Spatial Duopoly with Cost Subsidy*

Ferenc Szidarovszky†and Akio Matsumoto‡

Abstract

The effects of government subsidies are examined in a spatial duopoly with a conventional and an electronic retailer. The Nash equilibrium is first determined and then the optimal cost subsidy rates are computed that maximize social welfare. The stability of the equilibrium is also investigated and it is shown that in the case of delayed information there is the possibility of cyclic behavior.

Keywords: stability, e-commerce, dynamic systems, time delay

1 Introduction

The theory of oligopoly has a large literature. Both static and dynamic models were discussed by many authors. A comprehensive summary of earlier studies is given in Okuguchi and Szidarovszky (1999) and that of recent studies is in Bischi et al. (2009). In this paper, we will consider a spatial duopoly model between conventional and electronic retailers, which is a modified version of the model earlier introduced and examined by Ahmad and Hegazi (2007). Instead of investigating different taxation strategies, we will focus on the effects of cost subsidies given by the government, when the two firms may have different subsidy rates. Government subsidies are used to control markets as well as to increase social welfare. Finding optimal subsidy rates is a fundamental component of the policy of the government. We will first compute the Nash equilibrium as a function of the government strategy, and then the optimal government policy will be determined that maximizes the social welfare of the country. Dynamic extensions will be then introduced and examined with both discrete and continuous time scales. In addition to finding conditions for the local asymptotic stability of the equilibrium we will also investigate the effect of optimal policies based on delayed information which might lead to cyclic behavior.

*The paper was prepared when the first author visited the Department of Systems and Industrial Engineering of the University of Arizona. He appreciates its hospitality over his stay. The usual disclaimer applies.

†University of Arizona, Department of Systems and Industrial Engineering, Tucson, Arizona, 85724-0020, USA.

‡University of Arizona, Department of Systems and Industrial Engineering and Chuo University, Department of Economics, 742-1, Higashi-Nakano, Hachiji, Tokyo, 192-0393, Japan.

2 The mathematical model

A spatial duopoly model is considered between conventional and electronic retailers. The market is assumed to consist of a line of fixed length X with a fixed population N distributed uniformally. For the sake of mathematical simplicity we may select $X = 1$. The best, cost minimizing location of the retailer would be the midpoint of the segment $[0, 1]$. We place it however in the origin in order to be consistent with earlier models and results, so a direct comparison of our findings with the earlier results becomes much easier. We will investigate the dependence of our results on the location of the conventional retailer in a future study. The conventional retailer is placed in the origin, and faces a transportation cost t per unit distance. The electronic retailer pays a fixed delivery charge F. Let furthermore m_c and m_e denote the mill price for the conventional and electronic retailers, respectively, and let c_c and c_e be their marginal costs. It is also assumed that the government gives subsidies to the retailers, and let s_c and s_e denote the percentages of cost subsidies. The delivery price for the conventional retailer is

$$P_c = m_c + tx \tag{1}$$

where x is its market share, and the delivery price for the electronic retailer is

$$P_e = m_e + F. \tag{2}$$

Equating the two prices gives the market share for the conventional retailer

$$x = \frac{m_e - m_c + F}{t}, \tag{3}$$

so the sold quantities are

$$q_c = Nx \text{ and } q_e = N(1 - x). \tag{4}$$

The profits of the retailers can be obtained as

$$\Pi_c = (m_c - c_c(1 - s_c))\, Nx - f_c \tag{5}$$

and

$$\Pi_e = (m_e - c_e(1 - s_e))\, N(1 - x) - f_e \tag{6}$$

where f_c and f_e denote the fixed costs.

With fixed subsides s_c and s_e, the Nash equilibrium can be obtained from the first-order conditions by assuming interior equilibrium. Combining equations (5) and (6) with (3),

$$\Pi_c = (m_c - c_c(1 - s_c))\, N \frac{m_e - m_c + F}{t} - f_c \tag{7}$$

and

$$\Pi_e = (m_e - c_e(1 - s_e))\, N \frac{t - m_e + m_c - F}{t} - f_e. \tag{8}$$

Both profit functions are concave parabolas, so the first-order conditions are sufficient and necessary for optimality. The first-order conditions are as follows:

$$\frac{\partial \Pi_c}{\partial m_c} = \frac{N}{t}\left(m_e - m_c + F - m_c + c_c(1 - s_c)\right) = 0$$

and

$$\frac{\partial \Pi_e}{\partial m_e} = \frac{N}{t}\left(t - m_e + m_c - F - m_e + c_e(1 - s_e)\right) = 0.$$

The best response of the conventional retailer is given by

$$R_c(m_e) = \frac{m_e + c_c(1 - s_c) + F}{2}$$

which is always positive for $m_e \geq 0$. Solving the first-order condition of the electronic retailer yields

$$m_e = \frac{m_c + c_e(1 - s_e) + t - F}{2}$$

if the numerator is positive. The best response of the electronic retailer therefore becomes piecewise linear:

$$R_e(m_c) = \begin{cases} \frac{m_c + c_e(1-s_e) + t - F}{2} & \text{if } m_c + c_e(1 - s_e) + t - F > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Assume first that both responses are positive. Then solving the linear equations

$$2m_c - m_e = F + c_c(1 - s_c) \tag{9}$$

and

$$m_c - 2m_e = F - t - c_e(1 - s_e) \tag{10}$$

for the mill prices gives the solutions

$$m_c(s_c, s_e) = \frac{1}{3}\left(F + t + 2c_c(1 - s_c) + c_e(1 - s_e)\right) \tag{11}$$

and

$$m_e(s_c, s_e) = \frac{1}{3}\left(-F + 2t + c_c(1 - s_c) + 2c_e(1 - s_e)\right). \tag{12}$$

Both are positive if $F < 2t + c_c(1 - s_c) + 2c_e(1 - s_e)$. Otherwise we have a corner equilibrium

$$m_c(s_c, s_e) = \frac{c_c(1 - s_c) + F}{2} \text{ and } m_e(s_c, s_e) = 0.$$

Notice that the value of $m_c(s_c, s_e)$ strictly decreases if at least one cost subsidy percentage increases. If $F \leq 2t$, then the same holds for $m_e(s_c, s_e)$. Otherwise it strictly decreases until reaching zero value, then it remains zero for all larger values of s_c and s_e. In Ahmed and Hegazi (2007) it is shown that any increase in the taxation rate results in higher delivery prices for both retailers.

3 Optimal government policy

The government wants to select subsidy rates that maximize social welfare, which is the difference of the social benefits and the total amount of subsidies. Social benefits can be calculated as the sum of the profits of the retailers and the total savings of the consumers. The profit functions are given in (7) and (8), and the consumers' savings can be computed in the following way. Without subsides (that is, with $s_c = s_e = 0$) the mill prices (11) and (12) would be

$$m_c(0,0) = \frac{1}{3}\left(F + t + 2c_c + c_e\right) \text{ and } m_e(0,0) = \frac{1}{3}\left(-F + 2t + c_c + 2c_e\right)$$

so the delivery price would become

$$m_e(0,0) + F = \frac{1}{3}\left(2F + 2t + c_c + 2c_e\right) \tag{13}$$

where we used equation (2). With subsidies the delivery price is

$$m_e(s_c, s_e) + F = \frac{1}{3}\left(2F + 2t + c_c(1 - s_c) + 2c_e(1 - s_e)\right). \tag{14}$$

A decrease in the market price is a gain or saving of the consumers. Therefore the total saving of the consumers is the following:

$$\frac{1}{3}(c_c s_c + 2c_e s_e)N. \tag{15}$$

The total government subsides are

$$c_c s_c N x + c_e s_e N(1 - x), \tag{16}$$

so the social welfare has the following form:

$$W = (m_c - c_c)Nx - f_c + (m_e - c_e)N(1 - x) - f_e + \frac{1}{3}(c_c s_c + 2c_e s_e)N. \tag{17}$$

Notice that from (3), (11) and (12),

$$x = \frac{1}{3t}\left(F + t - c_c(1 - s_c) + c_e(1 - s_e)\right), \tag{18}$$

and

$$X - x = \frac{1}{3t}\left(2t - F + c_c(1 - s_c) - c_e(1 - s_e)\right), \tag{19}$$

so the social welfare has the expression,

$$W = \frac{N}{9t}W^* \tag{20}$$

with

$$\begin{aligned} W^* = \; & (F + t - c_c(1 + 2s_c) + c_e(1 - s_e))(F + t - c_c(1 - s_c) + c_e(1 - s_e)) \\ & +(-F + 2t + c_c(1 - s_c) - c_e(1 + 2s_e))(2t - F + c_c(1 - s_c) - c_e(1 - s_e)) \\ & +3t(c_c s_c + 2c_e s_e). \end{aligned} \tag{21}$$

Maximizing W is the same as maximizing W^*. By simple differentiation

$$\frac{\partial W^*}{\partial s_c} = c_c \left(F - 2t - c_c - 2c_c s_c + c_e + 2c_e s_e\right) \tag{22}$$

and

$$\frac{\partial W^*}{\partial s_e} = c_e \left(-F + 2t + c_c + 2c_c s_c - c_e - 2c_e s_e\right). \tag{23}$$

Furthermore

$$\frac{\partial^2 W^*}{\partial s_c^2} = -2c_c^2, \ \frac{\partial^2 W^*}{\partial s_c \partial s_e} = 2c_c c_e \text{ and } \frac{\partial^2 W^*}{\partial s_e^2} = -2c_e^2.$$

The Hessian is

$$H = \begin{pmatrix} -2c_c^2 & 2c_c c_e \\ 2c_c c_e & -2c_e^2 \end{pmatrix} \tag{24}$$

with characteristic polynomial

$$\varphi_H(\lambda) = \lambda^2 + 2\lambda(c_c^2 + c_e^2) \tag{25}$$

and eigenvalues

$$\lambda_1 = -2(c_c^2 + c_e^2) < 0 \text{ and } \lambda_2 = 0.$$

So the Hessian is negative semidefinite. Consequently, the first-order conditions give maximal social welfare. Notice that

$$\frac{\partial W^*}{\partial s_c} = -\frac{c_c}{c_e}\frac{\partial W^*}{\partial s_e}. \tag{26}$$

If we define a new continuous function by $\varphi(s_c, s_e) = 2c_c s_c - 2c_e s_e$, then a condition to have solutions between 0 and 1 are

$$\varphi(0,1) \leq F - 2t - c_c + c_e \leq \varphi(1,0)$$

where $\varphi(0,1)$ is the minimum of $\varphi(s_c, s_e)$ and $\varphi(1,0)$ is the maximum of $\varphi(s_c, s_e)$. Therefore, if the above condition or $2t + c_c - 3c_e \leq F \leq 2t + 3c_c - c_e$ holds, then all pairs (s_c, s_e) which satisfy equation

$$2c_c s_c - 2c_e s_e = F - 2t - c_c + c_e \tag{27}$$

give maximal social welfare. So there are infinitely many solutions. If the conditions are not satisfied, then a constrained optimization problems has to be solved with the constraints

$$0 \leq s_c \leq 1 \text{ and } 0 \leq s_e \leq 1.$$

Assume first that the government decides on the ratio of the subsidy rates of the two types of firms, $s_e = a s_c$ with given a. Then, from (27),

$$s_c = \frac{F - 2t - c_c + c_e}{2(c_c - a c_e)} \tag{28}$$

by assuming that $c_c \neq ac_e$. To confirm $0 \leq s_c \leq 1$, we identify the following three cases. In case I in which $c_c > ac_e$, $2t + c_c - c_e \leq F \leq 2t + 3c_c - (2a+1)c_e$ is the required condition. In case II in which $c_c < ac_e$, $2t + 3c_c - (2a+1)c_e \leq F \leq 2t + c_c - c_e$ is the required condition. Lastly in case III in which $c_c = ac_e$, a solution exists if $F - 2t - c_c + c_e = 0$. In this case any arbitrary value of s_c in the unit interval $[0, 1]$ can be a solution. Accordingly, any $s_e = as_c$ in the unit interval is also a solution. Thus there are infinitely many solutions in this case.

Assume next that the government decides to subsidize the firms and wants to maximize social welfare with minimum subsidy. In this case, the optimal government policy is the solution of the nonlinear programming problem,

$$\begin{aligned} &\text{minimize } \left(c_c s_c N \tfrac{F+t-c_c(1-s_c)+c_e(1-s_e)}{3t} + c_e s_e N \tfrac{-F+2t+c_c(1-s_c)-c_e(1-s_e)}{3t} \right) \\ &\text{subject to } s_c \geq 0,\ s_e \geq 0 \text{ and} \\ &\qquad 2c_c s_c - 2c_e s_e = F - 2t - c_c + c_e, \end{aligned} \tag{29}$$

where we used relations, (16), (18), (19) and (27). From the constraint we get

$$s_e = \frac{c_c}{c_e} s_c - \frac{F - 2t - c_c + c_e}{2c_e} \tag{30}$$

and by substituting this relation into the objective function, a single-variable problem is obtained. After dividing the objective function by $N/3t$, the resulting objective function becomes

$$3tc_c s_c + \frac{3}{4} \left(c_c - (c_e + F - 2t) \right)^2 . \tag{31}$$

This is a linear function in s_c with slope

$$3tc_c \tag{32}$$

which is always positive. So the smallest possible subsidy rates,

$$s_c = 0 \text{ and } s_e = \frac{2t + c_c - F - c_e}{2c_e},$$

minimize the government cost. The value of s_e is in the interval $[0, 1]$ if

$$2t + c_c - 3c_e \leq F \leq 2t + c_c - c_e.$$

Ahmed and Hegazi (2007) derive a simple formula for the tax revenue at the Nash equilibrium, but no analysis of the social welfare is offered. The optimal taxation rate which maximizes social welfare can be also determined along the lines of our analysis presented above. Higher delivery prices to the consumers have to be compared to the higher tax revenues of the government.

4 Dynamic extensions

Assume first that the government subsidy rates are fixed, and the firms adjust their mill prices in proportion to the gradients of their profits. This process is known as gradient adjustment. Since

$$\frac{\partial \Pi_c}{\partial m_c} = \frac{N}{t}\left(m_e - 2m_c + F + c_c(1-s_c)\right)$$

and

$$\frac{\partial \Pi_e}{\partial m_e} = \frac{N}{t}\left(m_c - 2m_e + t - F + c_e(1-s_e)\right),$$

the gradient adjustment model with continuous time scales is

$$\dot{m}_c = K_c\left(m_e - 2m_c + F + c_c(1-s_c)\right)\frac{N}{t} \tag{33}$$

and

$$\dot{m}_e = K_e\left(m_c - 2m_e + t - F + c_e(1-s_e)\right)\frac{N}{t} \tag{34}$$

where K_c and K_e are the speeds of adjustment of the firms.

If discrete time scales are assumed, then the model becomes

$$m_c' = m_c + K_c\left(m_e - 2m_c + F + c_c(1-s_c)\right)\frac{N}{t} \tag{35}$$

and

$$m_e' = m_e + K_e\left(m_c - 2m_e + t - F + c_e(1-s_e)\right)\frac{N}{t} \tag{36}$$

where $'$ indicates a unit-time advancement operator. Both systems are linear, so local asymptotical stability implies global asymptotical stability.

In the case of the continuous system (33)-(34), the Jacobian has the special form

$$\mathbf{J}_C = \begin{pmatrix} -\frac{2K_cN}{t} & \frac{K_cN}{t} \\ \frac{K_eN}{t} & -\frac{2K_eN}{t} \end{pmatrix} \tag{37}$$

with characteristic polynomial

$$\varphi_C(\lambda) = \lambda^2 + \lambda\frac{2N}{t}(K_c + K_e) + \frac{3K_cK_eN^2}{t^2}. \tag{38}$$

Since the linear and constant coefficients are positive, the roots have negative real parts, so the process is always asymptotically stable.

In the case of the discrete system (35)-(36), the Jacobian is

$$\mathbf{J}_D = \begin{pmatrix} 1-\frac{2K_cN}{t} & \frac{K_cN}{t} \\ \frac{K_eN}{t} & 1-\frac{2K_eN}{t} \end{pmatrix} \tag{39}$$

with characteristic polynomial

$$\varphi_D(\lambda) = \lambda^2 + \lambda\left(-2 + \frac{2N}{t}(K_c + K_e)\right) + \left(1 - \frac{2N}{t}(K_c + K_e) + \frac{3K_cK_eN^2}{t^2}\right). \tag{40}$$

The roots are inside the unit circle if and only if

$$1 - \frac{2N}{t}(K_c + K_e) + \frac{3K_cK_eN^2}{t^2} < 1 \tag{41}$$

$$-2 + \frac{2N}{t}(K_c + K_e) + 1 - \frac{2N}{t}(K_c + K_e) + \frac{3K_cK_eN^2}{t^2} + 1 > 0 \tag{42}$$

$$2 - \frac{2N}{t}(K_c + K_e) + 1 - \frac{2N}{t}(K_c + K_e) + \frac{3K_cK_eN^2}{t^2} + 1 > 0. \tag{43}$$

Relation (42) always holds. Introducing new variables

$$k_c = \frac{K_cN}{t} > 0 \text{ and } k_e = \frac{K_eN}{t} > 0 \tag{44}$$

reduces conditions (41) and (43) to

$$2(k_c + k_e) - 3k_ck_e > 0 \tag{45}$$

and

$$4 - 4(k_c + k_e) + 3k_ck_e > 0. \tag{46}$$

The first condition clearly holds for all $0 < k_c < 1$ and $0 < k_e < 1$. The domain Ω in the (k_c, k_e) plain satisfying the second stability condition is shown in Figure 1. It is easy to see that

$$\Omega = \left\{(k_c, k_e) \mid 0 < k_c < 1,\ 0 < k_e < 1 \text{ and } k_e < \frac{4 - 4k_c}{4 - 3k_c}\right\}. \tag{47}$$

Both values of k_c and k_e have to be sufficiently small. By (44), it can be seen that this is the case when the market is small enough, the adjustment speeds are slow enough or the transportation cost is large enough.

Insert Figure 1 about here.

(caption, Stability region for system (35)-(36))

In Ahmad and Hegazi (2007), no analysis is offered for continuous models, however it can be shown that the continuous version of their model is always asymptotically stable. In the case of discrete time scale, they show that the equilibrium is asymptocially stable if the speeds of adjustment in the gredient adjustment process are sufficiently small. These results are in line with our analysis as well as with the general stability analysis for dynamic oligopolies (Bischi et al., 2009).

Next we assume that the government also uses gradient adjustment with respect to the social welfare under the assumption that the firms form Cournot equilibrium at each time period. The firms also use gradient adjustment with respect to their profit functions.

Assuming continuous time scales, this concept can be described by the following four-dimensional system, which is based on (22), (23), (33) and (34):

$$\dot{s}_c = K_{gc}\frac{Nc_c}{9t}\left(F - 2t - c_c - 2c_c s_c + c_e + 2c_e s_e\right) \tag{48}$$

$$\dot{s}_e = K_{ge}\frac{Nc_e}{9t}\left(-F + 2t + c_c + 2c_c s_c - c_e - 2c_e s_e\right) \tag{49}$$

$$\dot{m}_c = K_c\frac{N}{t}\left(m_e - 2m_c + F + c_c(1 - s_c)\right) \tag{50}$$

$$\dot{m}_e = K_e\frac{N}{t}\left(m_c - 2m_e + t - F + c_e(1 - s_e)\right). \tag{51}$$

In the case of discrete time scales, the corresponding dynamic equations become

$$s_c^{'} = s_c + K_{gc}\frac{Nc_c}{9t}\left(F - 2t - c_c - 2c_c s_c + c_e + 2c_e s_e\right) \tag{52}$$

$$s_e^{'} = s_e + K_{ge}\frac{Nc_e}{9t}\left(-F + 2t + c_c + 2c_c s_c - c_e - 2c_e s_e\right) \tag{53}$$

$$m_c^{'} = m_c + K_c\frac{N}{t}\left(m_e - 2m_c + F + c_c(1 - s_c)\right) \tag{54}$$

$$m_e^{'} = m_e + K_e\frac{N}{t}\left(m_c - 2m_e + t - F + c_e(1 - s_e)\right). \tag{55}$$

The Jacobian of the continuous system is

$$\mathbf{J}_C^* = \begin{pmatrix} \mathbf{J}_{gc} & \mathbf{0} \\ \mathbf{A} & \mathbf{J}_C \end{pmatrix} \tag{56}$$

where

$$\mathbf{J}_{gc} = \begin{pmatrix} -\frac{2K_{gc}c_c^2 N}{9t} & \frac{2K_{gc}c_c c_e N}{9t} \\ \frac{2K_{ge}c_c c_e N}{9t} & -\frac{2K_{ge}c_e^2 N}{9t} \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} -\frac{K_c N c_c}{t} & 0 \\ 0 & -\frac{K_e N c_e}{t} \end{pmatrix}$$

and $\mathbf{J}_C$ is given in (37). Clearly the eigenvalues of $\mathbf{J}_C^*$ are the eigenvalues of $\mathbf{J}_{gc}$ and $\mathbf{J}_C$. The characteristic polynomial of $\mathbf{J}_{gc}$ is

$$\varphi_{gc}(\lambda) = \lambda^2 + \lambda\frac{2N}{9t}\left(K_{gc}c_c^2 + K_{ge}c_e^2\right) \tag{57}$$

with a negative and a zero eigenvalue. We have already seen that both eigenvalues of $\mathbf{J}_C$ have negative real parts, so the system is marginally stable and the stability is not asymptotic. It means that starting close enough from the steady state, the trajectory will remain close enough to the steady state for all future times.

In the case of the discrete system (52)-(55), the Jacobian has the form

$$\mathbf{J}_D^* = \begin{pmatrix} \mathbf{I} + \mathbf{J}_{gc} & \mathbf{0} \\ \mathbf{A} & \mathbf{J}_D \end{pmatrix} \tag{58}$$

where $\mathbf{I}$ is the identity matrix, $\mathbf{J}_{gc}$, $\mathbf{A}$ are as before and $\mathbf{J}_D$ is given by (39). The eigenvalues of $\mathbf{I} + \mathbf{J}_{gc}$ are

$$1 \text{ and } 1 - \frac{2N}{9t}\left(K_{gc}c_c^2 + K_{ge}c_e^2\right).$$

The second eigenvalue is inside the unit circle if

$$\frac{2N}{9t}\left(K_{gc}c_c^2 + K_{ge}c_e^2\right) < 2. \tag{59}$$

Similarly to (44) we can introduce the new variables

$$k_{gc} = \frac{K_{gc}N}{t} \text{ and } k_{ge} = \frac{K_{ge}N}{t},$$

then (59) holds if and only if (k_{gc}, k_{ge}) belongs to the stability region

$$\Omega_g = \{(k_{gc}, k_{ge}) \mid 0 < k_{gc} < \frac{9}{c_c^2} \text{ and } k_{ge} < \frac{9 - k_{gc}c_c^2}{c_e^2}\}, \tag{60}$$

which is illustrated in Figure 2. Hence system (52)-(55) is marginally stable if

$$(k_c, k_e) \in \Omega \text{ and } (k_{gc}, k_{ge}) \in \Omega_g.$$

Insert Figure 2 about here.

(caption, Stability region for government in system (48)-(51))

The continuous best reply dynamics with adaptive adjustments is

$$\dot{m}_c = \bar{K}_c(R_c(m_e) - m_c)$$

and

$$\dot{m}_e = \bar{K}_e(R_e(m_c) - m_e).$$

The discrete best reply dynamics is

$$m_c' = m_c + \bar{K}_c(R_c(m_e) - m_c)$$

and

$$m'_e = m_e + \bar{K}_e(R_e(m_c) - m_e).$$

Gradient dynamics is always linear, but best reply dynamics is nonlinear, since $R_e(m_c)$ is only piecewise linear.

In the neighborhood of the equilibrium,

$$R_c(m_e) - m_c = \frac{m_e - 2m_c + F + c_c(1 - s_c)}{2}$$

and

$$R_e(m_c) - m_e = \frac{m_c - 2m_e + t - F + c_e(1 - s_e)}{2}.$$

So the above models are the same as dynamics with gradient adjustments by selecting

$$\bar{K}_c = \frac{2K_cN}{t}$$

and

$$\bar{K}_e = \frac{2K_eN}{t}.$$

Therefore the local stability of these models is also the same. Global stability properties are however different, since far from the interior equilibrium, we might have breakpoints and nonlinearities.

5 Delayed dynamics

In this section we assume that the firms react to delayed information since collecting and implementing information in their decision process needs some time. Similar situation occurs if they want to react to average information rather than following sudden changes. Assuming continuously distributed time delays the 2D dynamic system becomes

$$\dot{m}_c = \frac{K_cN}{t}\left(\bar{m}_e - 2\bar{m}_c + F + c_c(1 - s_c)\right), \tag{61}$$

and

$$\dot{m}_e = \frac{K_eN}{t}\left(\bar{m}_c - 2\bar{m}_e + t - F + c_e(1 - s_e)\right), \tag{62}$$

where

$$\bar{m}_c = \int_0^\tau w(\tau - s, T, m)m_c(s)ds,$$

and

$$\bar{m}_e = \int_0^\tau w(\tau - s, S, \ell)m_e(s)ds.$$

Here we assume that the weighting function has the special form

$$w(\tau - s, T, m) = \begin{cases} \dfrac{1}{T} e^{-\frac{\tau - s}{T}} & \text{if } m = 0, \\ \dfrac{1}{m!} \left(\dfrac{m}{T}\right)^{m+1} (\tau - s)^m e^{-\frac{(\tau - s)m}{T}} & \text{if } m \geq 1. \end{cases}$$

We look for solutions of the homogeneous equations in forms

$$m_c = \nu_c e^{\lambda \tau} \text{ and } m_e = \nu_e e^{\lambda \tau}.$$

Then

$$\lambda \nu_c e^{\lambda \tau} = \frac{K_c N}{t} \left(\int_0^\tau w(\tau - s, S, \ell) e^{\lambda s} ds \nu_e - 2 \int_0^\tau w(\tau - s, T, m) e^{\lambda s} ds \nu_c \right), \tag{63}$$

and

$$\lambda \nu_e e^{\lambda \tau} = \frac{K_e N}{t} \left(\int_0^\tau w(\tau - s, T, m) e^{\lambda s} ds \nu_c - 2 \int_0^\tau w(\tau - s, S, \ell) e^{\lambda s} ds \nu_e \right). \tag{64}$$

Introducing the new variable $x = \tau - s$ and noting that

$$\int_0^\tau w(\tau - s, T, m) e^{\lambda s} ds = \int_0^\tau w(x, T, m) e^{\lambda(\tau - x)} dx,$$

we have

$$\lambda \nu_c = \frac{K_c N}{t} \left(\int_0^\tau w(x, S, \ell) e^{-\lambda x} dx \nu_e - 2 \int_0^\tau w(x, T, m) e^{-\lambda x} dx \nu_c \right), \tag{65}$$

and

$$\lambda \nu_e = \frac{K_e N}{t} \left(\int_0^\tau w(x, T, m) e^{-\lambda x} dx \nu_c - 2 \int_0^\tau w(x, S, \ell) e^{-\lambda x} dx \nu_e \right). \tag{66}$$

The introduction of the new variable

$$z = \left(\lambda + \frac{m}{T} \right) x$$

shows that

$$\int_0^\tau w(x, T, m) e^{-\lambda x} dx == \frac{1}{m!} \left(1 + \frac{\lambda T}{m} \right)^{-(m+1)} \int_0^\tau z^m e^{-z} dz.$$

So as $\tau \to \infty$, the system becomes

$$\left[\lambda + 2 \frac{K_c N}{t} \left(1 + \frac{\lambda T}{q} \right)^{-(m+1)} \right] \nu_c - \frac{K_c N}{t} \left(1 + \frac{\lambda S}{r} \right)^{-(\ell+1)} \nu_e = 0 \tag{67}$$

and

$$-\frac{K_eN}{t}\left(1+\frac{\lambda T}{q}\right)^{-(m+1)}\nu_c+\left[\lambda+2\frac{K_eN}{t}\left(1+\frac{\lambda S}{r}\right)^{-(\ell+1)}\right]\nu_e=0, \qquad (68)$$

where

$$q=\begin{cases}1 & \text{if } m=0\\ m & \text{if } m\geq 1\end{cases}, \text{ and } r=\begin{cases}1 & \text{if } \ell=0\\ \ell & \text{if } \ell\geq 1.\end{cases}$$

Nonzero solution exits if and only if

$$\det\begin{pmatrix}\lambda+2\frac{K_cN}{t}\left(1+\frac{\lambda T}{q}\right)^{-(m+1)} & -\frac{K_cN}{t}\left(1+\frac{\lambda S}{r}\right)^{-(\ell+1)}\\ -\frac{K_eN}{t}\left(1+\frac{\lambda T}{q}\right)^{-(m+1)} & \lambda+2\frac{K_eN}{t}\left(1+\frac{\lambda S}{r}\right)^{-(\ell+1)}\end{pmatrix}=0$$

or

$$\left[\lambda+2\frac{K_cN}{t}\left(1+\frac{\lambda T}{q}\right)^{-(m+1)}\right]\left[\lambda+2\frac{K_eN}{t}\left(1+\frac{\lambda S}{r}\right)^{-(\ell+1)}\right]-\frac{K_cK_eN^2}{t^2}\left(1+\frac{\lambda T}{q}\right)^{-(m+1)}\left(1+\frac{\lambda S}{r}\right)^{-(\ell+1)}=0. \qquad (69)$$

It is very hard to examine the locations of the roots of this equation in general. Therefore we will consider some important special cases.

If no delay is assumed, then $T=S=0$. In this case, this equation reduces to (38), and there we have proved that the equilibrium is asymptotically stable.

Assume $S=0, m=1$ and $T>0$ (Note that the symmetric case of $T=0, \ell=1$ and $S>0$ is similar). Equation (69) becomes

$$\left[\lambda+2\frac{K_cN}{t(1+\lambda T)}\right]\left[\lambda+2\frac{K_eN}{t}\right]-\frac{K_cK_eN^2}{t^2(1+\lambda T)}=0, \qquad (70)$$

which can be reduced to the cubic equation

$$a_3\lambda^3+a_2\lambda^2+a_1\lambda+a_0=0$$

with

$$a_3=Tt^2,$$
$$a_2=2K_eNTt+t^2,$$
$$a_1=2K_eNt+2K_cNt$$

and

$$a_0=3K_cK_eN^2.$$

All coefficients are positive, so the system is asymptotically stable if and only if $a_1a_2>a_0a_3$, that is,

$$a_1a_2-a_0a_3=4K_e^2NT+2K_et+K_cK_eNT+2tK_c>0$$

which is always the case. So if at most one of the firms has delayed information, then stability holds.

Assume next that $m = \ell = 0$, $S > 0$ and $T > 0$. In this case equation (69) is the following:

$$\left[\lambda + 2\frac{K_c N}{t(1+\lambda T)}\right]\left[\lambda + 2\frac{K_e N}{t(1+\lambda S)}\right] - \frac{K_c K_e N^2}{t^2(1+\lambda T)(1+\lambda S)} = 0. \qquad (71)$$

This is equivalent to the following 4th order equation:

$$\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0 \qquad (72)$$

where

$$a_3 = \frac{T+S}{TS},$$

$$a_2 = \frac{2K_e NT + t + 2K_c NS}{TSt},$$

$$a_1 = \frac{2N(K_c + K_e)}{TSt},$$

and

$$a_0 = \frac{3K_c K_e N^2}{TSt^2}.$$

Since all coefficients are positive, the Routh-Hurwitz condition is reduced to $a_3 a_2 a_1 > a_1^2 + a_0 a_3^2$. In this case it can be written as

$$\frac{T+S}{TS}\frac{2K_e NT + t + 2K_c NS}{TSt}\frac{2N(K_c+K_e)}{TSt} > \left(\frac{2N(K_c+K_e)}{TSt}\right)^2 + \frac{3K_c K_e N^2}{TSt^2}\left(\frac{T+S}{TS}\right)^2.$$

Simple calculation shows that this inequality always holds, therefore the system remains asymptotically stable.

We will finally examine the case when $S = 0$, $m = 1$ and $T > 0$. In this case equation (69) can be simplified as

$$\left(\lambda + \frac{2K_c N}{t(1+\lambda T)^2}\right)\left(\lambda + \frac{2K_e N}{t}\right) - \frac{K_c K_e N^2}{t^2(1+\lambda T)^2} = 0, \qquad (73)$$

which is again a fourth-order equation (72) with coefficients

$$a_3 = \frac{2(t + K_e NT)}{Tt}$$

$$a_2 = \frac{t + 4K_e NT}{T^2 t}$$

$$a_1 = \frac{2N(K_c + K_e)}{T^2 t}$$

and

$$a_0 = \frac{3K_c K_e N^2}{T^2 t^2}.$$

Similarly to the previous case, the Routh-Hurwitz criterion shows that the system is asymptotically stable if and only if

$$\frac{2(t+K_eNT)}{Tt}\frac{t+4K_eNT}{T^2t}\frac{2N(K_c+K_e)}{T^2t} > \left(\frac{2N(K_c+K_e)}{T^2t}\right)^2 + \frac{3K_cK_eN^2}{T^2t^2}\left(\frac{2(t+K_eNT)}{Tt}\right)^2.$$

After simple calculation the condition can be rewritten as

$$K_c^2T\frac{t^2}{N^2}+K_c\left(2T^2K_c^2\frac{t}{N}+3T^3K_e^3-\frac{t^3}{N^3}\right)-\left(K_e\frac{t^3}{N^3}+4K_e^2T^2\frac{t}{N}+4K_e^3T^2\frac{t}{N}\right)<0. \quad (74)$$

The left hand side has two real roots, one is positive and the other is negative. Let K_c^* denote the positive root. The system is asymptotically stable if and only if $K_c < K_c^*$.

Let us select K_c as the bifurcation parameter. We will finally show the birth of limit cycle at the critical value K_c^* of K_c.

The existence of limit cycles is guaranteed if there is a pair of pure complex eigenvalues, while all other eigenvalues have negative real parts and the derivative of the common real part of this pair of eigenvalues with respect to the bifurcation parameter is nonzero at the critical value. After dividing by the leading coefficient, the characteristic equation becomes (72) with coefficients given above.

Assume that $\lambda = i\alpha$ is a root, then $\lambda^2 = -\alpha^2$, $\lambda^3 = -i\alpha^3$ and $\lambda^4 = \alpha^4$, so

$$\alpha^4 - i\alpha^3a_3 - \alpha^2a_2 + i\alpha a_1 + a_0 = 0.$$

Assuming $\alpha \neq 0$ and equating the imaginary part to zero gives

$$\alpha^2 = \frac{a_1}{a_3},$$

which is substituted into the real part to obtain

$$\left(\frac{a_1}{a_3}\right)^2 - \frac{a_1}{a_3}a_2 + a_0 = 0,$$

or

$$a_1a_2a_3 - a_1^2 - a_0a_3^2 = 0.$$

Solving the last equation for a_0 yields

$$a_0 = \frac{a_1a_2a_3 - a_1^2}{a_3^2}.$$

The characteristic polynomial is therefore

$$\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + \frac{a_1a_2a_3 - a_1^2}{a_3^2} = \left(\lambda^2 + \frac{a_1}{a_3}\right)\left(\lambda^2 + a_3\lambda + \frac{a_2a_3 - a_1}{a_3}\right).$$

So we have a pair of pure complex eigenvalues

$$\lambda_{1,2} = \pm i\sqrt{\frac{a_1}{a_3}},$$

and the roots of the second factor have negative real parts since its coefficients are all positive since $a_0 > 0$.

Differentiating the characteristic equation with respect to the bifurcation parameter K_c yields

$$\dot{\lambda}(4\lambda^3 + 3a_3\lambda^2 + 2a_2\lambda + a_1) + \dot{a}_3\lambda^3 + \dot{a}_2\lambda^2 + \dot{a}_1\lambda + \dot{a}_0 = 0$$

where the dot over variables indicates derivative with respect to K_c. Substituting $\lambda = i\alpha$ and solving for $\dot{\lambda}$ with noticing that $\dot{a}_0 = \frac{3KeN^2}{t^2T^2}$, $\dot{a}_1 = \frac{2N}{tT^2}$, $\dot{a}_2 = \dot{a}_3 = 0$, and $\alpha^2 = a_1/a_3$ give

$$\dot{\lambda} = \frac{-\dot{a}_1\alpha i - \dot{a}_0}{-4i\alpha^3 - 3a_3\alpha^2 + 2a_2 i\alpha + a_1}.$$

Substituting a_1, a_2, a_3, $\dot{a}_0$ and $\dot{a}_1$, arranging terms gives

$$sgn[\mathrm{Re}\,\dot{\lambda}] = sgn[12(K_eNT)^3 + 8(K_eNT)^2t + 8(K_cNT)t^2 - 4t^3].$$

Simple calculation shows that at the positive root of equation (72), this expression is always positive:

$$\mathrm{Re}\,\dot{\lambda} > 0,$$

which proves the birth of limit cycles at the critical value. Figure 3 illustrates a limit cycle in the case of $S = 0$, $m = 1$ and $T > 0$.

Insert Figure 3 about here.

(caption, Birth of a limit cycle)

6 Concluding Remarks and Future Extension

This paper examines the effects of government subsidies in a spatial duopoly with a conventional retailer and an electronic retailer. It constructs a static Hoteling linear market model. Then it determines the Nash equilibrium and computes the optimal government cost subsidy rates to maximize social welfare. It also investigates the stability of the equilibrium and shows the possibility of cyclic behavior when firms have delays in collecting and implementing information in their decision process.

This paper can be extended in several directions. The delivery cost is considered in this paper to be the main difference between a conventional and an electronic retailer. It is observed in e-commerce that psychic cost and information gap also play important roles, so they have to be taken into account. A conventional retailer can also introduce e-commerce into its business practice. So it will be interesting to examine its effect and its profitability.

References

Ahmad, E. and A. S. Hegazi (2007) A Dynamic Model for e-commerce Taxation. *Applied Mathematics and Comp*utation, 187, 965-967.

Okuguchi, K. and F. Szidarovszky (1999) *The Theory of Oligopoly with Multi-producted Firms*, Springer-Verlag, Berlin/New York. (2nd edition)

Bischi, G-I. C. Chiarella, M. Kopel and F. Szidarovszky (2009) *Nonlinear Oligopolies: Stability and Bifurcations*, Elsevier, Amsterdam/New York (to appear).

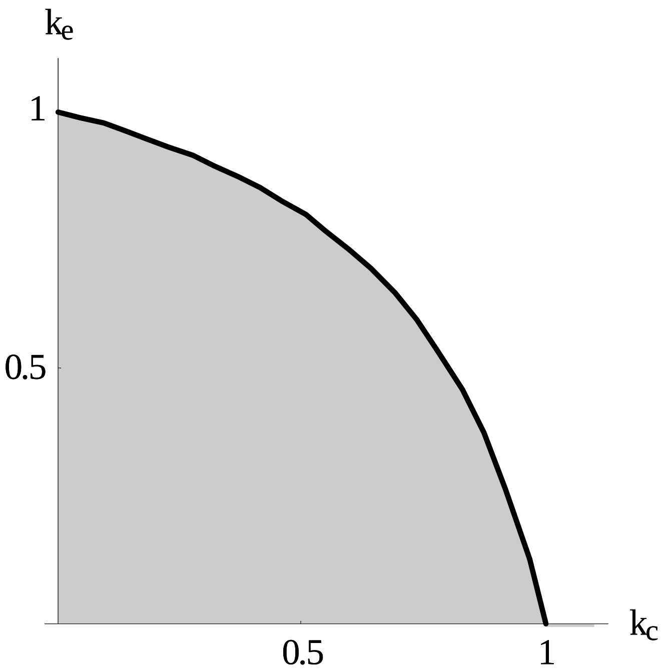

Figure 1 Stability region for system (35)-(36)

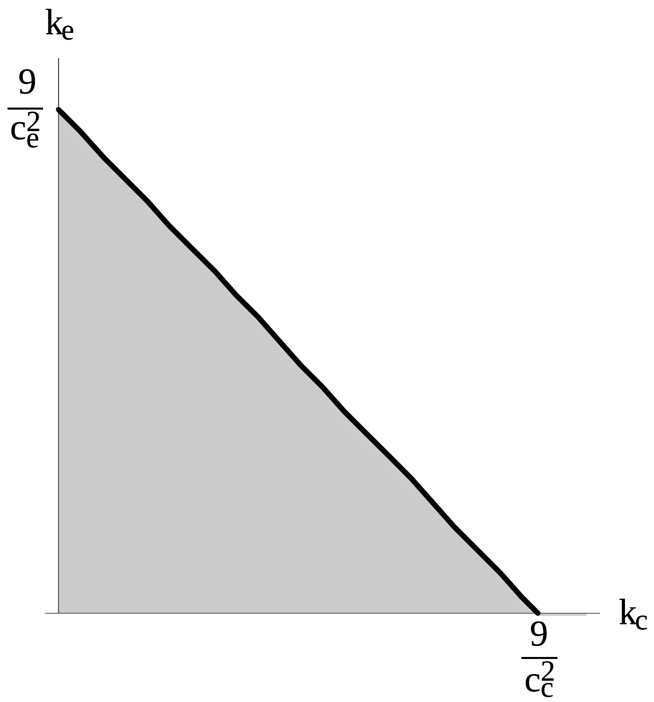

Figure 2 Stability region for government in system (48)-(51)

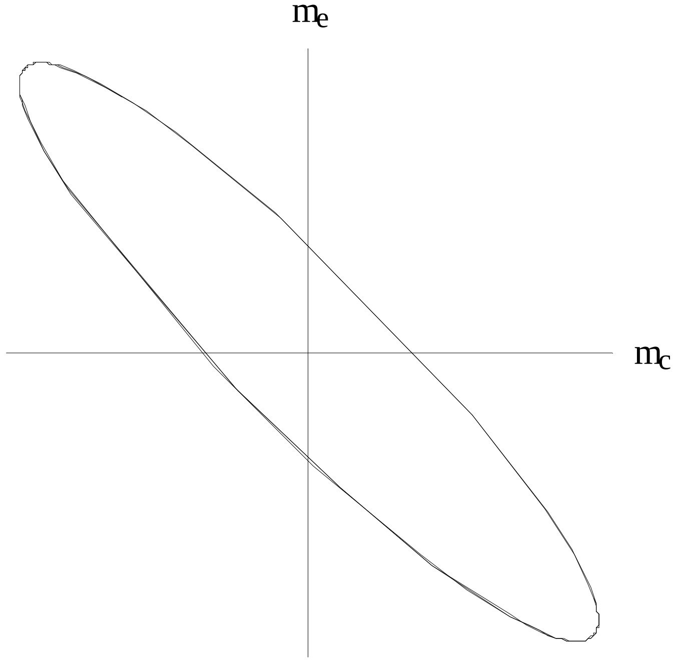

Figure 3 Birth of a limit cycle