*), (u^*, k^*), can be redefined as a solution of the following simultaneous equations:

$$0 = \delta + \nu + \frac{\varphi_o \exp(\eta_u u - \eta_r r^* - \eta_0) - \varphi_p}{1 + \exp(\eta_u u - \eta_r r^* - \eta_0)} - \sigma u + \frac{a}{k},$$

$$0 = \frac{\varphi_o \exp(\eta_u u - \eta_r r^* - \eta_0) - \varphi_p}{1 + \exp(\eta_u u - \eta_r r^* - \eta_0)}.$$

The unique equilibrium point of System (K*), provided it exists, can easily be calculated as follows:

$$(u^*, k^*) = \Big(\frac{\eta_0 + \eta_r r^* + \ln \varphi_p - \ln \varphi_o}{\eta_u}, \frac{\eta_u a}{\sigma(\eta_0 + \eta_r r^* + \ln \varphi_p - \ln \varphi_o) - \eta_u(\delta + \nu)}\Big). \tag{17}$$

To make sure that the unique equilibrium point (u^*, k^*) lies on the economic meaningful domain of $\mathbb{R}^2_{++}$, I impose the following assumption.

Assumption 1. *The following condition is satisfied:*

$$r^* > \frac{1}{\eta_r}\Big[\frac{\eta_u(\delta + \nu)}{\sigma} + (\ln \varphi_o - \ln \varphi_p) - \eta_0\Big]. \tag{18}$$

Condition (18) holds if the rate of interest r^* is sufficiently large or if the parameter η_r is sufficiently large (i.e., if investment is elastic enough to a change in the rate of interest). Under Assumption 1, I can ensure the existence and uniqueness of an economically meaningful equilibrium point of System (K*).

I confirm that the values of $u(t)$ and of $k(t)$ are positive all the time for every solution path of System (K*) with its initial condition $(u(0), k(0)) \in \mathbb{R}^2_{++}$. As for $k(t)$, it is easily known from (14) to be positive for all $t \geq 0$ provided that $k(0) > 0$. Then, it suffices to check that $u(t)$ is positive all the time along every solution path of System (K*). For this purpose, we only have to see that $\dot{u} > 0$ for $u = 0$ and for every $k > 0$:

$$\dot{u}|_{u=0} = \alpha\Big[\frac{f_o + f_p \exp(\eta_r r^* + \eta_0)}{1 + \exp(\eta_r r^* + \eta_0)} + \frac{a}{k}\Big] > 0.$$

This inequality turns out to hold because $0 \leq f_p < f_o$. I can thus find that $(u(t), k(t)) \in \mathbb{R}^2_{++}$ along every solution path of System (K*) with $(u(0), k(0)) \in \mathbb{R}^2_{++}$.

To facilitate the analysis, I reformulate System (K*) by introducing the following new variables x and y:

$$x = u - u^*, \tag{19}$$

$$y = \ln k^* - \ln k, \tag{20}$$

where (u^*, k^*) is the unique equilibrium defined by (17). Substituting (19) and (20) in System (K*), I can obtain the following system:

$$\dot{x} = \phi(y) - F(x), \tag{21}$$

$$\dot{y} = -g(x), \tag{22}$$

where

$$g(x) = \frac{\varphi_o \varphi_p}{\varphi_o + \varphi_p \exp(\eta_u x)}[\exp(\eta_u x) - 1], \tag{23}$$

$$F(x) = \alpha[\sigma x - g(x)], \tag{24}$$

$$\phi(y) = \alpha\Big[\frac{\sigma}{\eta_u}(\eta_0 + \eta_r r^* + \ln \varphi_p - \ln \varphi_o) - (\delta + \nu)\Big][\exp(y) - 1]. \tag{25}$$

In what follows, the system of (21) and (22) with (23)-(25) is denoted by System (L*).[17] To investigate the properties of System (K*), I have a closer look at System (L*) instead of System (K*). Note that the unique equilibrium point of System (L*) is given by $(x^*, y^*) = (0, 0)$.

Now I explore the possibility of emergence of persistent growth cycles, represented by a periodic orbit, in System (K*). To begin, I turn to local asymptotic stability of the unique equilibrium in System (L*). The Jacobian matrix of System (L*) evaluated at the unique equilibrium $(0, 0)$, denoted by J^*, is given by

$$J^* = \begin{pmatrix} \alpha[\eta_u \varphi_o \varphi_p/(\varphi_o + \varphi_p) - \sigma] & \alpha[\sigma(\eta_0 + \eta_r r^* - \ln \varphi_o + \ln \varphi_p)/\eta_u - (\delta + \nu)] \\ -\eta_u \varphi_o \varphi_p/(\varphi_o + \varphi_p) & 0 \end{pmatrix}.$$

The trace and determinant of J^* are given as follows:

$$\text{tr } J^* = \alpha\Big(\frac{\eta_u \varphi_o \varphi_p}{\varphi_o + \varphi_p} - \sigma\Big),$$

$$\det J^* = \alpha\frac{\varphi_o \varphi_p}{\varphi_o + \varphi_p}[\sigma(\eta_0 + \eta_r r^* - \ln \varphi_o + \ln \varphi_p) - \eta_u(\delta + \nu)] > 0,$$

[17] System (L*) can be regarded as a generalized Liénard system. For generalized Liénard systems, see, for instance, Xiao and Zhang (2003).

where the inequality follows from Assumption 1. Then, the unique equilibrium point is not a saddle point. For the emergence of a periodic orbit, I make the following assumption.

Assumption 2. *The following condition is satisfied:*

$$\frac{\eta_u \varphi_o \varphi_p}{\varphi_o + \varphi_p} > \sigma. \tag{26}$$

The economic implication of this assumption can be expounded as follows. The left hand side of (26) is the marginal propensity to invest, while the right hand side is that to save with income tax taken into consideration. In this sense, condition (26) requires that the marginal propensity to save be smaller than that to invest at the unique equilibrium, i.e., that the Keynesian stability condition (cf. Marglin and Bhaduri 1990) be violated at this point. As Kaldor (1940) insisted, the violation of the Keynesian stability condition plays a pivotal role for the emergence of persistent business cycles in the Keynesian system.[18]

Next, I detect a (nonempty) compact and positively invariant region in System (L*). To this end, I draw the phase diagram of System (L*). First, I have a look at the loci of $\dot{x} = 0$ and of $\dot{y} = 0$ of System (L*). As regards the locus of $\dot{y} = 0$, I can see from (22) and (23) that it is given by $x = 0$. As for the locus of $\dot{x} = 0$, it is given by $\phi(y) = F(x)$. To scrutinize this locus, I first consider the characteristics of $F(x)$. I can easily calculate $F'(x)$ as follows:

$$F'(x) = \alpha\Big\{\sigma - \eta_u \varphi_o \varphi_p (\varphi_o + \varphi_p)\frac{\exp(\eta_u x)}{[\varphi_o + \varphi_p \exp(\eta_u x)]^2}\Big\}. \tag{27}$$

I can then calculate the two roots of $F'(x) = 0$, denoted by $\underline{x}'$ and $\overline{x}'$, as follows:

$$\underline{x}' = \frac{1}{\eta_u}\ln\Big(\frac{\varphi_o\{\eta_u(\varphi_o + \varphi_p) - 2\sigma - \sqrt{\eta_u(\varphi_o + \varphi_p)[\eta_u(\varphi_o + \varphi_p) - 4\sigma]}\}}{2\sigma\varphi_p}\Big) < 0, \tag{28}$$

$$\overline{x}' = \frac{1}{\eta_u}\ln\Big(\frac{\varphi_o\{\eta_u(\varphi_o + \varphi_p) - 2\sigma + \sqrt{\eta_u(\varphi_o + \varphi_p)[\eta_u(\varphi_o + \varphi_p) - 4\sigma]}\}}{2\sigma\varphi_p}\Big) > 0, \tag{29}$$

where the inequalities follow from Assumption 2. It then follows that $F'(x) > 0$ for $x < \underline{x}'$ or $x > \overline{x}'$ and $F'(x) < 0$ for $x \in (\underline{x}', \overline{x}')$. Thus, I have $F(\underline{x}') > F(0) = 0 > F(\overline{x}')$. Since $F(\pm\infty) = \pm\infty$, I can find that $F(x) = 0$ has exactly two roots $\underline{x}_0$ and $\overline{x}_0$ besides $x = 0$ with $\underline{x}_0 < \underline{x}' < 0 < \overline{x}' < \overline{x}_0$. Thus, I can draw the graph of $F(x)$ as in figure 2.

[18]Recently, Skott (2012) and Murakami (2018b) reported that the marginal propensity to save is less than that to invest in reality.

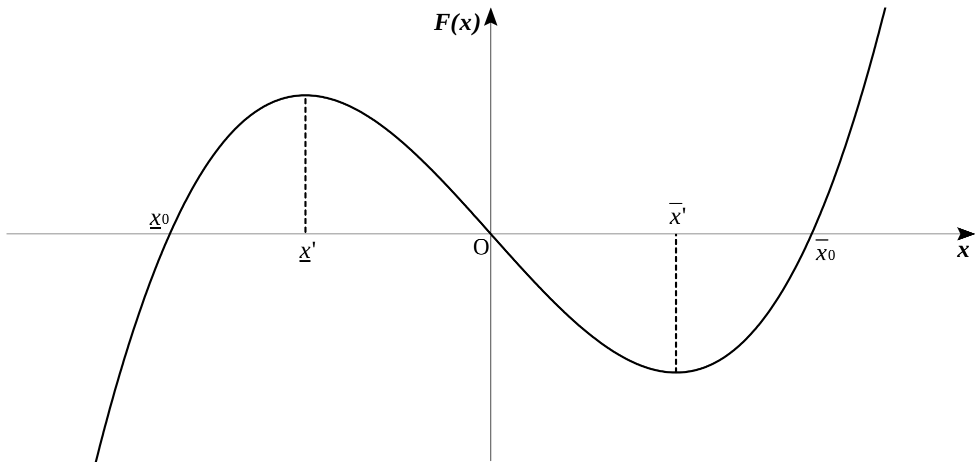

Figure 2: Graph of $F(x)$

Second, I ensure that the locus of $\dot{x} = 0$ or of $\phi(y) = F(x)$ is well defined at least for $x \in [\underline{x}_0, \overline{x}_0]$.[19] It is easy to see from figure 2 that, for $x \in [\underline{x}_0, \overline{x}_0]$, the minimum of F is given by $F(\overline{x}')$. According to the definition of ϕ given by (25), ϕ is strictly increasing in y and satisfies $\phi(\infty) = \infty$. Thus, I impose the following condition for the locus of $\dot{x} = 0$ to be well defined for $x \in [\underline{x}_0, \overline{x}_0]$:

$$F(\overline{x}') > \phi(-\infty). \tag{30}$$

For this condition to hold, I make the following additional assumption.

Assumption 3. *The following condition is satisfied:*

$$\begin{aligned} \eta_0 >& \frac{\eta_u(2\delta + 2\nu + \varphi_o - \varphi_p) + \sqrt{\eta_u(\varphi_o + \varphi_p)[\eta_u(\varphi_o + \varphi_p) - 4\sigma]}}{2\sigma} \\ &- \ln\Big(\frac{\eta_u(\varphi_o + \varphi_p) + \sqrt{\eta_u(\varphi_o + \varphi_p)[\eta_u(\varphi_o + \varphi_p) - 4\sigma]}}{2\sigma} - 1\Big). \end{aligned} \tag{31}$$

Condition (31) is satisfied if η_0 is relatively large. Condition (30) is actually fulfilled under Assumption 3. Under Assumptions 2 and 3 along with Assumption 1, the phase diagram can be drawn as in figure 3. Note that the locus of $\dot{x} = 0$ is wholly located on the right side of the line of $x = -u^*$, which corresponds to $u = 0$ in System (K*), because $\dot{u} > 0$ for every $u = 0$ and $k > 0$.

[19]This is one of the requirements of Theorem 1, which is used for the uniqueness of a limit cycle in System (K*).

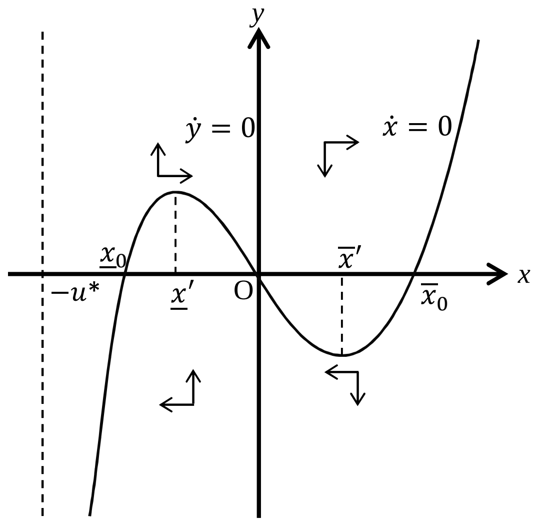

Figure 3: Phase diagram of System (L*)

Now I am ready to detect a positively invariant region with respect to System (L*) by making use of the phase diagram illustrated in figure 3. To this end, I consider the solution path of System (L*) with the initial condition $(x(0), y(0)) = (-u^*, 0)$. I can see that it starts at P$(-u^*, 0)$ for $t = 0$ and intersects with the y axis on the positive part at Q$(0, y_q)$ for the first time. I denote by R(x_r, y_q) the intersection point of the locus of $\dot{x} = 0$ and the line through Q and parallel with the x axis. I next consider the solution path of System (L*) with the initial condition $(x(0), y(0)) = (x_r, 0)$. I can find that it starts at S$(x_r, 0)$ for $t = 0$ and intersects with the y axis on the positive part at T$(0, y_t)$ for the first time. I denote by D the region enclosed by the arcs of PQ and of ST and by the line segments of QR, of RS, of SU and of UP, where U$(-u^*, y_t)$ (cf. figure 4). Since the arc of PQ or of ST turns out to be uniquely determined as a solution path of System (L*),[20] no solution path can cross either of them. Then, it is easily seen from the phase diagram drawn in figure 4 that D is a positively invariant compact region.

[20]It is possible to check that the Lipschitz condition (cf. Coddington and Levinson 1955, chap. 1) holds to guarantee the uniqueness of a solution path of System (L*) with respect to an initial condition. For the Lipschitz condition in related Keynesian models, see Murakami (2014, 2017, 2018a).

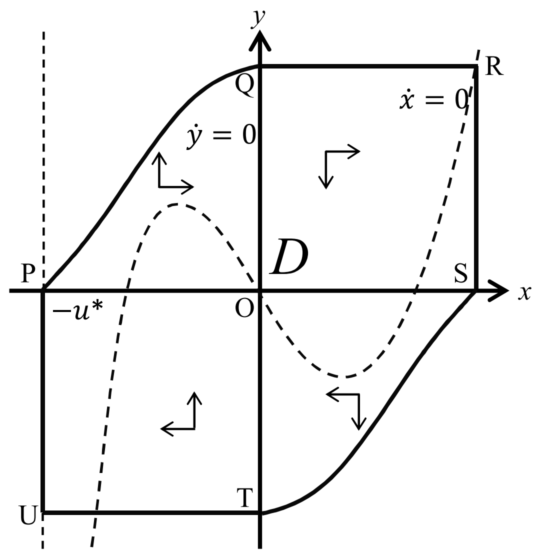

Figure 4: Positively invariant region D

I am in a position to verify the existence of a periodic orbit in System (K*) or (L*) with the help of the Poincaré-Bendixson theorem.

Proposition 1. *Let Assumptions 1-3 hold. Then, System (K*) has at least one periodic orbit on* $\mathbb{R}^2_{++}$.

Proof. It follows from Assumptions 1 and 2 that the unique equilibrium of System (L*), $(0, 0)$, is locally asymptotically totally unstable. It is then possible to construct a new positively invariant compact set, denoted by D', by eliminating the interior of a sufficiently small rectangle surrounding the unique equilibrium from the positively invariant compact set D illustrated in figure 4. I can apply the Poincaré-Bendixson theorem (cf. Coddington and Levinson 1955, chap. 16) to the compact set D' to conclude that there exists at least one periodic orbit of System (L*) on this set. Since $\dot{x} > 0$ for $x = -u^*$ and every real y, such a periodic orbit is wholly located on the domain of $x > -u^*$. I can thus draw the conclusion of this proposition due to (19) and (20). □

Proposition 1 implies that persistent growth cycles, represented by a periodic orbit, are observed in System (K*).

Now I proceed to discuss the uniqueness of a periodic orbit (or of persistent growth cycles) in System (K*). For this purpose, I make use of the theorem established by Xiao and Zhang (2003), reproduced as Theorem 1 in Appendix.

To establish the uniqueness of a periodic orbit (or of a limit cycle), I examine whether System (L*) satisfies the requirements of Theorem 1, Assumptions 5-7 in Appendix. First, it is easy to find from (23)-(25) and figure 2 that Assumptions 5 and 6 are fulfilled. Second, the former half of Assumption 7 has already been proved to hold by Assumption 3. In what follows, I check if the latter half of Assumption 7 is met in System (L*).

Now I claim that the latter half of Assumption 7, condition (56), holds if α is sufficiently large. To see this, I begin with confirming that condition (56) is satisfied if the following condition is fulfilled because $\underline{x}_0 < \underline{x}' < 0 < \overline{x}' < \overline{x}_0$:

$$\begin{cases} G(\overline{x}') + \Phi(\phi^{-1}(F(\overline{x}'))) \geq G(\underline{x}_0) \text{ if } G(\underline{x}_0) > G(\overline{x}_0), \\ G(\underline{x}') + \Phi(\phi^{-1}(F(\underline{x}'))) \geq G(\underline{x}_0) \text{ if } G(\overline{x}_0) > G(\underline{x}_0), \end{cases}$$

or

$$\begin{cases} \Phi(\phi^{-1}(F(\overline{x}'))) \geq G(\underline{x}_0) - G(\overline{x}') \text{ if } G(\underline{x}_0) > G(\overline{x}_0), \\ \Phi(\phi^{-1}(F(\underline{x}'))) \geq G(\overline{x}_0) - G(\underline{x}') \text{ if } G(\overline{x}_0) > G(\underline{x}_0). \end{cases} \tag{32}$$

Next, I confirm that $\Phi(\phi^{-1}(F(x)))$ is positive and proportional to α for $x \in (\underline{x}_0, 0)$ or $x \in (0, \overline{x}_0)$, where $\Phi(y) = \int_0^y \phi(s)ds$. It is easy to see from (25) that the inverse function of ϕ is given as follows:[21]

$$\phi^{-1}(y) = \ln\Big(1 + \frac{\eta_u}{\alpha[\sigma(\eta_0 + \eta_r r^* + \ln\varphi_p - \ln\varphi_o) - \eta_u(\delta+\nu)]} y\Big).$$

Also, I have

$$\Phi(y) = \int_0^y \phi(s)ds = \alpha \frac{\sigma(\eta_0 + \eta_r r^* + \ln\varphi_p - \ln\varphi_o) - \eta_u(\delta+\nu)}{\eta_u}[\exp(y) - y - 1].$$

Then, I find that

$$\Phi(\phi^{-1}(F(x))) = \alpha \frac{\sigma(\eta_0 + \eta_r r^* + \ln\varphi_p - \ln\varphi_o) - \eta_u(\delta+\nu)}{\eta_u}[v(x) - \ln(1 + v(x))], \tag{33}$$

where

$$v(x) = \frac{\eta_u}{\sigma(\eta_0 + \eta_r r^* + \ln\varphi_p - \ln\varphi_o) - \eta_u(\delta+\nu)}[\sigma x - g(x)]. \tag{34}$$

Since $F(x) = \alpha[\sigma x - g(x)] \neq 0$ for $x \in (\underline{x}_0, 0)$ or $x \in (0, \overline{x}_0)$ and $v(x) > \ln(1 + v(x))$ for $v(x) \neq 0$, it is easily seen from (33) that $\Phi(\phi^{-1}(F(x)))$ is positive and proportional to α for $x \in (\underline{x}_0, 0)$ or $x \in (0, \overline{x}_0)$.

Finally, I verify that condition (32), a sufficient condition for (56), holds if α is sufficiently large. To this end, it suffices to confirm that the left hand side of (32) is positive and proportional to α and the right hand side is independent from α. As is easily seen from (28), (29), (33) and (34), the left hand side of (32) is positive and proportional to α in each of the cases, while the right hand side is independent from α because $\underline{x}_0$ and $\overline{x}_0$ are independent from α. Thus, I can assert that condition (32) is fulfilled for α large enough.

With the help of Theorem 1, I can establish the uniqueness of a periodic orbit (or of a limit cycle) in System

[21] It follows from Assumption 3 that $\phi^{-1}(F(x))$ is well defined for $x \in [\underline{x}_0, \overline{x}_0]$.

(K*) or (L*).

Proposition 2. *Let Assumptions 1-3 hold. Then, System (K*) has a unique and (periodically) stable limit cycle on $\mathbb{R}^2_{++}$, if α is sufficiently large.*

Proof. As I have confirmed above, Assumptions 5-7 are all satisfied if α is large enough. It then follows from Theorem 1 that System (L*) possesses a unique limit cycle and so does System (K*). □

This proposition indicates the fact that there exists a "unique" growth cycle in System (K*), in which the rate of interest is kept constant by the monetary authority, if the speed of quantity (or utilization) adjustment α is sufficiently large. Since it is fast quantity adjustment that is one of the distinguished features of the Keynesian theory (cf. Leijonfuhvud 1968; Tobin 1993), the conclusion of Proposition 2 implies that a persistent and unique growth cycle is a universal phenomenon in the Keynesian system (when the rate of interest is kept constant).

3.2 The monetary policy rule for the rate of interest: System (K)

Now I explore the possibility that the monetary authority can mitigate persistent economic fluctuations along the unique growth cycle. For this purpose, I first define the "natural" rate of interest as one of the criteria in monetary policy implementations and then formalize a monetary policy rule for the rate of interest.

To begin, I define the natural rate of interest r_n as the rate of interest for which the equilibrium value of the rate of utilization of System (K*), u^*, is equal to the level of full utilization u_f. It then follows from (17) that the natural rate of interest r_n is defined as follows:

$$r_n = \frac{1}{\eta_r}(\eta_u u_f - \eta_0 + \ln \varphi_o - \ln \varphi_p), \tag{35}$$

where u_f is a positive constant that stands for the output-capital ratio (rate of utilization) in full utilization.

Next, I turn to the monetary authority's counter-cyclical policy rule. Specifically, I postulate that the monetary authority adopts the following rule in setting the rate of interest r:

$$r = r_n + \beta(u - u_f), \tag{36}$$

where β is a positive constant. In (36), β is a policy parameter that represents the willingness of the monetary authority to pursue the level of full utilization. Equation (36) means that the monetary authority follows a kind of feedback policy rule similar to Taylor's (1993) one.[22] In this section, I examine the effect of changes in the monetary policy parameter β on the stability of the unique equilibrium and on the unique growth cycle in my post Keynesian system.

[22] Unlike in Taylor's (1993) rule, the rate of inflation is not taken into consideration because price changes are not considered in my analysis.

I can thus complete our post Keynesian system by substituting (36) in (10) and (14):

$$\dot{u} = \alpha\Big[\delta + \nu + \frac{\varphi_o \exp(\eta_u u - \eta_r[r_n + \beta(u - u_f)] - \eta_0) - \varphi_p}{1 + \exp(\eta_u u - \eta_r[r_n + \beta(u - u_f)] - \eta_0)} - \sigma u + \frac{a}{k}\Big], \tag{37}$$

$$\frac{\dot{k}}{k} = \frac{\varphi_o \exp(\eta_u u - \eta_r[r_n + \beta(u - u_f)] - \eta_0) - \varphi_p}{1 + \exp(\eta_u u - \eta_r[r_n + \beta(u - u_f)] - \eta_0)}. \tag{38}$$

In what follows, the system of equations (37) and (38) is redefined as "System (K)."

An equilibrium point of System (K), (u^*, k^*), is defined as a point $(u, k) \in \mathbb{R}^2_{++}$ at which we have $\dot{u} = \dot{k} = 0$ or

$$0 = \delta + \nu + \frac{\varphi_o \exp(\eta_u u - \eta_r[r_n + \beta(u - u_f)] - \eta_0) - \varphi_p}{1 + \exp(\eta_u u - \eta_r[r_n + \beta(u - u_f)] - \eta_0)} - \sigma u + \frac{a}{k},$$

$$0 = \frac{\varphi_o \exp(\eta_u u - \eta_r[r_n + \beta(u - u_f)] - \eta_0) - \varphi_p}{1 + \exp(\eta_u u - \eta_r[r_n + \beta(u - u_f)] - \eta_0)}.$$

Then, the unique equilibrium point of System (K), provided that it exists, can be given as follows:

$$(u^*, k^*) = \Big(u_f, \frac{a}{\sigma u_f - (\delta + \nu)}\Big). \tag{39}$$

To guarantee the existence of the unique equilibrium point on $\mathbb{R}^2_{++}$, I make the following assumption.

Assumption 4. *The following condition is satisfied:*

$$u_f > \frac{\delta + \nu}{\sigma}. \tag{40}$$

Condition (40) requires that the rate of full utilization u_f be large enough. In this respect, Assumption 4 is not a stringent assumption.

As in the previous subsection, I re-formalize System (K) by introducing the variables x and y defined by (19) and (20), where the equilibrium values, u^* and k^*, are redefined by (39). Making use of these x and y, System (K) can be transformed into the following system:

$$\dot{x} = \phi(y) - F(x), \tag{41}$$

$$\dot{y} = -g(x), \tag{42}$$

where F is defined by (24) and g and ϕ are redefined by

$$g(x) = \frac{\varphi_o \varphi_p}{\varphi_o + \varphi_p \exp((\eta_u - \beta\eta_r)x)}[\exp((\eta_u - \beta\eta_r)x) - 1], \tag{43}$$

$$\phi(y) = \alpha[\sigma u_f - (\delta + \nu)][\exp(y) - 1]. \tag{44}$$

In what follows, the system of (41) and (42) with (24), (43) and (44) is denoted by System (L). The unique equilibrium point of System (L) is, of course, given by $(x^*, y^*) = (0, 0)$.

I am now in a position to examine the stability of the unique equilibrium of System (L). I can easily calculate the Jacobian matrix of System (L), denoted by J, as follows:[23]

$$J = \begin{pmatrix} \alpha\Big\{[\varphi_o\varphi_p(\varphi_o+\varphi_p)(\eta_u-\beta\eta_r)\exp((\eta_u-\beta\eta_r)x)]/[\varphi_o+\varphi_p\exp((\eta_u-\beta\eta_r)x)]^2-\sigma\Big\} & \alpha[\sigma u_f-(\delta+\nu)]\exp(y) \\ -[\varphi_o\varphi_p(\varphi_o+\varphi_p)(\eta_u-\beta\eta_r)\exp((\eta_u-\beta\eta_r)x)]/[\varphi_o+\varphi_p\exp((\eta_u-\beta\eta_r)x)]^2 & 0 \end{pmatrix}.$$

The trace and determinant of J are as follows:

$$\text{tr } J = \alpha\Big\{\frac{(\eta_u-\beta\eta_r)\varphi_o\varphi_p(\varphi_o+\varphi_p)\exp((\eta_u-\beta\eta_r)x)}{[\varphi_o+\varphi_p\exp((\eta_u-\beta\eta_r)x]^2}-\sigma\Big\}, \tag{45}$$

$$\det J = \alpha(\eta_u-\beta\eta_r)[\sigma u_f-(\delta+\nu)]\frac{\varphi_o\varphi_p(\varphi_o+\varphi_p)\exp((\eta_u-\beta\eta_r)x+y)}{[\varphi_o+\varphi_p\exp((\eta_u-\beta\eta_r)x]^2}. \tag{46}$$

It is easily seen from (46) that if β is large enough to satisfy $\beta > \eta_u/\eta_r$, the determinant of J is negative and the unique equilibrium is a saddle point and locally asymptotically unstable. If β is close to 0, on the other hand, I can find from (45) that the unique equilibrium is locally asymptotically totally unstable and from the conclusion of the previous section that a periodic orbit is generated.

I explore the possibility that the monetary authority can provide the unique equilibrium of System (L) with "global" asymptotic stability by choosing a suitable value of the policy parameter β.[24] According to the well-established Olech theorem, the unique equilibrium of System (L) is globally asymptotically stable if the following conditions are satisfied for every (x, y):[25]

$$\text{tr } J = \alpha\Big\{\frac{(\eta_u-\beta\eta_r)\varphi_o\varphi_p(\varphi_o+\varphi_p)\exp((\eta_u-\beta\eta_r)x)}{[\varphi_o+\varphi_p\exp((\eta_u-\beta\eta_r)x]^2}-\sigma\Big\} < 0, \tag{47}$$

$$\det J = \alpha(\eta_u-\beta\eta_r)[\sigma u_f-(\delta+\nu)]\frac{\varphi_o\varphi_p(\varphi_o+\varphi_p)\exp((\eta_u-\beta\eta_r)x+y)}{[\varphi_o+\varphi_p\exp((\eta_u-\beta\eta_r)x]^2} > 0. \tag{48}$$

I can immediately know that condition (48) holds for every (x, y) if $\beta < \eta_u/\eta_r$. In what follows, I examine the case in which condition (47) is satisfied. As for this condition, it is seen that

$$\begin{aligned}\frac{\exp((\eta_u-\beta\eta_r)x)}{[\varphi_o+\varphi_p\exp((\eta_u-\beta\eta_r)x)]^2} &= \Big(2\varphi_o\varphi_p+\frac{\varphi_o}{\exp((\eta_u-\beta\eta_r)x)}+\varphi_p\exp((\eta_u-\beta\eta_r)x)\Big)^{-1} \\ &\leq \Big(2\varphi_o\varphi_p+2\varphi_o\varphi_p\Big)^{-1} = \frac{1}{4\varphi_o\varphi_p},\end{aligned}$$

[23] Note that the Jacobian matrix J is not confined to the unique equilibrium.

[24] The unique equilibrium point of the dynamical system under consideration is said to be globally asymptotically stable if every solution path with an arbitrary initial condition converges to this point as $\to \infty$. The concept of global asymptotic stability is, of course, stronger than that of local asymptotic stability.

[25] Olech's (1963, p. 395, Theorem 4) theorem also requires that either the product of the (1,1) and (2,2) elements or that of the (1,2) and (2,1) elements of the Jacobian matrix J is nonzero for every (x, y), but this requirement is met in our case if the determinant is positive.

where the equality in the above inequality holds if $x = (\ln \varphi_o - \ln \varphi_p)/[2(\eta_u - \beta\eta_r)]$. Then, we obtain

$$\frac{(\eta_u - \beta\eta_r)\varphi_o\varphi_p(\varphi_o + \varphi_p)\exp((\eta_u - \beta\eta_r)x)}{[\varphi_o + \varphi_p \exp((\eta_u - \beta\eta_r)x]^2} \leq \frac{\varphi_o + \varphi_p}{4}(\eta_u - \beta\eta_r).$$

I can thus find that conditions (47) and (48) are both fulfilled for every (x, y) if

$$\frac{1}{\eta_r}\left(\eta_u - \frac{4\sigma}{\varphi_o + \varphi_p}\right) < \beta < \frac{\eta_u}{\eta_r}. \tag{49}$$

It is thus possible to draw the following conclusion on the stability of the unique equilibrium of System (K).

Proposition 3. *Let Assumptions 3 and 4 hold. Then, the unique equilibrium point of System (K) is globally asymptotically stable if β satisfies (49).*

Proof. I have seen that, if condition (49) holds, the unique equilibrium of System (L) possesses global asymptotic stability and then so does that of System (K). □

Proposition 3 implies that the monetary authority has to set the monetary policy parameter β, measuring the intensiveness of its monetary policy, to a "medium" value, not too small or too large, so as to achieve the objective of global asymptotic stability. I can thus assert that the monetary authority has to conduct a "moderate" counter-cyclical policy to attain the economic stability in our post Keynesian system, i.e., that the monetary authority should not be too passive or too active in its monetary policy implementations.

4 Conclusion

I am now in a position to summarize my analysis.

I have set up a post Keynesian system to discuss the existence and uniqueness of a growth cycle and evaluate the effectiveness of monetary policy for macroeconomic stability. Following Murakami (2018b), I have first demonstrated that a persistent growth cycle, represented by a limit cycle, uniquely exists in our post Keynesian system if the speed of quantity adjustment is high enough. I have also clarified that the monetary authority can mitigate (or eliminate) cyclical fluctuations along the unique growth cycle by its "moderate," not too active or too passive, counter-cyclical policy. These consequences indicate that the (post) Keynesian system, characterized by fast quantity or utilization adjustment, is, without policy interventions, exposed to persistent economic fluctuations (with constant amplitudes) induced by the unique growth cycle but that this system can be stabilized (in the global sense) by the monetary authority's proper policy implementations. In this respect, it is possible to state that our analysis confirms the vulnerability of the laissez-faire regime and the need for policy interventions.

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Appendix

I briefly introduce the theorem established by Xiao and Zhang (2003) on the uniqueness of a (stable) limit cycle in generalized Liénard systems.

I consider the following generalized Liénard system:

$$\dot{x} = \phi(y) - F(x), \tag{50}$$

$$\dot{y} = -g(x). \tag{51}$$

In what follows, the system of equations (50) and (51) is denoted by System (GL).

Following Xiao and Zhang (2003), I make the following assumptions concerning System (GL).

Assumption 5. *The real valued functions $g(x)$ and $F(x)$ are, respectively, continuous and continuously differentiable on $(\underline{x}, \overline{x})$, and the real valued function $\phi(y)$ is continuously differentiable on $(\underline{y}, \overline{y})$ with $-\infty \leq \underline{x} < 0 < \overline{x} \leq \infty$ and $-\infty \leq \underline{y} < 0 < \overline{y} \leq \infty$. Furthermore, the following conditions are satisfied:*

$$xg(x) > 0 \text{ for } x \neq 0, \tag{52}$$

$$\phi(0) = 0, \ \phi'(y) > 0 \text{ for } y \in (\underline{y}, \overline{y}). \tag{53}$$

Assumption 6. *There exist $\underline{x}_0$ and $\overline{x}_0$ with $\underline{x} < \underline{x}_0 < 0 < \overline{x}_0 < \overline{x}$ such that the following conditions are satisfied:*

$$F(\underline{x}_0) = F(0) = F(\overline{x}_0) = 0, \tag{54}$$

$$\begin{cases} xF(x) \leq 0 \text{ for } x \in (\underline{x}_0, \overline{x}_0), \\ xF(x) > 0, \ F'(x) \geq 0 \text{ for } x \in (\underline{x}, \underline{x}_0) \text{ or } x \in (\overline{x}_0, \overline{x}). \end{cases} \tag{55}$$

Furthermore, $F(x)$ is not identically equal to 0 for x sufficiently close to 0.

Assumption 7. *The curve of $\phi(y) = F(x)$ is well defined for $x \in [\underline{x}_0, \overline{x}_0]$.*[26] *Furthermore, the following condition*

[26] Xiao and Zhang (2003) assumed that the curve of $\phi(y) = F(x)$ is well defined for $x \in (\underline{x}, \overline{x})$, but my assumption suffices for the proof of their theorem (cf. Xiao and Zhang 2003, pp. 1187-1190).

is satisfied:

$$\begin{cases} \sup_{x\in[0,\overline{x}_0]}(G(x)+\Phi(\phi^{-1}(F(x)))) \geq G(\underline{x}_0) \text{ if } G(\underline{x}_0) \geq G(\overline{x}_0), \\ \sup_{x\in[\underline{x}_0,0]}(G(x)+\Phi(\phi^{-1}(F(x)))) \geq G(\overline{x}_0) \text{ if } G(\overline{x}_0) > G(\underline{x}_0), \end{cases} \tag{56}$$

where $G(x) = \int_0^x g(s)ds$ *and* $\Phi(x) = \int_0^x \phi(s)ds$.

Assumption 5 is a typical one in the theory of generalized Liénard systems, while Assumptions 6 and 7 concern the shape of $F(x)$ and those of $g(x)$ and $\phi(x)$, respectively.

Concerning the uniqueness of a periodic orbit (or of a limit cycle) in System (GL), the following theorem was established by Xiao and Zhang (2003).

Theorem 1. *Let Assumptions 5-7 hold. Then, System (GL) has at most one limit cycle, and it is (periodically) stable if it exists.*

Proof. See Xiao and Zhang (2003, p. 1187, Theorem 2.2). □

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