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of a Liquidity Trap

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Abstract

The main purpose of this study is to show the existence of a 2D torus in a macrodynamic model of a liquidity trap. The model developed here has two steady-state points: a targeted steady state and a liquidity trap. When a 2D torus exists around a targeted steady state, an economy will keep fluctuating on a torus and never reach the state, unless the variables jump. We develop an analysis using the continuous-time New Keynesian dynamic general equilibrium model proposed by Benhabib et al. (2003). Although their model takes the form of “money in the production function,” we reconstruct it into the more typical “money in the utility function.”

JEL Classification: E32; E52

Keywords: new Keynesian DGE model, backward-looking interest-rate rules, 2D torus, homoclinic orbit, Hopf bifurcation

1 Introduction

Two types of New Keynesian dynamic general equilibrium (DGE) model exist: one attributes the cause of liquidity traps to a decrease in the natural rate of interest, while

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the other to the emergence of multiple steady states. Eggertsson and Woodford (2003) is representative of the former, while Benhabib et al. (2001) is that of the latter. This study is based on the latter, in which the liquidity trap is regarded as the result of the emergence of multiple steady states.

Benhabib et al. (2001) demonstrate that a steady state other than that targeted by the central bank can emerge when a zero lower bound on nominal interest rates is considered. If the central bank conducts an active interest rate policy that satisfies the Taylor principle in the neighborhood of a targeted steady state, the economy will be trapped in a low-inflation (and sometimes deflationary) state.¹ They call this low-inflation steady state a liquidity trap.

Benhabib et al.'s (2001) model compounds a two-dimensional system of differential equations. They completely analyze the dynamics and demonstrate the existence of periodic solutions, a homoclinic orbit, and a path connecting the targeted steady state with the liquidity trap (i.e., a saddle connection). Further, Benhabib et al. (2002a) propose a model that compounds a system of difference equations to show the existence of chaos. They suggest that fiscal policies that can prevent an economy from entering a liquidity trap, considered by Benhabib et al. (2002b), are ineffective in avoiding chaos.

Moreover, Benhabib et al. (2003) propose a continuous-time model of liquidity traps that incorporates a backward-looking interest-rate rule under which the central bank adjusts the nominal interest rate in response to a weighted mean of past inflation rates. Their model compounds a three-dimensional system of differential equations. As in Benhabib et al. (2001), Benhabib et al. (2003) demonstrate that multiple steady states (i.e., a targeted steady state and a liquidity trap) can emerge and that periodic solutions, homoclinic orbits, and saddle connections exist. However, they make little reference to a torus.

This study examines the existence of a 2D torus under plausible parameter values in a continuous-time New Keynesian DGE model, which incorporates a backward-looking interest-rate rule with zero lower bound on the nominal interest rate.

Applying Kopell and Howard's (1975) theorem, Benhabib et al. (2003) demonstrate that a homoclinic orbit can exist around a targeted steady state. If a 2D torus does not exist, a solution originating from inside the orbit will converge to the steady state, because it is a stable spiral point (indeterminate). However, if a 2D torus exists, a solution

¹For general arguments about the effects of interest-rate policies using the New Keynesian DGE model, see Woodford (2003), Galí (2008), and Walsh (2010).

originating within the orbit will never attain the steady state unless the variables jump, because solutions keep fluctuating on a torus. Therefore, alongside the liquidity trap, there exists another possibility that an economy cannot reach a targeted steady state. Moreover, as Behnabib et al. (2002a) suggest, these fluctuations cannot be avoided by fiscal policies that can prevent an economy from falling into liquidity traps.

This paper is organized as follows. Section 2 proposes a continuous-time new Keynesian DGE model incorporating a backward-looking interest-rate rule with zero lower bound on the nominal interest rate. Section 3 presents a theorem needed to verify the existence of a homoclinic orbit and applies it to the model proposed in Section 2. Section 4 set the parameters at plausible values and numerically examines the existence of a 2D torus. Section 5 presents the conclusions.

2 The model

Although Benhabib et al.’s (2003) model takes the form of the “money in the production function,” we reconstruct it into the more typical “money in the utility function.”

The model comprises three economic agents: firms, households, and the consolidated government (central bank + government). We assume a perfectly competitive goods market, a monopolistically competitive labor market, sticky nominal wages, and flexible prices.

Firms

Firms aggregate a differentiated labor force l_i ($i \in [0, 1]$) supplied by households according to the Dixit-Stiglitz function as follows:²

$$l = \left[\int_0^1 l_i^{\frac{\phi-1}{\phi}} di \right]^{\frac{\phi}{\phi-1}}, \quad (1)$$

where l is the composite labor and ϕ (> 1) is the elasticity of labor supply.

Firms produce goods using composite labor l and a constant volume of input k that they possess (e.g., land). We specify the form of the production function as a constant elasticity of substitution:

$$y = [\zeta l^\delta + (1 - \zeta)k^\delta]^{\frac{1}{\delta}}, \quad (2)$$

²See Dixit and Stiglitz (1977) and Blanchard and Kiyotaki (1987) for details.

where $0 < \zeta < 1$ and δ is the technical substitutability between l and k .³ For simplicity, we employ the following assumption:

Assumption 1 l and k are perfectly substitutable ($\delta \rightarrow 1$).

Given the nominal wage rate of labor i W_i and the volume of composite labor l , firms determine the volume of l_i to minimize total cost $\int_0^1 W_i l_i di$. The first-order condition for optimality is given by

$$l_i = \left(\frac{W_i}{W} \right)^{-\phi} l, \quad (3)$$

where W is the nominal wage rate of composite labor defined as

$$W = \left[\int_0^1 W_i^{1-\phi} di \right]^{\frac{1}{1-\phi}}. \quad (4)$$

Because the goods market is perfectly competitive, firm profits are zero. Further, we assume that the value created by input k is distributed to households in a lump sum. Then $\zeta p = W$ holds, where p is the price of goods.

Households

Households obtain utility from consumption c and real money balances m ,⁴ and disutility from labor l_i and wage negotiations ω_i (reflected by the change rate in nominal wage).

We specify the form of the instantaneous utility function as

$$\ln c + \ln m - \frac{l_i^{1+\psi}}{1+\psi} - \frac{\eta}{2}(\omega_i - \bar{\omega})^2,$$

where ψ (> 0) denotes the elasticity of marginal disutility of labor and η (> 0) denotes the stickiness of nominal wages; if $\eta > 0$, nominal wages are sticky, and if $\eta \rightarrow 0$, they are flexible.⁵ $\bar{\omega}$ is a constant that reflects households' subjective views about inflation. They take wage adjustment costs when the rate of change in the nominal wages differs from $\bar{\omega}$.

³In Benhabib et al.'s (2003) model, real money balances m enter Eq. (2) in place of k as an input.

⁴For precision, we must write c as c_i and m as m_i ; however, because households are identical in consumption and money holdings, we simply denote them by c and m , respectively.

⁵We followed Rotemberg's (1982) formulation of wage adjustment cost. $\frac{\eta}{2}(\omega_i - \bar{\omega})^2$ can be regarded as psychological stresses caused by wage negotiations.

Assuming that assets A are composed of money M and bonds B

$$A = M + B,$$

we obtain

$$\dot{A} = RB + W_i l_i + p(1 - \zeta)k - pc - p\tau, \quad (5)$$

where R is the nominal interest rate of bonds. Eq. (5) indicates that income $p(1 - \zeta)k$ as well as capital and wage income increase assets and that consumption and tax $p\tau$ decrease them. Rewriting Eq. (5) in terms of real variables, we obtain

$$\dot{a} = ra + w_i l_i + (1 - \zeta)k - c - \tau - Rm, \quad (6)$$

where $a = \frac{A}{p}$ is real asset balances, $r = R - \pi$ is the real interest rate, and $w_i = \frac{W_i}{p}$ is the real wage rate of labor i .

Household i determines the volume of c , m , and ω_i subject to the demand function for labor i (3) and the budget constraint (6) to maximize the discounted present value of the stream of utility, given by

$$U_i = \int_0^\infty \left[\ln c + \ln m - \frac{l_i^{1+\psi}}{1+\psi} - \frac{\eta}{2}(\omega_i - \bar{\omega})^2 \right] e^{-\rho t} dt,$$

where ρ (> 0) is the subjective discount rate.

First-order conditions for optimality are given by (see Appendix A.1)

$$\frac{\dot{c}}{c} + \pi + \rho = R = \frac{c}{m}, \quad (7)$$

$$\dot{\omega} = \rho(\omega - \bar{\omega}) - \frac{\phi l^{1+\psi}}{\eta} - \frac{(1 - \phi)\zeta l}{\eta c}. \quad (8)$$

Consolidated government

We formulate the central bank's policy rule as follows:

$$R = R^* e^{\frac{D}{R^*}(\pi^p - \pi^*)}, \quad (9)$$

which represents a backward-looking interest-rate rule with zero lower bound on the nominal interest rate.⁶ D (> 0) denotes the elasticity of the interest rate with respect to the

⁶The original Taylor rule by Taylor (1993) assumes that the nominal interest rate responds both to the inflation rate and the output (GDP gap). However, there is a positive correlation between the two variables; hence, there are no qualitative differences in the results of two cases where only inflation is considered and where both inflation and output are considered. For the sake of simplicity, we consider a special case where the nominal interest rate responds only to the inflation rate.

inflation rate. If $D > 1$, the central bank conducts an active policy around the targeted steady state (π^*); however, it conducts a passive policy if $D < 1$. π^p is a weighted mean of past inflation rates, defined as

$$\pi^p(t) = b \int_{-\infty}^t \pi(s) e^{-b(t-s)} ds, \quad (10)$$

where $b (> 0)$ denotes the degree of backward looking of the central bank. As Benhabib et al. (2003) show, if the central bank conducts an active policy around the targeted steady state, another steady state with low inflation (i.e., a liquidity trap) emerges.

Differentiating Eq. (10) with respect to time t , we obtain

$$\dot{\pi}^p = b(\pi - \pi^p). \quad (11)$$

Government expenditures are assumed to be zero. Then the budget constraint of the consolidated government is given by

$$\dot{B} + \dot{M} = RB - p\tau.$$

Rewriting this equation in terms of real variables, we obtain

$$\dot{a} = ra - \tau - Rm.$$

The present discounted value of total government liabilities must converge to zero:

$$\lim_{t \rightarrow \infty} a(t) e^{-\int_0^t r(s) ds} = 0.$$

This type of fiscal-monetary regime is referred to the Ricardian policy (see Benhabib et al. 2002b).

System of differential equations

Using the goods market equilibrium condition $y = c$, our model can be summarized in the following system of differential equations:

$$\begin{aligned} \dot{c} &= [R^* e^{\frac{D}{R^*}(\pi^p - \pi^*)} - \pi - \rho]c, \\ \dot{\pi} &= \rho(\pi - \bar{\pi}) - \frac{\phi(\frac{c}{\zeta} - \frac{1-\zeta}{\zeta}k)^{1+\psi}}{\eta} - \frac{(1-\phi)[1 - (1-\zeta)\frac{k}{c}]}{\eta}, \\ \dot{\pi}^p &= b(\pi - \pi^p), \end{aligned} \quad (12)$$

where $\bar{\pi} = \bar{\omega}$ because $\zeta p = W$.

Steady states

The steady-state value of π and π^p in the targeted steady state is given by $\pi^* = (\pi^p)^* = R^* - \rho$.

Define that

$$h(c) \equiv -\phi \left(\frac{c}{\zeta} - \frac{1-\zeta}{\zeta} k \right)^{1+\psi} - (1-\phi) \left[1 - (1-\zeta) \frac{k}{c} \right],$$

we then obtain

$$\dot{\pi} = \rho(\pi - \bar{\pi}) + h(c)/\eta.$$

Hence, the long-run Phillips curve is given by $\pi - \bar{\pi} = -h(c)/(\rho\eta)$. Given the value of π , the steady-state value of c is determined to satisfy this equation. Because the volume of employment $l = \frac{1}{\zeta}(y - (1-\zeta)k) = \frac{1}{\zeta}(c - (1-\zeta)k)$ must be non-negative, c must satisfy the condition that $c \geq \underline{c} \equiv (1-\zeta)k$.

Function $h(c)$ can be rewritten as follows:

$$h(c) = \left[-\frac{\phi}{\zeta^{1+\psi}}(c - \underline{c})^\psi - (1-\phi)\frac{1}{c} \right] (c - \underline{c}) \equiv \hat{h}(c)(c - \underline{c}).$$

Therefore, the value of c that satisfies the equation $\hat{h}(c) = g(c) \equiv \frac{-\rho\eta(\pi^* - \bar{\pi})}{c - \underline{c}}$ is the steady-state value.

- $\hat{h}(c)$ is monotonically decreasing for $c > \underline{c}$:

$$\hat{h}'(c) = -\frac{\phi\psi}{\zeta^{1+\psi}}(c - \underline{c})^{\psi-1} - (\phi - 1)c^{-2} < 0.$$

- $\hat{h}(\underline{c}) = (\phi - 1)\frac{1}{\underline{c}} > 0$ and $\lim_{c \rightarrow \infty} \hat{h}(c) = -\infty$.

In addition, if $\pi^* < \bar{\pi}$, function $g(c)$ is drawn as shown in Fig. 1; if $\pi^* > \bar{\pi}$, it is drawn as shown in Fig. 2. Whenever $\pi^* < \bar{\pi}$ and the value of $\bar{\pi}$ is sufficiently close to π^* , $\hat{h}(c) = g(c)$ has two real roots (c^* and \hat{c}_1) in the range of $c > \underline{c}$ (see Fig. 1). On the other hand, if $\pi^* > \bar{\pi}$, $\hat{h}(c) = g(c)$ has a unique real root (\hat{c}_2) in the range of $c > \underline{c}$ independent of parameter values.⁷

⁷If $\pi^* = \bar{\pi}$, \underline{c} becomes a steady-state value. The steady-state value \underline{c} represents an equilibrium at which employment is zero. In our model, k and l are perfectly substitutable; hence, an economy can produce and continue to consume goods using k only (without labor).

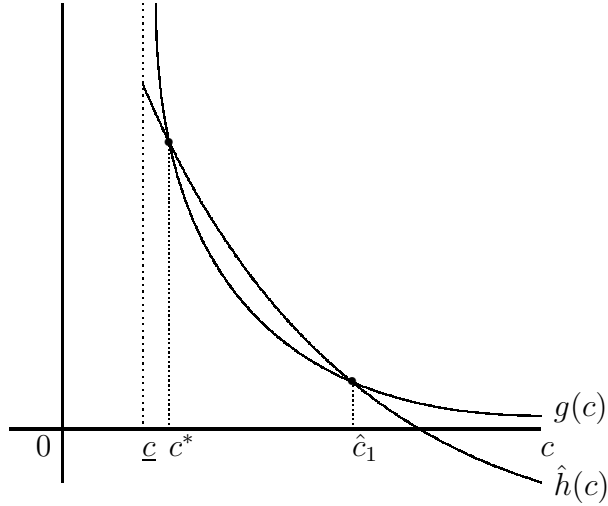


Figure 1: Case where $\pi^* < \bar{\pi}$ (two real roots exist)

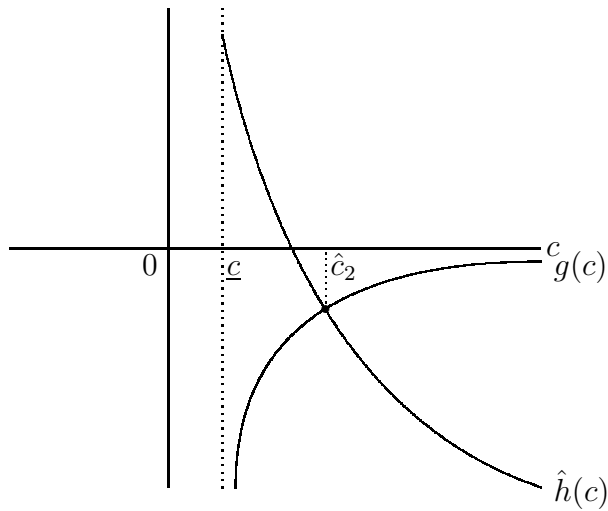


Figure 2: Case where $\pi^* > \bar{\pi}$

3 Existence of a homoclinic orbit

We present Kopell and Howard's (1975) theorem that provides conditions for assuring the existence of a homoclinic orbit.

Theorem 1 (Kopell and Howard, 1975; Theorem 7.1 and Corollary 7.1) : *Let*

$\dot{X} = F_{\mu,\nu}(X)$ be a two-parameter family of ordinary differential equations on \mathfrak{R}^n , F smooth in all its arguments, such that $F_{\mu,\nu}(\mathbf{0}) = \mathbf{0}$. Using a Taylor expansion, $\dot{X} = F_{\mu,\nu}(X)$ can be written

$$\dot{X} = (A + \mu A_1 + \nu A_2)X + Q(X, X) + R_1(X, \mu, \nu),$$

where A, A_1, A_2 are $n \times n$ matrices, the vector $Q(X, X)$ contains terms quadratic in x_i and is independent of (μ, ν) , and $R_1(X, \mu, \nu) = o(\mu x_i, \nu x_i, x_i x_j)$. Also assume

1. $dF_{0,0}(\mathbf{0})$ has a double-zero eigenvalue corresponding to a single eigenvector e .
2. The mapping $(\mu, \nu) \rightarrow (\det dF_{\mu,\nu}(\mathbf{0}), \sigma(dF_{\mu,\nu}(\mathbf{0})))$ has a nonzero Jacobian at $(\mu, \nu) = (0, 0)$, where $\sigma(dF_{\mu,\nu}(\mathbf{0}))$ is the sum of the principal minors of $dF_{\mu,\nu}(\mathbf{0})$.
3. $[dF_{0,0}(\mathbf{0})Q(e, e)]$ has rank n .

Then there is a curve $f(\mu, \nu) = 0$ such that if $f(\mu_0, \nu_0) = 0$, $\dot{X} = F_{\mu_0, \nu_0}(X)$ has a homoclinic orbit. This one-parameter family of homoclinic orbits (in X, μ, ν space) is on the boundary of a two-parameter family of periodic solutions. For all sufficiently small values of $|\nu|, |\mu|$, if $\dot{X} = F_{\mu,\nu}(X)$ has neither a homoclinic orbit nor a periodic solution, there is a unique trajectory joining the critical (fixed) points (i.e., a saddle connection).

Applying Theorem 1

Consider the three-dimensional system of differential equations (12). We transform variables so as to make the origin a fixed point:

$$x_1 = -\ln(c/c^*), \quad x_2 = \pi - \pi^*, \quad x_3 = \pi^p - \pi^*$$

Then System (12) can be rewritten as

$$\begin{aligned} \dot{x}_1 &= -R^* e^{\frac{D}{R^*} x_3} + (x_2 + \pi^*) + \rho, \\ \dot{x}_2 &= \rho(x_2 + \pi^* - \bar{\pi}) - \frac{\phi}{\eta} \left(\frac{c^*}{\zeta} e^{-x_1} - \frac{1-\zeta}{\zeta} k \right)^{1+\psi} - \frac{(1-\phi)[1 - (1-\zeta)\frac{k}{c^*} e^{x_1}]}{\eta}, \\ \dot{x}_3 &= b(x_2 - x_3). \end{aligned} \quad (13)$$

Further, we assume that $(\mu, \nu) = (b + \frac{A_{21}}{\rho}, 1 - D)$,⁸ where $A_{21} \equiv \frac{\phi(1+\psi)}{\eta} \left(\frac{c^*}{\zeta} - \frac{1-\zeta}{\zeta} k \right)^\psi \frac{c^*}{\zeta} + \frac{(1-\phi)(1-\zeta)\frac{k}{c^*}}{\eta}$.

⁸Changing the value of μ produces a corresponding change in the value of b .

Denote that $X = [x_1, x_2, x_3]'$, then Eq. (13) can be written as

$$\dot{X} = F_{\mu,\nu}(X) = (F1(X, \mu, \nu), F2(X, \mu, \nu), F3(X, \mu, \nu)).$$

Because F is smooth and $F_{\mu,\nu}(\mathbf{0}) = \mathbf{0}$, the premises of Theorem 1 are satisfied. We can assert the following proposition.

Proposition 1 *If $A_{21} \neq 0$, there is a homoclinic orbit around the targeted steady state for a certain set of parameter values.*

Proof: See Appendix A.2.

4 Existence of a 2D torus

This section set the parameters at plausible values and demonstrates that a 2D torus can surround the targeted steady state in the system of differential equations (13).

Following Benhabib et al. (2003), we set parameter values as shown in Table 1.⁹ In the case where $\bar{\pi} = 0.08$ ($\bar{\pi} > \pi^*$), we obtain $c^* = 0.2012$ and $\hat{c}_1 = 0.8840$ under Table 1.¹⁰ On the other hand, if $\bar{\pi} = 0.001$, we obtain $\hat{c}_2 = 0.8875$.

π^*	D	ϕ	ρ	ψ	η	ζ	k
0.01	1.5	21	0.005	1	350	0.8	1

Table 1: Parameter values

Steady state c^*

When the steady-state value of c is given by $c^* = 0.2012$, we obtain $A_{21} = -0.0568 \neq 0$. Therefore, System (13) satisfies the conditions of Theorem 1 and has a homoclinic orbit.

Taking b as a bifurcation parameter, the dynamics of the system around the targeted steady state c^* vary as shown in Table 2, depending on the value of b .

⁹We set $\zeta = 0.8$ and $k = 1$ here, but the main results in this study are not dependent on these values and hold for a relatively wide range of values for ζ and k .

¹⁰ $h'(0.2012) > 0$ and $h'(0.8840) < 0$. The long-run Phillips curve is given by $\pi - \bar{\pi} = -h(c)/(\rho\eta)$. Hence, the relation is negative between c and π when $h'(c) > 0$, whereas the relation is positive when $h'(c) < 0$. Thus, $c = 0.8840$ is viewed as the “usual” steady state.

Value of b	
$b \in (0, b_h)$	Unstable spiral (determinate).
$b = b_h \simeq 0.0033$	Hopf bifurcation.
$b \in (b_h, b_{sc})$	Stable spiral (indeterminate) & unstable cycles.
$b = b_{sc} \simeq 0.0054$	Homoclinic bifurcation.
$b_{sc} < b$	Saddle connection.

Table 2: Dynamics around the targeted steady state c^*

b_h is the value of b that satisfies the conditions for a Hopf bifurcation, provided in Appendix A.3. In the setting of Table 1, the Hopf bifurcation is subcritical, and therefore an unstable cycle exists around the stable spiral point. Moreover, we calculate the homoclinic bifurcation value b_{sc} as a value where cycles vanish.¹¹ Fig. 3 shows the numerical result when $b = b_{sc}$. A homoclinic orbit is evident.¹²

In addition, after plotting D on the abscissa and b on the ordinate, we can see the changes in b_h and b_{sc} when D increases (Fig. 4).

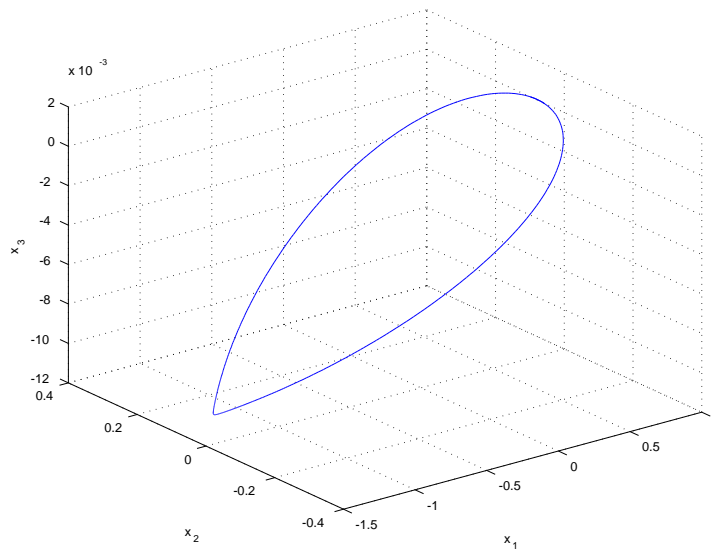


Figure 3: Homoclinic orbit

¹¹Technically, the homoclinic bifurcation value can be obtained using the Melnikov method. See Guckenheimer and Holmes (1983, p. 371) for details.

¹²Initial values of (x_1, x_2, x_3) are $(x_1, x_2, x_3) = (0.811, 0.02, 0.001)$.

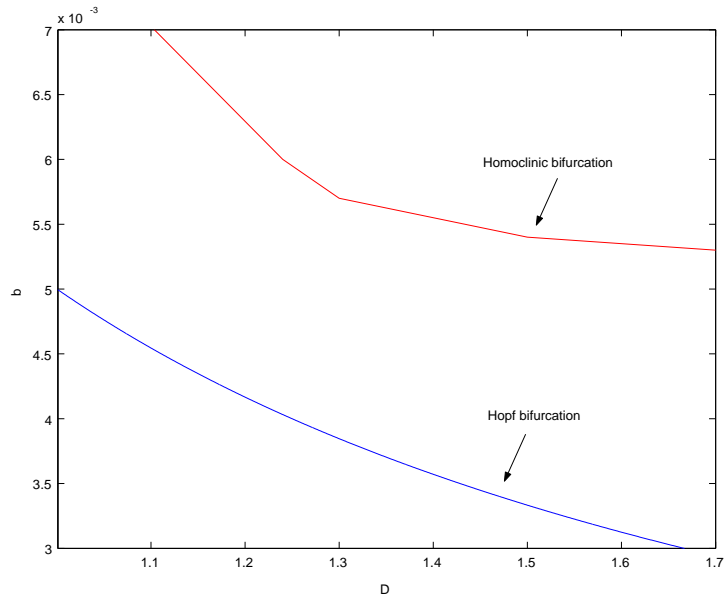


Figure 4: Changes in b_h and b_{sc}

Fig. 5 shows the dynamics of a solution starting from a point inside the homoclinic orbit. The two of three Lyapunov exponents converge to zero (Fig. 6¹³). Hence, a 2D torus exists inside the homoclinic orbit.

Such dynamics of a solution can also be seen for the values of b in the neighborhood of b_{sc} .

Steady states \hat{c}_1 and \hat{c}_2

When steady-state values of c are given by $\hat{c}_1 = 0.8840$ (where $\bar{\pi} > \pi^*$) and $\hat{c}_2 = 0.8875$ (where $\bar{\pi} < \pi^*$), the values of A_{21} are given by $A_{21} = 0.1004 \neq 0$ and $A_{21} = 0.1015 \neq 0$, respectively. Therefore, in these cases, System (13) also satisfies the conditions of Theorem 1 and has a homoclinic orbit.

We also confirmed that the two of three Lyapunov exponents converge to zero. Hence, a 2D torus exists.

¹³We construct this figure based on a Matlab code presented by Wolf et al. (1985).

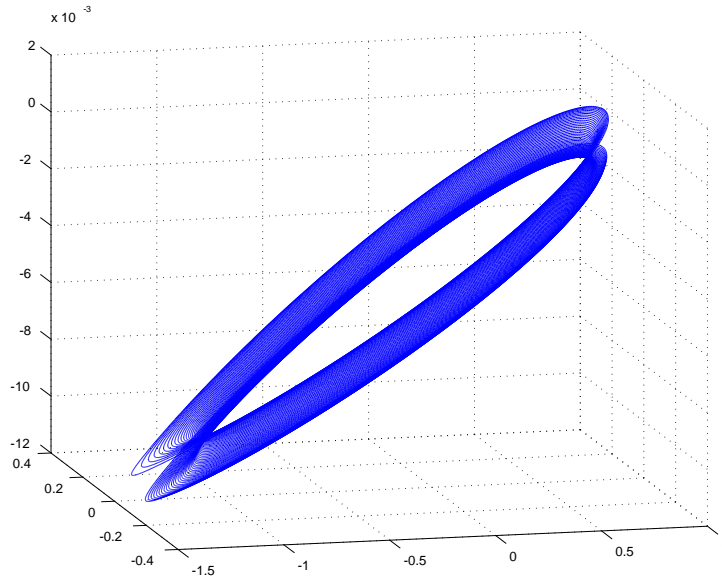


Figure 5: Dynamics of a solution starting from a point inside the homoclinic orbit

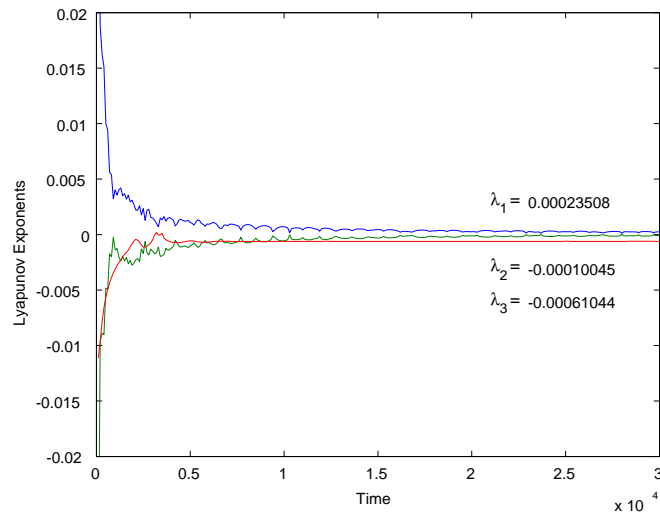


Figure 6: Dynamics of Lyapunov exponents

5 Conclusion

This study has analytically demonstrated the existence of a homoclinic orbit by applying Kopell and Howard's (1975) theorem to a variant of a continuous-time New Keynesian DGE model of a liquidity trap (MIUF version of Benhabib et al's 2003 model).

Under a plausible set of parameter values, we confirmed the existence of a 2D torus inside a homoclinic orbit. Therefore, it is possible that a solution originating from a point inside a homoclinic orbit keeps fluctuating on a torus and can never attain the targeted steady state.

A Appendix

A.1 Household optimization

The household optimization problem is described as

$$\text{Maximize}_{c,m,\omega_i} U_i = \int_0^\infty \left[\ln c + \ln m - \frac{l_i^{1+\psi}}{1+\psi} - \frac{\eta}{2}(\omega_i - \bar{\omega})^2 \right] e^{-\rho t} dt,$$

subject to

$$\dot{a} = ra + w_i l_i + (1 - \zeta)k - c - \tau - Rm,$$

$$\dot{W}_i = \omega_i W_i,$$

$$l_i = \left(\frac{W_i}{W} \right)^{-\phi} l.$$

The Hamiltonian function of this problem can be written as

$$\begin{aligned} \mathcal{H} = & \ln c + \ln m - \frac{[(\frac{W_i}{W})^{-\phi} l]^{1+\psi}}{1+\psi} - \frac{\eta}{2}(\omega_i - \bar{\omega})^2 \\ & + \mu_1 \left[ra + \frac{W_i}{p} \left(\frac{W_i}{W} \right)^{-\phi} l - c - Rm \right] + \mu_2 \omega_i W_i, \end{aligned}$$

where μ_1 and μ_2 are the costate variables of a and W_i , respectively. We obtain the

first-order conditions for optimality:

$$\frac{\partial \mathcal{H}}{\partial c} = \frac{1}{c} - \mu_1 = 0, \quad (\text{A.1})$$

$$\frac{\partial \mathcal{H}}{\partial m} = \frac{1}{m} - \mu_1 R = 0, \quad (\text{A.2})$$

$$\frac{\partial \mathcal{H}}{\partial \bar{\omega}} = -\eta(\omega_i - \omega^*) + \mu_2 W_i = 0, \quad (\text{A.3})$$

$$\dot{\mu}_1 = \rho \mu_1 - \frac{\partial \mathcal{H}}{\partial a} = (\rho - r)\mu_1, \quad (\text{A.4})$$

$$\dot{\mu}_2 = \rho \mu_2 - \frac{\partial \mathcal{H}}{\partial W_i} = \rho \mu_2 - \left[\frac{\phi l_i^{1+\psi}}{W_i} + \mu_1 (1 - \phi) \frac{l_i}{p} + \mu_2 \omega_i \right]. \quad (\text{A.5})$$

The transversality conditions are

$$\lim_{t \rightarrow \infty} a(t) e^{-\rho t} = 0,$$

$$\lim_{t \rightarrow \infty} W_i(t) e^{-\rho t} = 0.$$

The Hessian of $\mathcal{H}(c, m, \omega_i)$, which we denote as $|H|$, is given by

$$|H| = \begin{vmatrix} -\frac{1}{c^2} & 0 & 0 \\ 0 & -\frac{1}{m^2} & 0 \\ 0 & 0 & -\eta \end{vmatrix}.$$

The principal minors of $|H|$, which we denote $|H_1|$, $|H_2|$, and $|H_3|$, are given by

$$|H_1| = -\frac{1}{c^2}, \quad |H_2| = \begin{vmatrix} -\frac{1}{c^2} & 0 \\ 0 & -\frac{1}{m^2} \end{vmatrix} = \frac{1}{c^2 m^2}, \quad |H_3| = |H| = -\frac{1}{c^2 m^2 \eta}.$$

Because $|H_1| < 0$, $|H_2| > 0$, and $|H_3| < 0$, the second-order conditions for maximizing \mathcal{H} are satisfied.

Because households are symmetric, we can drop subscript i from l_i , W_i , and ω_i ; $l_i = l$, $W_i = W$, and $\omega_i = \omega$. Considering these expressions, Eqs. (A.1)–(A.5) are combined to obtain

$$\frac{\dot{c}}{c} + \pi + \rho = R = \frac{c}{m}, \quad (\text{A.6})$$

$$\dot{\omega} = \rho(\omega - \bar{\omega}) - \frac{\phi l^{1+\psi}}{\eta} - \frac{(1 - \phi)\zeta l}{\eta c}. \quad (\text{A.7})$$

A.2 Applying Theorem 1 to the system of differential equations (13)

Condition 1 The Jacobian matrix of F is given by

$$dF_{\mu,\nu}(X) = \mathbf{J} = \begin{bmatrix} 0 & 1 & -De^{\frac{D}{R^*}x_3} \\ Z_{21} & \rho & 0 \\ 0 & b & -b \end{bmatrix},$$

where $Z_{21} \equiv \frac{\phi(1+\psi)}{\eta} \left(\frac{c^*}{\zeta} e^{-x_1} - \frac{1-\zeta}{\zeta} k \right) \psi \frac{c^*}{\zeta} e^{-x_1} + \frac{(1-\phi)(1-\zeta) \frac{k}{c^*} e^{x_1}}{\eta}$.

Evaluating J at $X = \mathbf{0}$, we obtain

$$dF_{\mu,\nu}(\mathbf{0}) = \mathbf{J}^* = \begin{bmatrix} 0 & 1 & -D \\ A_{21} & \rho & 0 \\ 0 & b & -b \end{bmatrix}.$$

The characteristic equation of \mathbf{J} is written as

$$\begin{aligned} \Delta(\lambda) &= \det(\mathbf{J}^* - \lambda \mathbf{I}) \\ &= \det \begin{bmatrix} -\lambda & 1 & -D \\ A_{21} & \rho - \lambda & 0 \\ 0 & b & -b - \lambda \end{bmatrix} \\ &= \lambda^3 - (\rho - b)\lambda^2 - (\rho b + A_{21})\lambda - A_{21}b(1 - D). \end{aligned}$$

Because $b = \tilde{b} \equiv -A_{21}/\rho$ and $D = 1$ hold at $(\mu, \nu) = (0, 0)$, we obtain

$$dF_{0,0}(\mathbf{0}) = J_{0,0}^* = \begin{bmatrix} 0 & 1 & -1 \\ A_{21} & \rho & 0 \\ 0 & \tilde{b} & -\tilde{b} \end{bmatrix}.$$

The characteristic equation of this matrix is given by

$$\Delta_{0,0}(\lambda) = \lambda^3 - (\rho - \tilde{b})\lambda^2 = \lambda^2(\lambda - (\rho - \tilde{b})),$$

which has double-zero eigenvalues and one nonzero eigenvalue of $\rho - \tilde{b}$. Hence, Condition 1 is satisfied.

The eigenvector associated with zero eigenvalues $e = [e_1, e_2, e_3]'$ satisfies

$$\begin{aligned} dF_{0,0}(\mathbf{0})e &= 0e, \\ \begin{bmatrix} e_2 - e_3 \\ A_{21}e_1 + \rho e_2 \\ \tilde{b}e_2 - \tilde{b}e_3 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \end{aligned}$$

and therefore

$$e = q \begin{bmatrix} 1 \\ \tilde{b} \\ \tilde{b} \end{bmatrix},$$

where q is an arbitrary constant.

Condition 2

$$\begin{aligned} \det dF_{\mu,\nu}(\mathbf{0}) &= A_{21}b(1 - D) \\ &= A_{21} \left(\mu - \frac{A_{21}}{\rho} \right) \nu. \end{aligned}$$

Hence, the Jacobian matrix of the map $(\mu, \nu) \rightarrow (\det dF_{\mu,\nu}(\mathbf{0}), \sigma(dF_{\mu,\nu}(\mathbf{0})))$ is given by

$$\begin{bmatrix} A_{21}(1 - D) & A_{21}b \\ \sigma_\mu & \sigma_\nu \end{bmatrix}.$$

Evaluating this matrix at $(\mu, \nu) = (0, 0)$, we obtain

$$\begin{bmatrix} 0 & A_{21}\tilde{b} \\ \sigma_\mu(dF_{0,0}(\mathbf{0})) & \sigma_\nu(dF_{0,0}(\mathbf{0})) \end{bmatrix}.$$

This matrix cannot become zero as long as $\tilde{b} \neq 0$ (i.e., $A_{21} \neq 0$). Therefore, if $\tilde{b} \neq 0$, Condition 2 is satisfied. There is no need to calculate σ because it is only necessary to confirm that the above matrix is nonzero.

Condition 3 The Hessian matrix of $F1$ is given by

$$\mathbf{H}_1 = \frac{d^2 F1}{dX^2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{D^2}{R^*} e^{\frac{D}{R^*} x_3} \end{bmatrix},$$

the Hessian matrix of $F2$ is given by

$$\mathbf{H}_2 = \frac{d^2 F2}{dX^2} = \begin{bmatrix} -A_{21}(1 + \psi)e^{-x_1(1+\psi)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

and the Hessian matrix of $F3$ is given by

$$\mathbf{H}_3 = \frac{d^2 F3}{dX^2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Evaluating these matrices at $(X, \mu, \nu) = (\mathbf{0}, 0, 0)$, we obtain

$$\mathbf{H}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{R^*} \end{bmatrix}, \quad \mathbf{H}_2 = \begin{bmatrix} -A_{21}(1 + \psi) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{H}_3 = \mathbf{0}.$$

Using these expressions, we obtain

$$Q(X, X) = \begin{bmatrix} \frac{1}{2}X'H_1X \\ \frac{1}{2}X'H_2X \\ \frac{1}{2}X'H_3X \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\frac{x_3^2}{R^*} \\ -\frac{1}{2}A_{21}(1 + \psi)x_1^2 \\ 0 \end{bmatrix}.$$

Substitute the eigenvector $e = [1, \tilde{b}, \tilde{b}]'$ into X to obtain

$$Q(e, e) = \begin{bmatrix} -\frac{1}{2}\frac{\tilde{b}^2}{R^*} \\ -\frac{1}{2}A_{21}(1 + \psi) \\ 0 \end{bmatrix}.$$

Therefore,

$$[dF_{0,0}(\mathbf{0})Q(e, e)] = \begin{bmatrix} 0 & 1 & -1 & -\frac{1}{2}\frac{\tilde{b}^2}{R^*} \\ -\tilde{b}\rho & \rho & 0 & -\frac{1}{2}A_{21}(1 + \psi) \\ 0 & \tilde{b} & -\tilde{b} & 0 \end{bmatrix}.$$

As long as $\tilde{b} \neq 0$ ($A_{21} \neq 0$), the rank of this matrix is 3. Hence, if $\tilde{b} \neq 0$, Condition 3 is satisfied.

A.3 Hopf bifurcation theorem

The Hopf bifurcation theorem for a three-dimensional system of differential equations represented by the coefficients of a characteristic equation is proposed by Asada (1995).

Theorem 2 (Asada, 1995) : *There are periodic solutions that bifurcate from the fixed point X^* if and only if the characteristic equation $\Delta(\lambda; \mu) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$ satisfies the following conditions at $\mu = \mu_1$:*

- (i) $a_1 \neq 0$, $a_2 > 0$, $\Delta_1 \equiv a_1a_2 - a_3 = 0$,
- (ii) $\frac{d\Delta_1}{d\mu} \neq 0$.

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