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## **Abstract**

We study a Cournot duopoly dynamic model in which reaction functions are piecewise linear. Such a model typically generates ergodic chaos when it involves strong nonlinearities. To investigate statistical properties, we construct explicit forms of density functions associated with chaotic trajectories. We demonstrate that the long-run average behavior possesses regular properties although each chaotic trajectory exhibits irregular motions. In particular, the ratios of the average outputs as well as the average profits are the same as those of Cournot outputs as well as Cournot profits.

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# 1 Introduction

A Cournot duopoly model describes the output adjustment process in a market where a duopolist takes account of not only the behavior of consumers in terms of the market demand but also the behavior of the rival in terms of the forecasts of rival's output. The original model was introduced by Augustin Cournot in 1838 and has since been extended in various directions; see Puu [2002] for a historical account. The stability of the Cournot dynamics had been taken for granted for 140 years. However, Rand [1978] has shown that chaotic fluctuations can arise in the *non-linear* Cournot model where reaction functions are unimodal. Since Rand mathematically demonstrated the existence of chaotic economic dynamics, various modifications have been carried out by numerous economists. Rosser [2002] has a good review of the development of the theory of complex oligopoly dynamics since then. A lot of effort is still being devoted to investigate the dynamic structure of the nonlinear Cournot model, with recent developments summarized in Puu and Sushko [2002].

Chaotic dynamics has two salient features: irregularity of trajectory and extreme sensitivity to the initial conditions. The former implies that as a trajectory fluctuates erratically, it is difficult to distinguish chaotic behavior of a deterministic process from truly random behavior of a stochastic process. The latter implies that even a slightly different choice of initial conditions can drastically alter the whole future behavior of trajectories. Each chaotic trajectory moves in such a complicated way that it is difficult to predict its long-term behavior. Yet, Day and Pianigiani [1991] turn their attention to the statistical behavior that frequencies of trajectories can converge to a stable density function which characterize chaotic economic dynamics from the long-run point of view. Inspired by their results, Huang [1995] raises a provocative question of whether a complex (i.e., chaotic) dynamics is preferable to simple (i.e., stable) dynamics and provides an affirmative answer that perpetual fluctuations generated by a cautious cobweb model may be preferable to a stationary equilibrium. On the other hand, Kopel [1997] constructs a simple model of evolutionary dynamics and shows a negative result that various performance measures such as aggregate profits, aggregate sales revenues and mean sales indicate the inferiority of chaotic dynamics to equilibrium on the average. Matsumoto [2003a] shows an equivocal result in a simple exchange model with two agents and two goods that the long-run average performance measure taken along a chaotic trajectory is greater than the corresponding measure calculated at equilibrium for one agent but smaller for the other agent. Matsumoto [2003b] also considers a piecewise linear map to investigate qualitative implication of complex dynamics involving chaos.

It is important to investigate the long-run or statistical properties of chaotic dynamics so as to understand its economic implications.

So far, multidimensional chaos has been presented in variants of a Cournot model while the qualitative economic characterization of chaotic dynamics has been generated using a one dimensional map. So a natural question which we raise is about the statistical properties of multidimensional chaos. To answer this question is the purpose of this study. In particular, this study investigates the long-run average behavior of Cournot chaotic dynamics. To this end, we construct a piecewise linear duopoly model and derive, analytically as well as numerically, the long-run average outputs and the long-run average profits along chaotic trajectories.

This paper is organized as follows. Section 2 outlines a piecewise linear duopoly model, which allows us to construct an explicit form of density function of chaotic trajectory and thus to analytically study the statistical dynamics. Section 3 derives a two-dimensional Cournot adjustment process. Section 4 reduces the dimension of the dynamical system from two to one and shows that both system can generate qualitatively the same dynamics. Section 5, the main part of this study, demonstrates analytically as well as numerically that statistical dynamics has rather regular properties such that ratios of the average outputs and the average profits are constant, although each chaotic trajectory exhibits highly irregular motions. Section 6 provides conclusions.

## 2 Duopoly Model

Two possible microeconomic foundations of the unimodal reaction functions of a nonlinear Cournot model have been provided. One is by Puu [2000] who shows that linear production technologies and a hyperbolic market demand may result in unimodal reaction functions. The other is by Kopel [1996] who makes certain that the different assumptions of linear demand function and nonlinear cost function involving production externalities also result in unimodal reaction functions. In this study we follow Kopel's approach and construct a Cournot duopoly model with a production externality.

On the demand side, we assume that the inverse demand function is linear,

$$p = a - bQ, \quad a > 0 \text{ and } b > 0, \quad (1)$$

where  $Q$  is the industry demand. On the supply side, we assume that two firms, denoted by  $X$  and  $Y$ , produce homogenous outputs  $x$  and  $y$ . Provided demand equals supply,  $Q = x + y$ . Each firm forecasts the competitor's production and faces a production externality where the production possibilities

of one firm are influenced by the production of the other. Although there are various ways to introduce externalities, we confine our analysis to the case in which the cost of production depends on not only its own production but also its rival's production.

We say that a firm has a *positive production externality* if the firm's marginal cost, for an increment of the rival's output, is decreasing and a *negative production externality* if it is increasing. Forecasting the rival's output choice, each firm chooses a profit maximizing output for itself. The reaction function, thus, describes a functional relationship between the expected output of the rival and its own optimal choice. Due to the presence of production externality, the reaction function can be upward- or downward-sloping according to whether the external effect is positive or negative.

We call a firm *imitator* if its reaction function is upward-sloping and *accommodator* if downward-sloping. The imitator, who imitates the rival's behavior, expands or shrinks its own output if its rival expands or shrinks the rival's output. The accommodator is a text-book duopolist and adapts to reduce residual demand if the rival expands output. Further, we call a firm *dualist* if it has a dual reaction pattern, that is, if it changes its strategic profile from the imitator to the accommodator when the rival expands output beyond a pre-determined critical value.<sup>1</sup>

In what follows, we focus on the statistical behavior of the imitator and the dualist. Indeed, we examine the profit-maximizing behavior of, first, the imitator to derive a monotonic (invertible) reaction function and then, the behavior of the dualist to derive an asymmetric tent-shaped (i.e., noninvertible) reaction function.

## 2.1 Invertible Reaction Function

In this subsection, it is shown that in a situation involving a positive production externality, the profit-maximizing behavior of a firm is to follow the rival's behavior.

We suppose that firm  $X$  has a cost function,

$$c_i(y^e)x = \{(a - 2bA_i) - b(1 + 2\alpha_i)y^e\}x, \quad \alpha_i > 0, \quad (2)$$

where  $a - 2bA_i > 0$  is assumed to avoid the negative marginal cost and the subscript " $i$ " indicates the imitator.  $\alpha_i > 0$  implies that the more the rival firm is expected to produce, the less the marginal cost will be to produce an

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<sup>1</sup>We follow van Witteloostuijn and van Lier [1990] for the nomenclature, "imitator", "accommodator" and "dualist".

additional unit of its own production. The firm  $X$ 's profit is a revenue  $px$  minus cost  $c_i(y^e)x$ , which is, after arranging terms, spelled out as

$$\Pi_i(x, y^e) = \{2(A_i + \alpha_i y^e) - x\}bx. \quad (3)$$

The condition characterizing profit maximization is the equality between the marginal revenue and the marginal cost. Solving it for  $x$ , we have the reaction curve of firm  $X$ ,

$$r_i(y^e) = A_i + \alpha_i y^e. \quad (4)$$

This is a straight line with positive slope  $\alpha_i$  and an intercept  $A_i$ . It implies that the optimal strategy for firm  $X$  is to imitate the rival's strategy, in other words, firm  $X$  with (2) chooses to be an imitator.

From the profit function, we can also derive the zero-profit line which describes all combinations of  $x$  and  $y^e$  that yield a zero profit,

$$z_i(y^e) = 2(A_i + \alpha_i y^e). \quad (5)$$

Note that the zero-profit line has the same vertical intercept,  $-\frac{A_i}{\alpha_i}$ , as the reaction curve, but it is half as steep in the  $x$ - $y^e$  plane. This gives us an easy way to draw the zero-profit line. We know that the vertical intercept is  $-\frac{A_i}{\alpha_i}$ . To get the horizontal intercept, just take twice of the horizontal intercept of the reaction function. Drawing the straight line passing through these two intercepts gives us the zero-profit line. Thus the profit is negative for those combinations of  $x$  and  $y^e$  located below the zero-profit line. In Figure 1, the reaction curve (actually the bold straight line) is illustrated to pass through the lowest point of each  $U$ -shaped isoprofit curve, and all of the bundles,  $(x, y^e)$ , in the shaded area underneath the zero-profit line generate negative profits.

The cross-partial derivative of the imitator's marginal profit with respect to its rival's output is

$$\frac{\partial}{\partial y} \left( \frac{\partial \Pi_i}{\partial x} \right) = 2\alpha_i > 0. \quad (6)$$

Due to the presence of positive production externality, the imitator's marginal cost decreases as the rival increases its output so that it is preferable for the imitator to follow its rival's behavior. According to Bulow *et al.* [1985], this indicates that the imitator treats its own product as a *strategic complement* to the rival's output under the condition of a positive production externality.

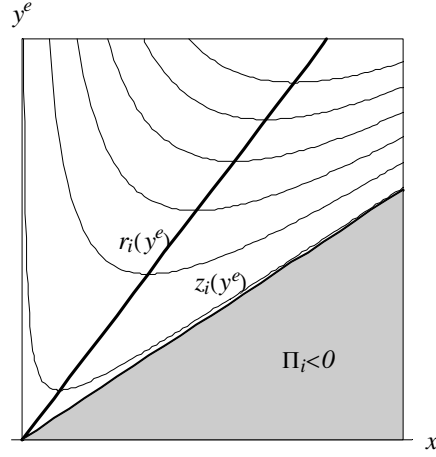


Figure 1. Upward sloping reaction and zero-profit curves.

## 2.2 Noninvertible Reaction Function

In this section, we consider a circumstance in which a firm can have an asymmetric reaction pattern. We assume that firm  $Y$ , which produces  $y$  and forecasts  $x^e$ , has a piecewise production cost function specified as

$$C_d(x^e, y) = \text{Max}[c_d^i(x^e)y, c_d^a(x^e)y], \quad (7)$$

where

$$c_d^i(x^e)y = \{(a - 2bA) - b(1 + 2\alpha)x^e\}y,$$

$$c_d^a(x^e)y = \{(a - 2bB) - b(1 - 2\beta)x^e\}y.$$

Here the subscript " $d$ " indicates the dualist. Depending on the scale of the rival's output, the dualist imitates as well as accommodates its rival.<sup>2</sup> Since  $\alpha > 0$  and  $\beta > 0$  are assumed, the marginal cost,  $c_d(x^e)$ , takes on a  $V$ -shaped profile which accounts for asymmetric externality.

The profit is also piecewise linear,

$$\Pi_d(x^e, y) = \text{Min}[\Pi_d^i(x^e, y), \Pi_d^a(x^e, y)], \quad (8)$$

<sup>2</sup>Kopel [1996] takes the book-buying-habit hypothesis to justify the dual reaction pattern.



where

$$\Pi_d^i(x^e, y) = \{2(A + \alpha x^e) - y\}by,$$

$$\Pi_d^a(x^e, y) = \{2(B - \beta x^e) - y\}by.$$

Given  $x^e$ , firm  $Y$  would maximize  $\Pi_d^i(x^e, y)$  with respect to  $y$  for  $0 < y < y_0$  and  $\Pi_d^a(x^e, y)$  for  $y_0 < y$  where  $y_0$  solves  $\Pi_d^i(x^e, y) = \Pi_d^a(x^e, y)$ , given  $x^e$ . Equating each partial derivative to zero, we can solve each for the reaction curve,

$$r_d(x^e) = \text{Min}[A + \alpha x^e, B - \beta x^e]. \quad (9)$$

Firm  $Y$  shows an asymmetric reaction pattern and its reaction curve is asymmetric tent-shaped. Firm  $Y$  with (7) chooses to be a dualist.  $r_d(x^e)$  is a map from  $[0, \frac{B}{\beta}]$  into  $[0, \frac{\alpha B + \beta A}{\alpha + \beta}]$ . For convenience, we reduce the reaction function to that of the special class by making the following assumptions.

**Assumption 1.**  $A = \frac{\alpha + \beta - \alpha\beta}{\beta}$  and  $B = \beta$  where  $\alpha > 0$  and  $\beta > 0$ .

Due to Assumption 1, the reaction function can be written in a simple form with two parameters,

$$r_d(x^e) = \text{Min}[r_d^i(x^e), r_d^a(x^e)], \quad (10)$$

where

$$r_d^i(x^e) = \alpha x^e + \frac{\alpha + \beta - \alpha\beta}{\beta} \text{ for } 0 \leq x^e \leq x_0,$$

$$r_d^a(x^e) = \beta(1 - x^e) \text{ for } x_0 \leq x^e \leq 1.$$

where  $x_0 = 1 - \frac{1}{\beta}$  is the turning point of the reaction function and solves  $r_d^i(x) = r_d^a(x)$ . For  $\alpha + \beta - \alpha\beta \geq 0$ ,  $r_d(x^e)$  maps the unit interval into itself. For  $\alpha + \beta - \alpha\beta < 0$ , it should be restricted to  $I' = [-\frac{A}{\alpha}, 1]$  because outside this range, the implied output would be negative, which is inadmissible. Alternatively, define  $r_d(x) \equiv 0$  for all  $x \in R \setminus I'$  then  $r_d(x)$  is extended to all positive real line.

From the definition of the profit, the zero-profit curves are also derived as

$$z_d^i(x^e) = 2\left(\frac{\alpha + \beta - \alpha\beta}{\beta} + \alpha x^e\right) \text{ for } 0 \leq x^e \leq x_0,$$

$$z_d^a(x^e) = 2\beta(1 - x^e) \text{ for } x_0 \leq x^e \leq 1.$$

A reaction curve as well as the zero-profit curve are illustrated in Figure 2 in which profits are negative for all the bundle  $(x^e, y)$  in the shaded areas. It can be seen that the dualist chooses the output level associated with the isoprofit line furthest to the right for  $0 \leq x^e \leq x_0$  and to the left for  $x_0 \leq x^e \leq 1$ .

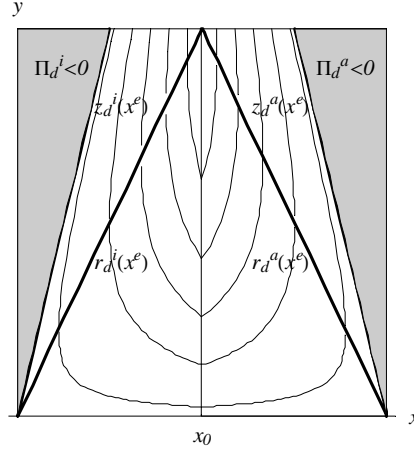


Figure 2. Piecewise linear reaction and zero-profit curves.

Returning to the cost function, it can be seen that marginal cost curve is V-shaped due to the presence of positive-negative production externality. It has a minimum at the turning point  $x_0$  and is strictly decreasing on  $[0, x_0]$  and strictly increasing on  $[x_0, 1]$ . Since the minimum marginal cost is attained for  $x_0$ ,

$$r_d^i(x_0) = r_d^a(x_0) = y \left\{ \frac{b}{\beta} + (a - 3b) \right\}, \quad (11)$$

which is non-negative if  $a \geq 3b$  holds. In the following, we make a stronger assumption and set  $a = 3b$  for convenience.

The cross-partial derivatives of firm Y's marginal profits with respect to its rival's output are

$$\begin{aligned} \frac{\partial}{\partial x} \left( \frac{\partial \Pi_d^i}{\partial y} \right) &= 2\alpha > 0 \text{ for } 0 \leq x \leq x_0, \\ \frac{\partial}{\partial x} \left( \frac{\partial \Pi_d^a}{\partial y} \right) &= -2\beta < 0 \text{ for } x_0 \leq x \leq 1. \end{aligned}$$

According to Bulow *et al.* [1985] these inequality conditions indicate that firm Y treats its own product as a *strategic complement* to the rival's output when the rival produces its output at less than the critical level,  $x_0$ , and as a *strategic substitute* when at greater than  $x_0$ .

### 3 Adjustment Process

We are interested in studying dynamic interactions between two firms. Since each of the two duopolist firms can be an imitator or dualist, depending on what production externality arises, there are four different duopoly markets. When both firms are imitators, the resultant dynamics are relatively simple as the reaction functions are monotonic. So we omit those simple case from further considerations. When both firms are dualists, the resultant dynamics is too complicated to yield analytical results and thus will be considered in a future study, based on the results obtained in this study.<sup>3</sup> So in this study we will limit ourselves to a market in which firm  $X$  is an imitator and firm  $Y$  is a dualist.

To consider the dynamic adjustment process of output, we lag the variables. Under the assumption of the naive expectation formation (i.e.,  $x_t^e = x_{t-1}$  and  $y_t^e = y_{t-1}$ ), a dynamics process is obtained by the iteration of the following two dimensional system,

$$T(x_t, y_t) = (r_i(y_t), r_d(x_t)) \quad (12)$$

where

$$r_i(y_t) = A_i + \alpha_i y_t,$$

$$r_d(x_t) = \text{Min}\left[\alpha x_t + \frac{\alpha + \beta - \alpha\beta}{\beta}, \beta(1 - x_t)\right].$$

Since the parameter  $A_i$ , which shifts the imitator's reaction function, does not affect the dynamical properties, we take it to be zero for the sake of analytical simplicity,

**Assumption 2.**  $A_i = 0$ .

A stationary state of this system is an intersection of the reaction functions of  $X$  and  $Y$ , and we call it the *Cournot point*. Due to the tent-shaped profile of dualist's reaction function, multiple Cournot points possibly exist if  $\alpha_i\alpha > 1$ . The left panel of Figure 3 below depicts the case in which there exist two Cournot points, denoted as  $C$  and  $C'$ , such that  $r_i^{-1}(x_t) = r_d^i(x_t)$  and  $r_i^{-1}(x_t) = r_d^a(x_t)$ . These points are, respectively, derived as

$$x^c = \frac{\alpha_i\beta}{1 + \alpha_i\beta} \text{ and } y^c = \frac{\beta}{1 + \alpha_i\beta}. \quad (13)$$

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<sup>3</sup>There are many studies on nonlinear duopoly as well as triopoly dynamics in which firms have unimodal reaction functions. See, for example, Puu and Sushko [2002] for a survey of recent contributions.

and

$$x^{c'} = \frac{\alpha_i(\alpha\beta - \alpha - \beta)}{\beta(\alpha_i\alpha - 1)} \text{ and } y^{c'} = \frac{(\alpha\beta - \alpha - \beta)}{\beta(\alpha_i\alpha - 1)}, \quad (14)$$

where  $\alpha\beta - \alpha - \beta > 0$  is necessary for a positive bundle of  $(x^{c'}, y^{c'})$ .

Since the adjustment process is piecewise linear, the local stability of Cournot point occurs when the product of the slopes of the reaction functions is less than unity in absolute value,

$$\left| \frac{dr_i(y)}{dy} \frac{dr_d(x)}{dx} \right| < 1. \quad (15)$$

The product at  $(x^{c'}, y^{c'})$  is  $\alpha_i\alpha$  and the existence condition is  $\alpha_i\alpha > 1$ , which makes this point locally unstable. Trajectories starting in the neighborhood of  $(x^{c'}, y^{c'})$  monotonically diverges away from it. On the contrary, dynamics in the neighborhood of  $(x^c, y^c)$  displays a more complex behavior. Since the product is  $\alpha_i\beta$  at  $(x^c, y^c)$ , the Cournot point is locally stable or unstable according to  $\alpha_i\beta$  is less or greater than unity. Even if the point is unstable, dynamics around  $(x^c, y^c)$  is rich as the tent-shaped profile of dualist's reaction curve prevents unbounded output oscillations. Indeed, the dynamics has a wide spectrum ranging from periodic cycles to chaos and thus the Cournot point is globally stable in the sense that the trajectories stay in a bounded region. In this study, we are interested in statistical properties of chaotic fluctuations around  $(x^c, y^c)$ , and thus we assume local instability of the Cournot point,

**Assumption 3.**  $\alpha_i\beta > 1$ .

We confine our analysis to a *Markov Perfect Equilibrium* (MPE) trajectory, along which the dynamic process proceeds as follows. At the beginning of a period, a duopolist expects that its rival is going to continue to keep its output at the level produced in the previous period and would want to choose the profit maximizing output given that expectation. At the beginning of the next period, the rival duopolist can reason the same way, and then the process repeats. Graphically, a MPE trajectory visits two reaction curves alternately as illustrated in the right panel of Figure 3. Each firm can maximize its profit on its own reaction curve but cannot on its rival's reaction curve. In consequence, it is probable that the profit taken at a point on the rival's reaction curve may be negative under some circumstance. However, negative profit will generate unfavorable outcome. Since it is always possible to produce a zero level of output, a rational firm, as described in any elementary text book on microeconomics, prefers the choice of doing nothing than

producing something to earn negative profit. It is, further, possible that the firm, forecasting its rival producing nothing, prefers to produce nothing at the next period when the optimal choice is in the shaded region. Then no economic activities are carried out anymore.

To avoid the occurrence of negative profit, we restrict the choice of parameters. The left panel of Figure 3 superimposes Figure 1 on Figure 2 for  $\alpha = \frac{2\beta}{2\beta-3}$  and  $\beta = 2$ . It depicts an illustrative example in which profits can be non-negative if the initial point is selected appropriately. The shaded area indicates that profits of either or both duopolists are negative for bundles in that area. The right panel of Figure 3 is an enlargement of the reaction curves restricted to  $[x_m, x_M] \times [y_m, y_M]$  in the left of Figure 3. In this enlarged diagram, parameters are selected so as to make the restricted reaction function of the dualist have a tent-shape profile.<sup>4</sup> Point  $C$  is unstable due to Assumption 3 and a trajectory starting near it gradually moves away from it but the upper bound makes it bounce back to a neighborhood of Point  $C$ , and then moves away again as illustrated. The trajectory keeps moving up and down chaotically but stays in the restricted area. As can be seen clearly, there is no shaded area in the right panel, which means that the profits are non-negative if a trajectory stays within this area. We investigate how to find the critical values,  $x_m$ ,  $x_M$ ,  $y_m$  and  $y_M$ .

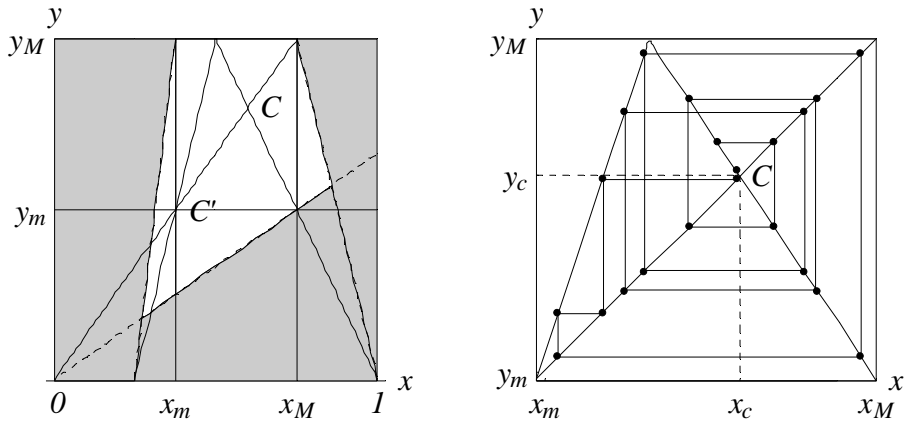


Figure 3. Restricted map (left panel) and its enlargement

We determine the parameter sets for which the following three conditions are simultaneously satisfied; (1) the imitator's reaction curve,  $x =$

<sup>4</sup>Note that the upward sloping line passing through point  $C$  is not the diagonal but the imitator's reaction function. The aspect ratio of the horizontal and vertical axes is adjusted to be 1 to 1.

$r_i(y)$ , crosses the dualist's negative sloping zero-profit curve,  $y = z_d^a(x)$ , at  $(x_M, y_M)$ ; (2) the imitator's zero-profit curve,  $x = z_i(y)$ , crosses the dualist's negative sloping reaction curve,  $y = r_d^a(x)$ , at  $(x_M, y_m)$ ; (3)  $r_d^a(x_m) \geq y_m$ . Condition (1) concerns a non-negative profit condition for a dualist. The zero-profit curve,  $y = z_d^a(x)$ , intersects the  $y = 1$  locus at  $x_d \equiv 1 - \frac{1}{2\beta}$  while the imitator's reaction curve,  $y = r_i(x)$ , intersects the  $y = 1$  locus at  $x_i \equiv \alpha_i$ . Thus if the inequality condition,

$$x_i \leq x_d \tag{16}$$

holds, then for  $x \geq x_m$  the imitator's reaction curve is not located in an area in which dualist's profit is negative,  $\Pi_d^a < 0$ .

Condition (2) is a non-negative profit condition for an imitator. Under Assumption 2, the positive-sloping zero-profit curve of the imitator starts at the origin and takes a positive value for  $x = 1$  while the downward-sloping reaction curve of the dualist passes through point  $(1, 0)$ . Therefore, these two curves must cross each other and, in consequence, some part of the dualist's reaction curve,  $y = r_d^a(x)$ , is inevitably located in a region in which the imitator's profit is negative. To prevent  $\Pi_i$  from being negative, we need to ensure that a MPE trajectory does not enter the region of  $\Pi_i(x) < 0$ . Solving  $x = z_i(y)$  and  $y = r_d^a(x)$  simultaneously for  $x$  gives the point at which the two curves intersect. We denote it by  $x_{id} = \frac{2\alpha_i\beta}{1+2\alpha_i\beta}$ . The imitator's profit is negative on the dualist's reaction curve for  $x > x_{id}$ . If the maximum output of the imitator can be designed to be not greater than  $x_{id}$ , then the maximum output along a MPE trajectory is less than or equal to  $x_{id}$  so that the imitator's profit can be non-negative along such a MPE trajectory. Denote the maximum output of the dualist by  $y_M$ , which is unity. For a given  $y_M$ , the maximum output of the imitator is  $x_i$ . Then we have

$$x_{id} - x_i = \frac{2\alpha_i\beta}{1 + 2\alpha_i\beta}(x_d - x_i) \tag{17}$$

which is non-negative if  $x_d \geq x_i$ . As seen above, this is the condition for which the dualist's profit is positive on the imitator's reaction curve. The same condition works to prevent the imitator's profit from being negative. For the sake of analytical simplicity, we assume  $x_d = x_i$  which is equivalent to

**Assumption 4.**  $\alpha_i = 1 - \frac{1}{2\beta}$ .

Condition (3) is imposed to eliminate the Cournot point of  $(x^c, y^c)$  from the restricted area. If it is in the restricted area, a trajectory sooner or later

escapes from the area as the point is unstable and then enters the shaded area in which profits are negative. Let  $y_m$  denote dualist's optimal output when its expectation is  $x^e = x_i$ . Since  $r_d^a(x)$  is negative-sloping,  $y_m = r_d^a(x_M)$  is the least output for the dualist corresponding to the largest output of the imitator. Further, let  $x_m$  denote imitator's optimal output when an expected output of the rival is  $y^e = y_m$ , so that  $x_m = r_i(y_m)$ . Finally if  $r_d^i(x_m) \geq y_m$  holds, then the minimum output set along a MPE trajectory is the greater of  $(x_m, y_m)$  so that such a MPE trajectory never enters an area in which  $\Pi_i < 0$  or  $\Pi_d^i < 0$ . Under Assumption 4,  $r_d^i(x_m) \geq y_m$  holds if

$$3\alpha + 2\beta - 2\alpha\beta \geq 0 \text{ or } \alpha < \frac{2\beta}{2\beta - 3}. \quad (18)$$

If we set  $x_M = 1 - \frac{1}{2\beta}$  (i.e.,  $x_M = x_d = x_i$ ), then we can determine, in addition to  $y_M = 1$ , other critical values as follows

$$x_m = \frac{1}{2}\left(1 - \frac{1}{2\beta}\right) \text{ and } y_m = \frac{1}{2}. \quad (19)$$

Since we use these critical values in the latter analysis, we formally define them.

**Assumption 5.**  $x_M = 1 - \frac{1}{2\beta}$ ,  $x_m = \frac{1}{2}x_M$ ,  $y_M = 1$  and  $y_m = \frac{1}{2}y_M$ .

The instability condition,  $\alpha_i\beta > 1$ , is transformed to  $\beta > \frac{3}{2}$  under Assumption 4. We summarize these results as follows.

**Theorem 1** *Suppose Assumptions 1, 2, 3, 4 and 5 hold. Then, for  $(\alpha, \beta)$  such as that  $3\alpha + 2\beta - 2\alpha\beta \geq 0$  and  $\beta > \frac{3}{2}$ , a MPE trajectory of  $T(x_t, y_t)$  starting at an initial point  $(x_0, y_0)$  selected in such a way that  $x_m < x_0 < x_M$  and  $y_0 = r_d(x_0)$  stays within a restricted area  $[x_m, x_M] \times [y_m, y_M]$  and thus profits of dualist and imitator are non-negative along such a MPE trajectory.*

## 4 Reduction of Dynamical System

The evolution of output is described by the two-dimensional dynamical system,  $T(x_t, y_t) = (r_i(y_t), r_d(x_t))$ . By Assumption 4, the Cournot point is unstable and thus a trajectory generated by the system neither converges nor diverges but keeps oscillating within a bounded region. Typical dynamics generated by  $T$  is chaos. One way to characterize such dynamics is to consider its long-run average (statistical) behavior. In this section, we reduce

the dimensionality of the dynamical system from two to one and investigate certain properties of the resulting complex output dynamics.

A MPE trajectory of output is given by iterating  $T(x_t, y_t)$  with an initial point which lies on either of the reaction curves. Since each firm moves alternately, their optimal movements are described by iterating the composite maps of reactions functions. Let

$$F(x) = r_i \circ r_d(x) \text{ and } G(y) = r_d \circ r_i(y). \quad (20)$$

Each composite map advances two periods from period  $t$  to period  $t + 2$ . Thus the evolution of the dualist is given by

$$y_{t+2} = G(y_t) : I_y \rightarrow I_y, \quad (21)$$

where  $I_y = [y_m, y_M]$  is the trapping interval and

$$G(y_t) = \text{Min}[\alpha_i \alpha y_t + \frac{\alpha + \beta - \alpha \beta}{\beta}, \beta(1 - \alpha_i y_t)]. \quad (22)$$

$G(y)$  is tent-shaped and has a turning point,  $y_0 = \frac{1}{\alpha_i}(1 - \frac{1}{\beta})$ . Similarly, the evolution of the imitator is given by

$$x_{t+2} = F(x_t) : I_x \rightarrow I_x, \quad (23)$$

where  $I_x = [x_m, x_M]$  is the trapping interval and

$$F(x_t) = \text{Min}[\alpha_i \alpha x_t + \alpha_i \frac{\alpha + \beta - \alpha \beta}{\beta}, \alpha_i \beta(1 - x_t)]. \quad (24)$$

$F(x)$  is also tent-shaped and has a turning point  $x_0 = 1 - \frac{1}{\beta}$  where  $x_0 = \alpha_i y_0$  holds.

As considered in Bischi, *et al.* [2000], dynamics generated by  $T(x, y)$  is strongly related to dynamics of  $F(x)$  and  $G(y)$ . Further, since the imitator's reaction function is invertible, two one-dimensional maps,  $F(x)$  and  $G(y)$ , are dynamically equivalent in the sense that they share the same stability properties, that is, if one gives rise stable dynamics, periodic cycles or complex dynamics, then so does the other. Hence, to understand dynamical properties of the two-dimensional map  $T(x, y)$ , it suffices to study dynamic properties of one of these one-dimensional maps. Both maps are single-peaked piecewise linear and have a turning point that divides the trapping interval into two subintervals. Maps are increasing on one subinterval and decreasing on the other. So the considerations of these two maps can be reduced to that of a single piecewise linear map with two parameters defined by

$$f_{\tilde{\alpha}, \tilde{\beta}}(u) = \text{Min}[\tilde{\alpha}u + \frac{\tilde{\alpha} + \tilde{\beta} - \tilde{\alpha}\tilde{\beta}}{\tilde{\beta}}, \tilde{\beta}(1 - u)] : [0, 1] \rightarrow [0, 1], \quad (25)$$



where  $\tilde{\alpha} = \alpha_i \alpha$  and  $\tilde{\beta} = \alpha_i \beta$ . We study the dynamical properties of  $F(x)$  and  $G(y)$  through the investigations of  $f_{\tilde{\alpha}, \tilde{\beta}}(u)$ .

Since  $F(x)$  as well as  $G(y)$  have a unique and unstable stationary state under Assumption 3 and 5,  $f_{\tilde{\alpha}, \tilde{\beta}}(u)$  should possess the same properties. To ensure this, we restrict combinations of parameters of  $f_{\tilde{\alpha}, \tilde{\beta}}(u)$  to the following set,  $D$ ,

$$D = \{(\tilde{\alpha}, \tilde{\beta}) \mid \tilde{\alpha} > 0, \tilde{\beta} > 1, \tilde{\alpha}\tilde{\beta} > 1 \text{ and } \tilde{\alpha} + \tilde{\beta} > \tilde{\alpha}\tilde{\beta}\}. \quad (26)$$

A fixed point of  $f_{\tilde{\alpha}, \tilde{\beta}}$  is  $u_E = \frac{\tilde{\beta}}{\tilde{\beta}+1}$  which is stable for  $\tilde{\beta} < 1$ .  $f_{\tilde{\alpha}, \tilde{\beta}}$  is reduced to  $1 - u$  and generates a period-two cycle for all initial points if  $\tilde{\beta} = 1$ . A fixed point of the second iterate that differs from  $u_E$  is identified with a periodic point with period two and is shown to be stable if  $\tilde{\alpha}\tilde{\beta} < 1$ . It is also verified that  $f_{\tilde{\alpha}, \tilde{\beta}}$  generates a stable periodic cycle with period four if  $\tilde{\alpha}\tilde{\beta} = 1$ . Hence we assume  $\tilde{\beta} > 1$  and  $\tilde{\alpha}\tilde{\beta} > 1$  to omit those stable dynamics and, further, assume  $\tilde{\alpha} + \tilde{\beta} - \tilde{\alpha}\tilde{\beta} > 0$  in order to eliminate multiple equilibria. Summing up, we see that for  $(\tilde{\alpha}, \tilde{\beta}) \in D$ ,  $f_{\tilde{\alpha}, \tilde{\beta}}(u)$  is continuous, maps the unit interval into itself, takes the extremum at the maximizer  $u_0 = \frac{\tilde{\beta}-1}{\tilde{\beta}}$ , is increasing on  $[0, u_0]$ , decreasing on  $[u_0, 1]$ , and has a unique unstable fixed point. The following theorem confirms that dynamics generated by  $F(x)$  as well as  $G(y)$  are essentially the same as the one generated by  $f_{\tilde{\alpha}, \tilde{\beta}}(u)$ .

**Theorem 2**  $F(x)$  as well as  $G(y)$  are linearly conjugate to  $f_{\tilde{\alpha}, \tilde{\beta}}(u)$ .

**Proof.** We start with  $F(x)$ .  $\varphi(x) = \frac{x-x_m}{x_M-x_m}$  can be a linear isomorphism from  $[x_m, x_M]$  onto the unit interval,  $[0, 1]$ , such that  $\varphi \circ F \circ \varphi^{-1}(u) = f_{\tilde{\alpha}, \tilde{\beta}}(u)$  which implies that  $F(x)$  is conjugate to  $f_{\tilde{\alpha}, \tilde{\beta}}(u)$ . With  $G(y)$ , we can also demonstrate that  $\psi(y) = \frac{y-y_m}{y_M-y_m}$  is a linear isomorphism from  $[x_m, x_M]$  onto the unit interval such that  $\psi \circ F \circ \psi^{-1}(u) = f_{\tilde{\alpha}, \tilde{\beta}}(u)$ . ■

For any pair of parameters in  $D$ , the typical dynamics of  $f_{\tilde{\alpha}, \tilde{\beta}}$  is ergodic chaos. If densities of chaotic trajectories could be constructed, it may be possible to capture statistical properties of such chaotic dynamics. Lasota and Yorke [1973] assure that an expansive map such as  $f_{\tilde{\alpha}, \tilde{\beta}}$  has a stable density function for ergodic chaotic fluctuations,<sup>5</sup> but do not say anything about its closed form. Although it is difficult to construct densities for general transformation maps, it is possible for some specific maps. In particular, Boyarsky and Scarowsky [1979] present a simple way to construct a closed-form expression for a stepwise density if a map is piecewise linear, continuous

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<sup>5</sup>A map is expansive if its slope in absolute value is greater than unity.

and *Markov*.<sup>6</sup> Here,  $f_{\tilde{\alpha},\tilde{\beta}}(u)$  is Markov if it has either a periodic point or an eventually-fixed point.<sup>7</sup>

According to their theorem, the following procedure is appropriate for constructing a density function which  $f_{\tilde{\alpha},\tilde{\beta}}(u)$  permits.

- (I) Find a parametric condition for which  $f_{\tilde{\alpha},\tilde{\beta}}$  possesses an eventually-fixed point or a periodic point with finite period.
- (II) Divide the unit interval into subintervals by the points of the corresponding periodic cycle.
- (III) Construct a matrix  $M = (m_{ij})$  whose entries are defined by

$$m_{ij} = \left| \frac{df_{\tilde{\alpha},\tilde{\beta}}(u)}{du} \right|^{-1} \delta_{ij}.$$

$\frac{df_{\tilde{\alpha},\tilde{\beta}}(u)}{dx}$  is the slope of  $f_{\tilde{\alpha},\tilde{\beta}}(u)$  on  $I_j$  where  $I_j$  is a subinterval of the unit interval, and  $\delta_{ij} = 1$  if  $I_j \subset f_{\tilde{\alpha},\tilde{\beta}}(I_j)$  and zero otherwise.

- (IV) Solve the matrix equation  $\Phi M = \Phi$  where  $\Phi = (\Phi_i) \in R^{N-1}$  and  $N$  is the number of period of a periodic cycle.
- (V) Since the ingegral of the density function over the trapping interval must be unity, elements of solution  $\Phi_i$  satisfying  $\sum_{i=1}^{N-1} \Phi_i \|I_i\| = 1$  are constant steps of a unique, absolutely continuous, invariant density function where  $\|I_i\|$  is the length of subinterval  $I_i$ .

Following each step of the above procedure, we give an illustrative example for constructing an explicit form of a density function in a case where  $f_{\tilde{\alpha},\tilde{\beta}}(u)$  generates a period-3 cycle.

- (I) Solving  $f_{\tilde{\alpha},\tilde{\beta}}^3(0) = 0$  for  $\alpha$  yields the period-3 condition,

$$\alpha = \frac{4\beta}{(2\beta - 3)(2\beta - 1)}, \quad (27)$$

for which the set  $\{0, f_{\tilde{\alpha},\tilde{\beta}}(0), f_{\tilde{\alpha},\tilde{\beta}}^2(0)\}$  is a period-3 cycle.

- (II) By Theorem 2,  $F(x)$  gives rise to a period-3 cycle,  $\{x_m, x_0, x_M\}$  under the same combination of  $(\alpha, \beta)$ . The periodic points of  $F(x)$  divide  $I_x$  into two subintervals,  $I_x^1 = [x_m, x_0)$  and  $I_x^2 = [x_0, x_M]$ .

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<sup>6</sup>See Theorem 3 of Boyarsky and Scarowsky[1979].

<sup>7</sup>See Matsumoto [2003b] that clarifies combinations of  $(\alpha, \beta)$  for which  $f_{\tilde{\alpha},\tilde{\beta}}$  is Markov.

(III) Since  $F(I_x^1) = I_x^2$  and  $F(I_x^2) = I_x^1 \cup I_x^2$  and the slope of  $F(x)$  is  $F'(x) = \tilde{\alpha}$  on  $I_x^1$  and  $F'(x) = \beta$  on  $I_x^2$ , the matrix  $M$  is defined by

$$M = \begin{pmatrix} 0 & \frac{1}{\tilde{\alpha}} \\ \frac{1}{\beta} & \frac{1}{\beta} \end{pmatrix}. \quad (28)$$

(IV) Solving  $zM = z$  where  $z = (z_1, z_2)$  gives the solution,  $z_2 = \tilde{\beta}z_1$ .

(V) To determine the value of  $z_1$ , we solve  $z_1 \|I_x^1\| + \tilde{\beta}z_1 \|I_x^2\| = 1$  for  $z_1$  where  $\|I_x^1\| = x_0 - x_m$  and  $\|I_x^2\| = x_M - x_0$  and obtain the solution  $z_1 = \frac{\beta}{\beta - 1}$ .

Therefore, in the period-3 case, the explicit form of the unique, invariant density that  $F(x)$  possesses is given by  $\Phi = (\Phi^1, \Phi^2)$  where

$$\begin{cases} \Phi^1 = \frac{\beta}{\beta - 1} & \text{on } I_x^1 = [x_m, x_0), \\ \Phi^2 = \frac{\beta(2\beta - 1)}{2(\beta - 1)} & \text{on } I_x^2 = [x_0, x_M]. \end{cases} \quad (29)$$

We also find that  $G(y)$  gives rise to a period-3 cycle,  $\{y_m, y_0, y_M\}$ , which divides  $I_y$  into  $I_y^1 = [y_m, y_0)$  and  $I_y^2 = [y_0, y_M]$ . Thus, by the same procedure, we can construct the explicit form of density,  $\Psi = (\Psi^1, \Psi^2)$ , for a chaotic trajectory of  $y$  such that

$$\begin{cases} \Psi^1 = \frac{2\beta - 1}{2(\beta - 1)} & \text{on } I_y^1 = [y_m, y_0), \\ \Psi^2 = \frac{(2\beta - 1)^2}{4(\beta - 1)} & \text{on } I_y^2 = [y_0, y_M]. \end{cases} \quad (30)$$

From (29) and (30), it can be checked that a step of the density of  $G(y)$  is  $\alpha_i$  times the step of the density of  $F(x)$ , i.e. smaller,

$$\Psi^k = \alpha_i \Phi^k \text{ for } k = 1, 2. \quad (31)$$

where  $\alpha_i = 1 - \frac{1}{2\beta}$  by Assumption 4.

This relationship connecting two densities generally holds regardless of the number of period whenever  $F(x)$  and  $G(y)$  are Markov. To see this, suppose that  $F(x)$  generates a periodic cycle with period  $N$ . Then there exist  $N$  points,  $\{\xi_k\}_{k=1,2,\dots,N}$  in interval  $I_x = [x_m, x_M]$  such that  $\xi_1 = x_m <$

$\xi_2 < \dots < \xi_{N-1} < \xi_N = x_M$ . These points divide  $I_x$  into  $N - 1$  mutually disjoint subintervals,  $I_x^k = [\xi_k, \xi_{k+1})$ ,  $k = 1, 2, \dots, N-2$  and  $I_x^{N-1} = [\xi_{N-1}, \xi_N]$ . The step function has a step  $\Phi^k$  on each  $I_x^k$  which satisfies

$$\sum_{k=1}^{N-1} \Phi^k \|I_x^k\| = 1, \quad (32)$$

where  $\|I_x^k\| = \xi_{k+1} - \xi_k$ . Since  $G(y)$  is conjugate to  $F(x)$ ,  $G(y)$  also possesses a period  $N$  cycle with periodic points  $\{\theta_k\}_{k=1,2,\dots,N}$  in the interval  $I_y = [y_m, y_M]$ . By the same principle,  $G(y)$  has a stepwise function with step  $\Psi^k$  on each subinterval  $I_y^k = [\theta_k, \theta_{k+1}]$  such that

$$\sum_{k=1}^{N-1} \Psi^k \|I_y^k\| = 1. \quad (33)$$

By Assumption 5,  $x_m = \alpha_i y_m$  and  $x_M = \alpha_i y_M$ , and it is also clear that  $\xi_k = \alpha_i \theta_k$  holds for all  $k$ . Then we have  $I_x^k = \alpha_i I_y^k$ . Subtracting (33) from (32) gives

$$\sum_{k=1}^{N-1} \Phi^k \|I_x^k\| - \sum_{k=1}^{N-1} \Psi^k \|I_y^k\| = \sum_{k=1}^{N-1} \{\Phi^k \alpha_i - \Psi^k\} \|I_y^k\| \quad (34)$$

which must be zero. Therefore we have  $\alpha_i \Phi^k = \Psi^k$  in the period- $N$  case. We summarize this result as follows.

**Theorem 3** *Let  $\Phi = (\Phi^k)$  and  $\Psi = (\Psi^k)$  for  $k = 1, 2, \dots, N - 1$  be stepwise densities that  $F(x)$  and  $G(y)$  possess. Then  $\alpha_i \Phi^k = \Psi^k$  for  $k = 1, 2, \dots, N - 1$ .*

## 5 Statistical Dynamics

### 5.1 Cournot Point

As the reference point of the long-run average behavior, we calculate the Cournot outputs and the Cournot profits, the profits obtained at the Cournot point. The Cournot point (13) under Assumption 4 is reduced to

$$x^c = \frac{2\beta - 1}{2\beta + 1} \text{ and } y^c = \frac{2\beta}{2\beta + 1}. \quad (35)$$

Substituting these outputs into  $\Pi_i$  in (3) and  $\Pi_d$  in (8) gives the Cournot profits of the imitator and dualist, respectively

$$\Pi_d^c = b \left( \frac{2\beta}{2\beta + 1} \right)^2 \text{ and } \Pi_i^c = b \left( \frac{2\beta - 1}{2\beta + 1} \right)^2, \quad (36)$$

where the superscript "c" means that the profit is evaluated at the Cournot point. We observe through (35) and (36) that Cournot outputs increase with  $\beta$  and so do Cournot profits. Further, we can verify that the ratio of  $x^c$  over  $y^c$  and the ratio of  $\Pi_i^c$  over  $\Pi_d^c$  also increase with  $\beta$ ,

$$\frac{x^c}{y^c} = \frac{2\beta - 1}{2\beta} < 1 \text{ and } \frac{\Pi_i^c}{\Pi_d^c} = \left(\frac{2\beta - 1}{2\beta}\right)^2 < 1. \quad (37)$$

Inequalities of (37) indicate that the dualist produces more output and earns more profit than the imitator for any feasible  $\beta > \frac{3}{2}$  at the Cournot point. By Assumption 4, we have  $\frac{2\beta-1}{2\beta} = \alpha_i$  where  $\alpha_i$  is the inverse of the slope of the imitator's reaction function. Given  $\beta$ , the constant output ratio is an alternative expression of the fact that the Cournot point is located on the imitator's reaction curve.<sup>8</sup> From (37), we also see that the profit ratio is equal to the square of the output ratio. These facts can be summarized as:

**Theorem 4** *At the Cournot point, an imitator produces output  $\alpha_i$  times less than dualist's output and earns profit  $\alpha_i^2$  times smaller than dualist's profit,*

$$x^c = \alpha_i y^c \text{ and } \Pi_i^c = \alpha_i^2 \Pi_d^c,$$

where  $\alpha_i = \frac{2\beta - 1}{2\beta} < 1$  for all positive values of  $\beta$  and in particular for  $\beta > \frac{3}{2}$ .

## 5.2 Markov Trajectory

Once a periodic cycle is found, we can construct an explicit form of density function and thus look for statistical characteristic of chaotic dynamics such as the long-run averages of output and profit with the help of the following Birkhoff-von Neuman mean ergodic theorem.<sup>9</sup>

**Theorem 5** *If a dynamical system of  $x$  is chaotic ergodic, then the time average of a function  $g(x)$  associated with a chaotic trajectory,  $\{x_t\}$ , equals to its space average, that is,*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} g(x_t) = \int_I g(x) \phi(x) dx$$

where  $g \in C^1$ ,  $\phi(x)$  is a density function and  $I$  is its support.

<sup>8</sup>As the Cournot point is the intersection of two reaction functions, it is also possible to show that the Cournot point is on the dualist's reaction function, that is,  $y^c = \beta(1 - x^c)$ .

<sup>9</sup>See, for example, Day[1994, Theorem 8.2 on p.142] for this theorem.

### 5.2.1 Long-run Average Output

According to the mean ergodic theorem, the time average output of  $x$  taken along a chaotic trajectory is given by

$$\bar{x} = \int_{x_m}^{x_M} x \Phi dx. \quad (38)$$

Similarly, the time average output of  $y$  is

$$\bar{y} = \int_{y_m}^{y_M} y \Psi dy. \quad (39)$$

Since  $F(x)$  and  $G(y)$  are conjugate, we can presume a certain connection between those long-run averages of  $x$  and  $y$ . Making the change of variable,  $x = \alpha_i y$ , we can find the connection such that the average output of  $x$  is equal to  $\alpha_i$  times the average output of  $y$ ,

$$\begin{aligned} \bar{x} &= \int_{x_m/\alpha_i}^{x_M/\alpha_i} \alpha_i y \Phi(\alpha_i dy) \\ &= \alpha_i \bar{y}, \end{aligned}$$

where  $x_m/\alpha_i = y_m$ ,  $x_M/\alpha_i = y_M$ , and  $\Psi = \alpha_i \Phi$  by Theorem 3. We state this as our first result on the long-run average behavior.

**Theorem 6** *The ratio of the average outputs is equal to the ratio of the Cournot outputs,*

$$\frac{\bar{x}}{\bar{y}} = \frac{x^c}{y^c} = \alpha_i.$$

From Theorem 6, we also have  $x^c - \bar{x} = \alpha_i(y^c - \bar{y})$ , that is, the ratio of the differences between the Cournot outputs and the average outputs is equal to  $\alpha_i$ . A chaotic trajectory of output oscillates around the Cournot point, in other words, firms produce less than the Cournot output in some periods and more in other periods. *So our next concern is about whether the average output is less than the Cournot output or more.*

We examine two special but simple cases; the tent-map case where  $F(x)$  as well as  $G(y)$  are the tent maps and the period-3 case where  $F(x)$  as well as  $G(y)$  generate a period-3 cycle.

#### Tent-map Case

Since  $f_{\tilde{\alpha}, \tilde{\beta}}$  becomes the tent map if its vertical intercept is zero, we solve  $f_{\tilde{\alpha}, \tilde{\beta}}(0) = 0$  for  $\alpha$  to find the tent-map condition,

$$\alpha = \frac{2\beta}{2\beta - 3}. \quad (40)$$

It is well known that the tent map has the uniform density.<sup>10</sup> Since  $F(x)$  and  $G(y)$  are conjugate to  $f_{\tilde{\alpha}, \tilde{\beta}}$  by Theorem 2, they also have uniform density on their trapping intervals,

$$\Phi^T = \frac{2}{\alpha_i} \text{ on } I_x = [x_m, x_M], \quad (41)$$

$$\Psi^T = 2 \text{ on } I_y = [y_m, y_M],$$

where the superscript "T" denotes the tent-map case. Then the long-run average outputs of  $x$  and  $y$  can be calculated as

$$\begin{aligned} \bar{x}^T &= \int_{x_m}^{x_M} x \Phi^T dx = \frac{3}{4} \left(1 - \frac{1}{2\beta}\right), \\ \bar{y}^T &= \int_{y_m}^{y_M} y \Psi^T dx = \frac{3}{4}, \end{aligned} \quad (42)$$

where  $\bar{x}^T = \alpha_i \bar{y}^T$  holds. From (35) and (42), differences between the Cournot outputs and the average outputs are also calculated as,

$$\begin{aligned} x^c - \bar{x}^T &= \frac{(2\beta - 1)(2\beta - 3)}{8\beta(1 + 2\beta)} > 0, \\ y^c - \bar{y}^T &= \frac{2\beta - 3}{3(1 + 2\beta)} > 0. \end{aligned} \quad (43)$$

where  $x^c - \bar{x}^T = \alpha_i(y^c - \bar{y}^T)$  also holds. We observe that the Cournot output is larger than the long-run average output, although the ratio of the differences is constant (i.e.,  $x^c - \bar{x}^T = \alpha_i(y^c - \bar{y}^T)$ ). In consequence, the Cournot total output,  $Q^c = x^c + y^c$ , is larger than the average total output,  $\bar{Q} = \bar{x} + \bar{y}$ , which in turn implies that the Cournot price  $p^c$  is higher than the average price  $\bar{p}$  where each price is determined through the demand function,  $p^c = a - bQ^c$  and  $\bar{p} = a - b\bar{Q}$ .

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<sup>10</sup>See, for example, Section 8.6.1 of Day [1994].

Since the density is uniform in the tent-map case, it is possible to show in another way why this result holds. Indeed the difference is written as

$$x^c - \bar{x}^T = \int_{x_m}^{x^c} \underbrace{(x^c - x)}_{(+)} \Phi^T dx + \int_{x^c}^{x_M} \underbrace{(x^c - x)}_{(-)} \Phi^T dx, \quad (44)$$

where  $x_m < x^c < x_M$  implies that  $x^c - x > 0$  for  $x_m < x < x^c$  and  $x^c - x < 0$  for  $x^c < x < x_M$  as indicated. Further, given the tent-map condition, (40), we have

$$\frac{x_M - x^c}{x^c - x_m} = \frac{2}{2\beta - 1} < 1, \quad (45)$$

where the direction of the last inequality is due to the instability condition,  $\beta > \frac{3}{2}$ . Thus, for any feasible value of  $\beta$ , the first term of (44) dominates the second term so that the total sum becomes positive, that is, the Cournot output is larger than the average output. The same reasoning applies to  $y^c > \bar{y}$ .

### Period-3 Case

Under (27), we have the period-3 case where (29) and (30) are densities. Thus the long-run average of output  $x$  is

$$\begin{aligned} \bar{x}^3 &= \int_{x_m}^{x_0} x \Phi^1 dx + \int_{x_0}^{x_M} x \Phi^2 dx \\ &= \frac{21 - 48\beta + 28\beta^2}{32\beta(\beta - 1)}, \end{aligned}$$

and the long-run average of output  $y$  is

$$\begin{aligned} \bar{y}^3 &= \int_{y_m}^{y_0} y \Psi^1 dy + \int_{y_0}^{y_M} y \Psi^2 dy \\ &= \frac{21 - 48\beta + 28\beta^2}{16(2\beta - 1)(\beta - 1)}, \end{aligned}$$

where the superscript "3" denotes the period-3 case. It can be checked that the ratio of average outputs is constant,  $\bar{x}^3 = \alpha_i \bar{y}^3$ . The differences between the Cournot outputs and the average outputs are calculated as

$$\begin{aligned} x^c - \bar{x}^3 &= \frac{-21 + 38\beta - 28\beta^2 + 8\beta^3}{32\beta(\beta - 1)(2\beta + 1)} > 0, \\ y^c - \bar{y}^3 &= \frac{-21 + 38\beta - 28\beta^2 + 8\beta^3}{16(\beta - 1)(2\beta - 1)(2\beta + 1)} > 0, \end{aligned} \quad (46)$$



where, again, the constant ratio of the differences is observed,  $x^c - \bar{x}^3 = \alpha_i(y^c - \bar{y}^3)$ . Equation (46) indicates that we have the same results in the period-3 case as in the tent-map case, that is, the Cournot outputs are larger than the average outputs, and the Cournot price is higher than the average price. It is possible to show, using mathematical induction, that the same results hold in any case where the density is explicitly constructed. Summarizing these results gives the following,

**Theorem 7** *If  $F(x)$  and  $G(y)$  are Markov, then the Cournot outputs are larger than the average outputs and are sold at the lower price on the average,*

$$(1) \ x^c > \bar{x} \text{ and } y^c > \bar{y},$$

$$(2) \ p^c < \bar{p}.$$

### 5.2.2 Long-run Average Profit

We now consider the long-run average profits. Since the MPE trajectory visits the imitator's reaction curve and the dualist's reaction curve alternately, the average profit taken along the MPE trajectory is the sum of the average profits taken on each reaction curve. If, for instance, a trajectory starts on the dualist's reaction curve, then it is on the dualist's reaction curve at every odd period and on the imitator's reaction function at every even period. Suppose  $M = 2N$  where  $T^M(x, y) = \{F^N(x), G^N(y)\}$ ,<sup>11</sup> and  $\Pi$  denotes a profit function, then the average  $\Pi$  over  $M$  periods is

$$\begin{aligned} \frac{1}{M} \sum_{t=1}^M \Pi(x_t, y_t) &= \frac{1}{2N} \{[\Pi(x_1, y_1) + \Pi(x_3, y_3) + \dots + \Pi(x_{M-1}, y_{M-1})] \\ &\quad + [(\Pi(x_2, y_2) + \Pi(x_4, y_4) + \dots + \Pi(x_M, y_M))]\} \\ &= \frac{1}{2} \left\{ \frac{1}{N} \sum_{t=1}^N \Pi(x_{2t-1}, r_d(x_{2t-1})) + \frac{1}{N} \sum_{t=1}^N \Pi(x_{2t}, r_i^{-1}(x_{2t})) \right\} \\ &= \frac{1}{2} \left\{ \frac{1}{N} \sum_{t=1}^N \Pi(r_i(y_{2t-1}), y_{2t-1}) + \frac{1}{N} \sum_{t=1}^N \Pi(r_d^{-1}(y_{2t}), y_{2t}) \right\} \end{aligned}$$

The second and third equations imply that the average profit can be defined in terms of  $x$  as well as  $y$ .

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<sup>11</sup>See Dana and Montrucchio [1986] and Bischi *et al.* [2000].

Transformation from the first to the second equality is carried out in the following way. Since the trajectory is on the dualist's reaction curve at odd periods, the dualist's output at period  $2t-1$  is the best reply to the imitator's action at period  $2(t-1)$ ,  $y_{2t-1} = r_d(x_{2(t-1)})$ . When the dualist moves, the imitator does not move so that the imitator's output at period  $2t-1$  is equal to the one in the previous period,  $x_{2t-1} = x_{2(t-1)}$ . Therefore  $y_{2t-1} = r_d(x_{2t-1})$  and the profit at period  $2t-1$  is written as  $\Pi(x_{2t-1}, r_d(x_{2t-1}))$ . By the same reasoning, when the trajectory is on the imitator's reaction curve at even period  $2t$ , the profit can be written as  $\Pi(x_{2t}, r_i^{-1}(x_{2t}))$ . Therefore as shown above, the average profit over  $M$  periods is the average of the sum of the average profit taken along each reaction curve. The same reasoning applies to the transformation from the first equality to the third.

When the MPE trajectory is chaotic, the average profit is the limiting value of the finite average,

$$\begin{aligned} \lim_{M \rightarrow \infty} \frac{1}{M} \sum_{t=1}^M \Pi(x_t, y_t) &= \frac{1}{2} \left\{ \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=1}^N \Pi(x_{2t-1}, r_d(x_{2t-1})) \right. \\ &\quad \left. + \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=1}^N \Pi(x_{2t}, r_i^{-1}(x_{2t})) \right\}. \end{aligned}$$

Since the adjustment process is ergodic, the mean ergodic theorem implies that the time average of ergodic trajectory converges to its space average,

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=1}^N \Pi(x_{2t-1}, r_d(x_{2t-1})) &= \int_{x_m}^{x_M} \Pi(x, r_d(x)) \Phi dx, \\ \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=1}^N \Pi(x_{2t}, r_i^{-1}(x_{2t})) &= \int_{x_m}^{x_M} \Pi(x, r_i^{-1}(x)) \Phi dx, \end{aligned} \tag{47}$$

where  $\Phi(x)$  is the invariant density function. Summing up our findings, we have

**Theorem 8** *For the profit function  $\Pi$  of  $(x, y)$ , the average profit taken along the MPE trajectories equals to the average of the sum of the average profits taken along each reaction function,*

$$\lim_{M \rightarrow \infty} \frac{1}{M} \sum_{t=1}^M \Pi(x_t, y_t) = \frac{1}{2} (\bar{\Pi}^i + \bar{\Pi}^d)$$

where  $\bar{\Pi}^i$  is the average profit taken along the imitator's reaction curve and  $\bar{\Pi}^d$  is the average profit taken along the dualist's reaction curve.

In Theorem 4, the ratio of the Cournot profits is constant, namely  $\alpha_i^2$ . Applying Theorem 8, we can demonstrate the following result concerning the ratio of the average profits.

**Theorem 9** *The ratio of the imitator's profit to the dualist's profit, and the ratio of the imitator's difference between the Cournot profit and the average profit to the dualist's difference are equal to the ratio of the imitator's Cournot profit to the dualist's Cournot profit,*

$$\frac{\bar{\Pi}_i}{\bar{\Pi}_d} = \frac{\Pi_i^c - \bar{\Pi}_i}{\Pi_d^c - \bar{\Pi}_d} = \frac{\Pi_i^c}{\Pi_d^c} = \alpha_i^2$$

**Proof.** See Appendix. ■

Numerical examples give an answer to our question on whether the long-run average profits are larger or smaller than the Cournot profit. As before, we consider the tent-map case and the period-3 case.

### Tent-map case

We can calculate the long-run average profits,

$$\begin{aligned} \bar{\Pi}_i^T &= \frac{b(2\beta - 1)(7\beta - 4)}{24\beta^2}, \\ \bar{\Pi}_d^T &= \frac{b(7\beta - 4)}{6(2\beta - 1)}, \end{aligned} \tag{48}$$

and then the difference between the Cournot profit and the average profits,

$$\begin{aligned} \Pi_i^c - \bar{\Pi}_i^T &= \frac{b(4 + 9\beta - 36\beta^2 + 20\beta^3)}{6(2\beta - 1)(2\beta + 1)^2} > 0, \\ \Pi_d^c - \bar{\Pi}_d^T &= \frac{b(4 + 9\beta - 36\beta^2 + 20\beta^3)}{24\beta(2\beta + 1)^2} > 0. \end{aligned} \tag{49}$$

which imply that the Cournot profit is larger than the average profit for both the imitator and dualist. This result is also demonstrated in another way. The long-run average profit of the imitator is

$$\begin{aligned} \bar{\Pi}_i^T &= \frac{1}{2} \left\{ \int_{x_m}^{x_0} \Pi_i(x, r_i(x)) \Phi^T dx + \int_{x_m}^{x_0} \Pi_i(x, r_i(x)) \Phi^T dx \right. \\ &\quad \left. + \int_{x_m}^{x_0} \Pi_i(x, r_d^i(x)) \Phi^T dx + \int_{x_0}^{x_M} \Pi_i(x, r_d^a(x)) \Phi^T dx \right\}, \\ &= \frac{1}{2} \left\{ \int_{x_m}^{x_0} \Pi_i^1(x) \Phi^T dx + \int_{x_0}^{x_M} \Pi_i^2(x) \Phi^T dx \right\}, \end{aligned}$$

where

$$\Pi_i^1(x) = \Pi_i(x, r_i(x)) + \Pi_i(x, r_d^i(x)),$$

$$\Pi_i^2(x) = \Pi_i(x, r_i(x)) + \Pi_i(x, r_d^a(x)).$$

Thus the difference between imitator's Cournot profit and average profit is,

$$\begin{aligned} \Pi_i^c - \bar{\Pi}_i^T &= \Pi_i^c \int_{x_m}^{x_M} \Phi^T dx - \frac{1}{2} \left\{ \int_{x_m}^{x_0} \Pi_i^1(x) \Phi^T dx + \int_{x_0}^{x_M} \Pi_i^2(x) \Phi^T dx \right\}, \\ &= \frac{1}{2} \left\{ \int_{x_m}^{x_1} \underbrace{(\Pi_i^c - \Pi_i^1(x))}_{(+)} \Phi^T dx + \int_{x_1}^{x_0} \underbrace{(\Pi_i^c - \Pi_i^1(x))}_{(-)} \Phi^T dx \right\} \\ &\quad + \frac{1}{2} \left\{ \int_{x_0}^{x_c} \underbrace{(\Pi_i^c - \Pi_i^2(x))}_{(-)} \Phi^T dx + \int_{x_c}^{x_M} \underbrace{(\Pi_i^c - \Pi_i^2(x))}_{(+)} \Phi^T dx \right\}, \end{aligned}$$

where  $\Pi_i^c = \Pi_i(x^c, y^c)$  and  $x_1$  solves  $\Pi_i^c = \Pi_i^1(x)$ . In Figure 4, the light grey area is the integral of the first and fourth terms while the dark is the integral of the second and third terms. It is obvious that the former dominates the latter. In consequence, the Cournot profit of the imitator is larger than the average profit.

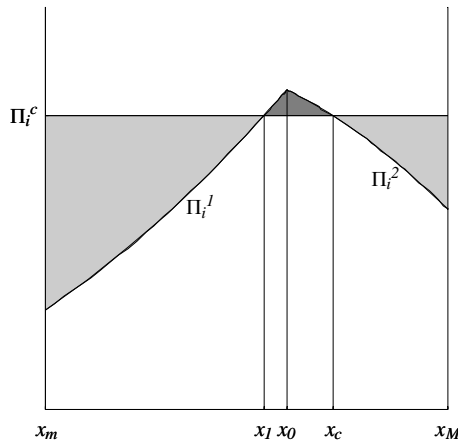


Figure 4. Difference between imitator's Cournot profit and average profit.

### Period-3 case

In the period-3 case, the long-run average profits are

$$\begin{aligned}\bar{\Pi}_i^3 &= \frac{b(-77 + 4\beta(71 - 87\beta + 36\beta^2))}{192(\beta - 1)\beta^2}, \\ \bar{\Pi}_d^3 &= \frac{b(-77 + 4\beta(71 - 87\beta + 36\beta^2))}{48(2\beta - 1)^2(\beta - 1)},\end{aligned}\tag{50}$$

and the differences between the Cournot profit and the average profits are

$$\begin{aligned}\Pi_d^c - \bar{\Pi}_d^3 &= \frac{b(7+4(\beta-2)\beta)(11+4\beta(4+3\beta(4\beta-7)))}{48(\beta-1)(1-4\beta^2)^2} > 0, \\ \Pi_i^c - \bar{\Pi}_i^3 &= \frac{192(1-2\beta)^2(\beta-1)\beta^2 - b(1+2\beta)^2b(-77+4\beta(71-87\beta+36\beta^2))}{192(\beta-1)\beta^2(2\beta+1)^2} > 0.\end{aligned}\tag{51}$$

Again we obtain that the Cournot profit is larger than the long-run average profits for both the imitator and the dualist. Calculations get longer and much more tedious as the number of period of periodic cycle gets larger, but it is possible, with mathematical induction, to confirm this result regardless of the number.

**Theorem 10** *The Cournot profits are larger than the average profits for both the imitator and the dualist,*

$$\Pi_i^c > \bar{\Pi}_i \text{ and } \Pi_d^c > \bar{\Pi}_d.$$

## 5.3 Non-Markov Trajectory

We have so far examined the statistical dynamics in a case where  $F(x)$  and  $G(y)$  are Markov. In this section, we investigate the statistical properties of the average profit in a case where  $F(x)$  and  $G(y)$  are not Markov. Since we are not able to use the mean ergodic theorem unless the density function is constructed, we perform numerical simulations in order to verify whether the results analytically derived for Markov case can hold in a non-Markov case.

For the numerical simulations, we fix the slope of the positive sloping part of the dualist's reaction curve  $\alpha = 2$  and increase the value of  $\beta$  from  $\frac{3}{2}$  to 3.  $\frac{3}{2}$  is the critical value at which the instability occurs and will be denoted as  $\beta_s$ . For  $\beta > 3$ , the map gives rise to multiple Cournot points and its trajectory escapes from the feasible region. Thus 3 is the maximum value that the parameter can take. We denote this critical value by  $\beta_T$  as  $F(x)$  and  $G(y)$  become the tent maps for  $\alpha = 2$  and  $\beta = 3$ .  $\beta$  measures the degree of strategic substitutability of the dualist as shown in (14). Note that

increasing this parameter value corresponds to pronounced steepness of the downward sloping part of the dualist's reaction curve.

Figure 5 illustrates the results of numerical simulations of the long-run average profits of the imitator (in the left panel) and the dualist (in the right panel) against variations of parameter  $\beta$ . For comparison, graphs of the Cournot profits are also depicted. In each panel, the parameter  $\beta$  has been increased in steps of 0.01, and for each of these  $\beta$ -values, the average is calculated from 5,000 iterations. For values of  $\beta$  indicated by  $\beta_3$  and  $\beta_1$ , the map generates the periodic 3 cycle and eventually-fixed cycle respectively. In other words, the map is Markov for these  $\beta$  values for which the analytical values of the average profit can be calculated. The heavy dots on the vertical lines at  $\beta_1$  and  $\beta_3$  indicate the analytical value of the average profit and the white circles indicate the corresponding Cournot profit. It is safe to say that the simulations approximate the analytical results for any other  $\beta$ -values as the simulation results plotted against these critical  $\beta$ -values pass through the points obtained from analytical results. We observe that for any value of  $\beta$ , the Cournot profit is higher than the long-run average.

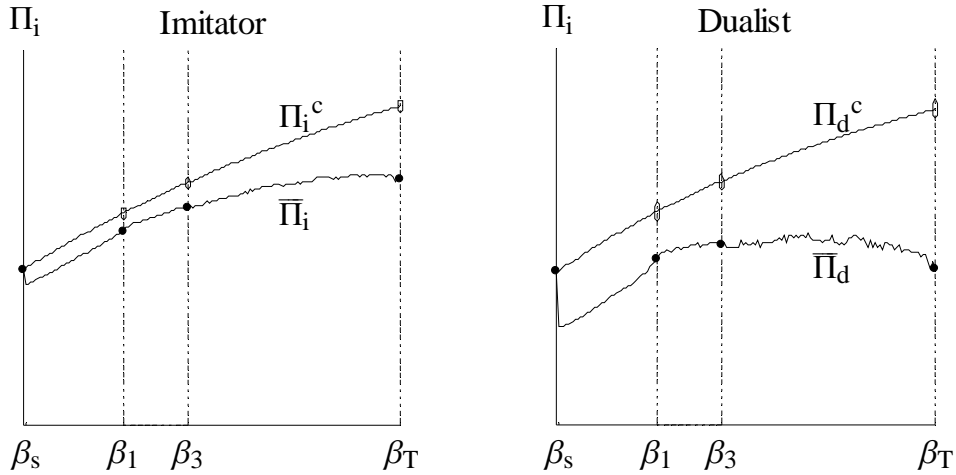


Figure 5. Numerical simulations of imitator's and dualist's profits.

We then check whether the ratio of the average profit is constant. Figure 6 shows plots of the ratio of the average profit of the imitator to the average profit of the dualist under the same initial conditions and the same parameter conditions but the number of iterations is 50 in the left panel and 5000 in the right panel. The generally upward-sloping locus but with small indentations

is the set of the ratios of average profits as a function of  $\beta$ . Comparing the left panel with the right, we can say that the locus gets smoother as the number of iterations gets larger. We also find that each calculated ratio is located on  $(\frac{2\beta-1}{2\beta})^2$  curve (which is not depicted in the panel) for the large number of iterations. The dots on the locus correspond the analytical results obtained from the calculations done in the previous section. From an examination of the graph, it is safe to surmise that the average profit ratio may converge to the constant value  $\alpha_i^2 = (\frac{2\beta-1}{2\beta})^2$  as the number of iterations is increased. These numerical results support the analytical results.

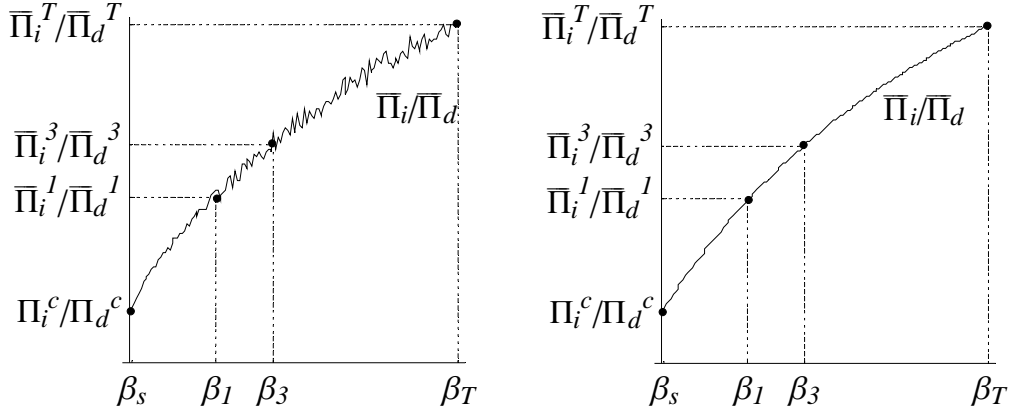


Figure 6. Average profit ratios (50 iterations in the left panel and 5000 in the right).

## 6 Concluding Remarks

In this study, we have constructed the nonlinear Cournot model in which the duopolists have production externality and examined the long-run or statistical behavior associated with chaotic dynamics of output. Although chaotic trajectories of  $x$  and  $y$  exhibit highly irregular motions, the ratio of the average output of  $x$  to that of  $y$  shows a predictable trend, that is, the ratio is equal to the ratio of Cournot outputs as summarized in Theorem 6. The imitator as well as the dualist produce more than the Cournot output in some periods and less in other periods but, on the average, produces more at the Cournot point. For the demand side, this result implies that consumers can afford to purchase a smaller amount of goods at the higher price on the

average. Similar statistical properties hold for the long-run average profit. Theorem 8 indicates that the ratio of the average profits is equal to the ratio of the Cournot profits, and Theorem 9 implies that the Cournot profit is larger than the average profit. These statistical properties are analytically as well as numerically confirmed. However, this does not mean that chaotic production is less favorable than the production at Cournot point because our analysis is limited to dynamics in the imitator-dualist market. We need to investigate other markets such as the accommodator-dualist market or the dualist-dualist market which may have interesting dynamics which have not yet been analyzed.



## Appendix

In this Appendix, we prove Theorem 9 in three steps; we derive the long-run average profit of imitator in the first step, then the average profit of dualist in the second step, and finally we clarify the relationship between these average profits.

**Step 1.** From Theorem 8 and (47), the long-run average profit of the imitator is defined as follows.

$$\begin{aligned}\bar{\Pi}_i &= \frac{1}{2} \left\{ \int_{x_m}^{x_0} \left[ \Pi_i \left( x, \frac{x}{\alpha_i} \right) + \Pi_i \left( x, r_d^i(x) \right) \right] \Phi_I dx \right. \\ &\quad \left. + \int_{x_0}^{x_M} \left[ \Pi_i \left( x, \frac{x}{\alpha_i} \right) + \Pi_i \left( x, r_d^a(x) \right) \right] \Phi_{II} dx \right\} \\ &= \alpha_i b \left\{ \int_{x_m}^{x_0} x \left( \frac{\alpha + \beta - \alpha\beta}{\beta} + \alpha x \right) \Phi_I dx + \int_{x_0}^{x_M} \beta(1-x)x \Phi_{II} dx \right\}\end{aligned}$$

where the steps of the density are divided into two,  $\Phi_I$  on  $[x_m, x_0]$  and  $\Phi_{II}$  on  $[x_0, x_M]$ .

**Step 2.** It is convenient to define the long-run average profit of the dualist in terms of  $y$  as follows.

$$\begin{aligned}\bar{\Pi}_d &= \frac{1}{2} \left\{ \left[ \int_{y_m}^{y_0} \Pi_d^i(\alpha_i y, y) \Psi_I dy + \int_{y_0}^{y_M} \Pi_d^a(\alpha_i y, y) \Psi_{II} dy \right] \right. \\ &\quad \left. + \int_{y_m}^{y_0} \left[ \Pi_d^i(r_d^{i-1}(y), y) \Psi_I dy + \int_{y_0}^{y_M} \Pi_d^a(r_d^{a-1}(y), y) \Psi_{II} dy \right] \right\}.\end{aligned}$$

Here the sum of the first two terms is the average profit along the imitator's reaction curve and the sum of the last two terms is the average profit along the dualist's reaction curve.  $r_d^{i-1}(y)$  and  $r_d^{a-1}(y)$  are the inverse of  $r_d^i(y)$  and  $r_d^a(y)$  respectively. As in Step 1 above, the steps of density  $\Psi$  are decomposed into two,  $\Psi_I$  on  $[y_m, y_0]$  and  $\Psi_{II}$  on  $[y_0, y_M]$ . Adding the first and the third term, and the second and the fourth term, and then arranging yield,

$$\bar{\Pi}_d = b \left\{ \int_{y_m}^{y_0} \left( \frac{\alpha + \beta - \alpha\beta}{\beta} + \alpha_i \alpha y \right) y \Psi_I dy + \int_{y_0}^{y_M} (\beta(1 - \alpha_i y) y) \Psi_{II} dy \right\}.$$

**Step 3.** In  $\bar{\Pi}_i$  substituting  $x = \alpha_i y$  and using the result of Theorem 3,  $\bar{\Pi}_i$  can be written as follows.

$$\begin{aligned}
\bar{\Pi}_i &= \alpha_i b \left\{ \int_{x_m/\alpha_i}^{x_0/\alpha_i} \alpha_i y \left( \frac{\alpha + \beta - \alpha\beta}{\beta} + \alpha \alpha_i y \right) \Phi_I(\alpha_i dy) \right. \\
&\quad \left. + \int_{x_0/\alpha_i}^{x_M/\alpha_i} \beta(1 - \alpha_i y) \alpha_i y \Phi_{II}(\alpha_i dy) \right\} \\
&= \alpha_i^2 \left\{ b \left\{ \int_{y_m}^{y_0} \left( \frac{\alpha + \beta - \alpha\beta}{\beta} + \alpha_i \alpha y \right) y \Psi_I dy + \int_{y_0}^{y_M} (\beta(1 - \alpha_i y) y) \Psi_{II} dy \right\} \right\} \\
&= \alpha_i^2 \bar{\Pi}_d.
\end{aligned}$$

Since the Cournot profits are constant, we can show  $\Pi_i^c - \bar{\Pi}_i = \alpha_i^2 \Pi_d^c - \alpha_i^2 \bar{\Pi}_d$  in the same way. This proves Theorem 9.

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