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DYNAMIC MONOPOLY WITH BOUNDED CONTINUOUSLY DISTRIBUTED DELAY

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ABSTRACT. A continuously distributed time lag is considered in a monopoly where the time window of past data is bounded from below and its length is fixed. The dynamic behavior of the resulting system is described by a special delayed differential equation with infinite spectrum. The location of the stability switches are determined and a simple rule is developed to determine which ones lead to the loss of stability or the regaining of stability. A simple computer example illustrates the theoretical findings. The dynamic model and the stability conditions are different from what is known from earlier studies on continuously distributed delays.

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1. Introduction

Dynamic economic models are one of the most frequently studied problem areas in the literature on mathematical economics. Dynamic monopolies and oligopolies as special models have been examined by many authors. The earlier developments up to the mid 70s are summarized in Okuguchi (1976), and their multi-product extensions with special model variants are discussed in Okuguchi & Szidarovszky (1999). Most of the models examined earlier are linear, where local asymptotic stability implies global stability. In recent decades increasing attention has been given to nonlinear dynamics. Bischi et al. (2010) give a comprehensive summary of the recent developments in nonlinear oligopolies. Most models discussed earlier in the literature assume complete and instantaneous information about the market and the behavior of the competitors. In real economies this assumption is unrealistic. The effect of misspecified demand functions on the oligopoly equilibria and on the dynamic properties of the associated dynamic models are discussed in Bischi et al. (2010) and in the references cited there. Repeated observations on the market and on the behavior of the competitors can also lead to different learning schemes. The data obtained about the market and the competitors are usually delayed, and obtaining and implementing decisions also need some time. Information lags can be modeled either by fixed or by continuously distributed delays. Fixed delays are described by difference-differential equations, the characteristic equations of which are mixed polynomial-exponential equations with infinite spectra. The classical book of Bellman & Cooke (1956) gives an excellent introduction to this area. If the delay is uncertain, or the firms want to react to average past information instead of following sudden changes, continuously distributed lags are used. With special weighting functions the resulting Volterra-type integro-differential equation is equivalent to a system of ordinary differential equations with a finite spectrum. Cushing (1977) offers the mathematical developments and applications to population dynamics. This methodology is introduced into economic modeling by Invernizzi & Medio (1991), and its applications to oligopolies are presented in Chiarella & Khomin (1996) and Chiarella & Szidarovszky (2002). The expected demand is a weighted average of all past demand observations from zero to the current time period, that is, the length of the considered time interval is increasing and converges to infinity when $t \to \infty$. A more realistic assumption is the consideration of a given time window, when the firms use observations only in the interval $[t - \Delta, t]$ with fixed value of Δ. In this case the firms do not go back very far in time with observed information. This idea is introduced in Chiarella & Szidarovszky (2003) and the asymptotic behavior of the dynamic model is investigated in simple special cases. A comparison of fixed and continuously distributed delays is presented in Cooke & Grossman (1982).

The first step in analyzing dynamic oligopolies is the examination of monopolies, where the models and the mathematical methodology are simpler. Recently Matsumoto & Szidarovszky (2012a,b,c) examined dynamic monopolies with discrete and continuously distributed lags. Stability conditions were derived and compared as well as global dynamics were shown by computer studies. The dynamics of the output can be described by using best responses or marginal profits. The best response dynamics has the advantage that the steady states are the same as the static equilibria, however global information is needed to determine the reaction functions. In applying gradient dynamics only local information is needed, however only interior equilibria can be obtained as steady states of the dynamic model.

In this paper the dynamic monopoly model of Matsumoto & Szidarovszky (2012*c*) is revisited, and the continuously distributed delay is modified to a bounded continuously distributed delay. After the mathematical model is introduced, a complete stability analysis is given, which is then illustrated with numerical examples. Conclusions and future research directions are outlined at the end of this paper.

2. The Mathematical Model

Consider a monopoly with price function $p(x) = A - Bx$ and cost function $C(x) = Cx + D$. It is assumed that these functions are not exactly known by the firm, and that it observes profit information with an uncertain delay. The profit of the firm is given as

$$\Pi(x) = x(A - Bx) - (Cx + D) \tag{1}$$

and its marginal profit is

$$\Pi'(x) = A - C - 2Bx.$$

In order to have positive maximizer we assume that $A > C$, so the profit maximizing output is $\bar{x} = \frac{A-C}{2B}$, which is also unknown by the firm. Based on past profit information it is able to estimate the marginal profit, for example by using a numerical differentiation scheme. The obtained marginal profit is a delayed value, when we omit the error of numerical differentiation. So the gradient adjustment process can be given as

$$\dot{x}(t) = K(A - C - 2Bx(t - \tau))$$

where K is the speed of adjustment and τ is the uncertain delay. By assuming a bounded exponential distribution of τ in interval $[t-\Delta, t]$ and taking expectation of the delayed term we have the integro-differential equation

$$\dot{x}(t) = K\left(A - C - \frac{2B}{\gamma}\int_{t-\Delta}^{t}\frac{1}{T}e^{-\frac{t-s}{T}}x(s)ds\right) \tag{2}$$

with

$$\gamma = \int_{0}^{\Delta}\frac{1}{T}e^{-\frac{u}{T}}du = 1 - e^{-\frac{\Delta}{T}}.$$

Introduce the new variable

$$z(t) = \int_{t-\Delta}^{t}\frac{1}{T}e^{-\frac{t-s}{T}}x(s)ds,$$

then

$$\dot{z}(t) = \frac{1}{T}(x(t) - z(t)) - \frac{1}{T}e^{-\frac{\Delta}{T}}x(t-\Delta) \tag{3}$$

and from (2),

$$\dot{x}(t) = K(A - C - \frac{2B}{\gamma}z(t)). \tag{4}$$

From this equation,

$$z(t) = \frac{1}{2KB}(KA\gamma - KC\gamma - \gamma\dot{x}(t)),$$

and so

$$\dot{z}(t) = -\frac{\gamma}{2KB}\ddot{x}(t).$$

Combining this equation with (3) yields

$$-\frac{\gamma}{2KB}\ddot{x}(t) = \frac{1}{T}\left(x(t) - \frac{1}{2KB}(KA\gamma - KC\gamma - \gamma\dot{x}(t))\right) - \frac{1}{T}e^{-\frac{\Delta}{T}}x(t-\Delta)$$

which can be simplified as

$$T\ddot{x}(t) + \dot{x}(t) + \frac{2KB}{\gamma}x(t) - \frac{2KB}{\gamma}e^{-\frac{\Delta}{T}}x(t-\Delta) + K(C-A) = 0. \tag{5}$$

By introducing the notation

$$a = \frac{2KB}{\gamma} \quad \text{and} \quad b = K(A - C)$$

this equation can be rewritten as

$$T\ddot{x}(t) + \dot{x}(t) + ax(t) - ae^{-\frac{\Delta}{T}}x(t-\Delta) - b = 0. \tag{6}$$

The steady state of this system is given by

$$\bar{x} = \frac{b}{a\left(1 - e^{-\frac{\Delta}{T}}\right)} = \frac{K(A-C)}{\frac{2KB}{\gamma}\left(1 - e^{-\frac{\Delta}{T}}\right)} = \frac{A-C}{2B}$$

which is the maximizer of the profit function $\Pi(x)$.

3. Stability Analysis

In order to analyze the asymptotic behavior of system (6) we need to find the possible stability switches, where $\lambda = iw$ with $w > 0$. This assumption does not restrict generality, since if λ is an eigenvalue, then its complex conjugate is also on eigenvalue. The characteristic equation of (6) is

$$T\lambda^2 + \lambda + a - ae^{-\frac{\Delta}{T} - \lambda\Delta} = 0. \tag{7}$$

If $\lambda = iw$, then[1]

$$-Tw^2 + iw + a - ae^{-\frac{\Delta}{T}}(\cos \Delta w - i \sin \Delta w) = 0$$

and by separating the real and imaginary parts, we have

$$ae^{-\frac{\Delta}{T}} \cos \Delta w = a - Tw^2 \tag{8}$$

[1]Note that i is the $\sqrt{-1}$.

$$ae^{-\frac{\Delta}{T}}\sin\Delta w = -w. \tag{9}$$

By adding the squares of these equations we have

$$a^2e^{-\frac{2\Delta}{T}} = (a - Tw^2)^2 + w^2. \tag{10}$$

From (10), inequality

$$a^2 > (a - Tw^2)^2 + w^2$$

should be satisfied, which can be rewritten as

$$w^2 < \frac{2aT - 1}{T^2}, \tag{11}$$

and in order to have feasible solution for w we also need to satisfy relation

$$aT > \frac{1}{2}. \tag{12}$$

Notice that (11) is a quadratic inequality in T,

$$T^2w^2 - 2aT + 1 < 0$$

which is satisfied if and only if

$$\frac{a - \sqrt{a^2 - w^2}}{w^2} < T < \frac{a + \sqrt{a^2 - w^2}}{w^2}. \tag{13}$$

The expression under the square root is positive by (10).

First we show that pure complex roots of the characteristic equation are single. Otherwise $\lambda = iw$ has to solve both the equations

$$T\lambda^2 + \lambda + a - ae^{-\frac{\Delta}{T}-\lambda\Delta} = 0$$

and

$$2T\lambda + 1 + \Delta ae^{-\frac{\Delta}{T}-\lambda\Delta} = 0.$$

By adding the Δ - multiple of the first equation to the other we get

$$\Delta T\lambda^2 + (\Delta + 2T)\lambda + \Delta a + 1 = 0$$

that is,

$$(-\Delta T w^2 + \Delta a + 1) + iw(\Delta + 2T) = 0.$$

The imaginary part is zero when $w = 0$, in which case the real part is positive. This is a contradiction.

From equation (10) we have

$$T^2 w^4 - 2aw^2 T + a^2\left(1 - e^{-\frac{2\Delta}{T}}\right) + w^2 = 0.$$

The discriminant is

$$D = 4w^4\left(a^2 e^{-\frac{2\Delta}{T}} - w^2\right),$$

so a solution exists only if $w \leqq ae^{-\frac{\Delta}{T}}$, in which case

$$T_1, T_2 = \frac{a \pm \sqrt{a^2 e^{-\frac{2\Delta}{T}} - w^2}}{w^2} \quad (T_1 < T_2). \tag{14}$$

Clearly both roots are positive and satisfy (13). At $w = 0$, both T_1 and T_2 converge to infinity, and at $w = ae^{-\frac{\Delta}{T}}$ there holds $T_1(w) = T_2(w) = \frac{1}{a}e^{\frac{2\Delta}{T}}$. These functions are shown in Figure 1, where $T_1(w)$ is shown with blue color and $T_2(w)$ in red.

It is also clear, that T_2 strictly decreases in w. By differentiation,

$$\frac{\partial}{\partial w}T_1 = \frac{1}{w^4}\left\{\frac{-1}{2\sqrt{a^2 e^{-\frac{2\Delta}{T}} - w^2}}(-2w)w^2 - 2w\left(a - \sqrt{a^2 e^{-\frac{2\Delta}{T}} - w^2}\right)\right\},$$

which has the same sign as

$$w^2 - 2a\sqrt{a^2 e^{-\frac{2\Delta}{T}} - w^2} + 2\left(a^2 e^{-\frac{2\Delta}{T}} - w^2\right) = -2a\sqrt{a^2 e^{-\frac{2\Delta}{T}} - w^2} + \left(-w^2 + 2a^2 e^{-\frac{2\Delta}{T}}\right).$$

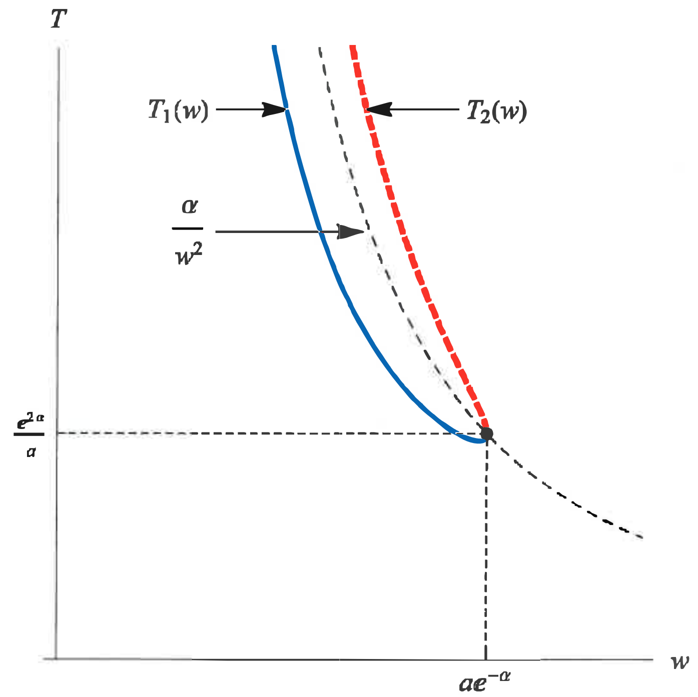

FIGURE 1. The graph of functions $T_1(w)$ and $T_2(w)$

The second term is clearly positive. This expression is positive, if

$$2a\sqrt{a^2e^{-\frac{2\Delta}{T}} - w^2} < 2a^2e^{-\frac{2\Delta}{T}} - w^2$$

that is if,

$$4a^4e^{-\frac{2\Delta}{T}} - 4a^2w^2 < 4a^4e^{-\frac{4\Delta}{T}} + w^4 - 4a^2e^{-\frac{2\Delta}{T}}w^2$$

or

$$w^4 + 4a^2w^2\left(1 - e^{-\frac{2\Delta}{T}}\right) - 4a^4e^{-\frac{2\Delta}{T}}\left(1 - e^{-\frac{2\Delta}{T}}\right) > 0.$$

The left hand side is a quadratic polynomial in w^2, one root is negative, the other is positive. Therefore there are two complex roots, furthermore one root is negative and the other is positive, which is

$$w^* = a\sqrt{2\left(\sqrt{1 - e^{-\frac{2\Delta}{T}}} - \left(1 - e^{-\frac{2\Delta}{T}}\right)\right)} \tag{15}$$

and T_1 increases if $w > w^*$ and decreases if $w < w^*$. It is easy to see that $w^* < ae^{-\frac{\Delta}{T}}$.

Notice that

$$T_1(w^*) = \frac{1 + \sqrt{1 - e^{-\frac{2\Delta}{T}}}}{2ae^{-\frac{2\Delta}{T}}}, \tag{16}$$

which satisfies (12). From (9),

$$\sin \Delta w = -\frac{w}{a} e^{\frac{\Delta}{T}}. \tag{17}$$

We consider $\alpha := \frac{\Delta}{T}$ fixed, and in this case we take T and w as variables. From (17),

$$\sin(\alpha T w) = -\frac{w}{a} e^{\alpha},$$

which can be rewritten as

$$\bar{T}_1^{(n)} = \frac{1}{\alpha w}\left(-\sin^{-1}\left(\frac{w}{a}e^{\alpha}\right) + 2n\pi\right) \quad (n \geqq 1) \tag{18}$$

if $a - \bar{T}_1^{(n)} w^2 \geqq 0$, or

$$\bar{T}_2^{(k)} = \frac{1}{\alpha w}\left(\sin^{-1}\left(\frac{w}{a}e^{\alpha}\right) + (2k+1)\pi\right) \quad (k \geqq 0) \tag{19}$$

if $a - \bar{T}_2^{(k)} w^2 < 0$. Figure 2 shows the graphs of $\bar{T}_1^{(n)}(w)$ and $\bar{T}_2^{(k)}(w)$ for $n = 1, 2, 3, 4, 5$ and $k = 0, 1, 2, 3, 4$. The graphs of $\bar{T}_1^{(n)}(w)$ are shown in blue color and the graphs of $\bar{T}_2^{(k)}(w)$ are given in red. It is clear that $\bar{T}_2^{(k)}(w)$ and $\bar{T}_1^{(k+1)}(w)$ have the same endpoint at $w = ae^{-\alpha}$.

The graph of $\bar{T}_1^{(n)}$ clearly strictly decreases in w, $\bar{T}_1^{(n)}(ae^{-\alpha}) = \frac{(2n-\frac{1}{2})\pi}{\alpha a e^{-\alpha}}$ and $\lim_{w \to 0} \bar{T}_1^{(n)}(w) = \infty$ for all $n \geqq 1$. Similarly,

$$\bar{T}_2^{(k)}(ae^{-\alpha}) = \frac{\left(2k + \frac{3}{2}\right)\pi}{\alpha a e^{-\alpha}}$$

and also

$$\lim_{w \to 0} \bar{T}_2^{(k)}(w) = \infty.$$

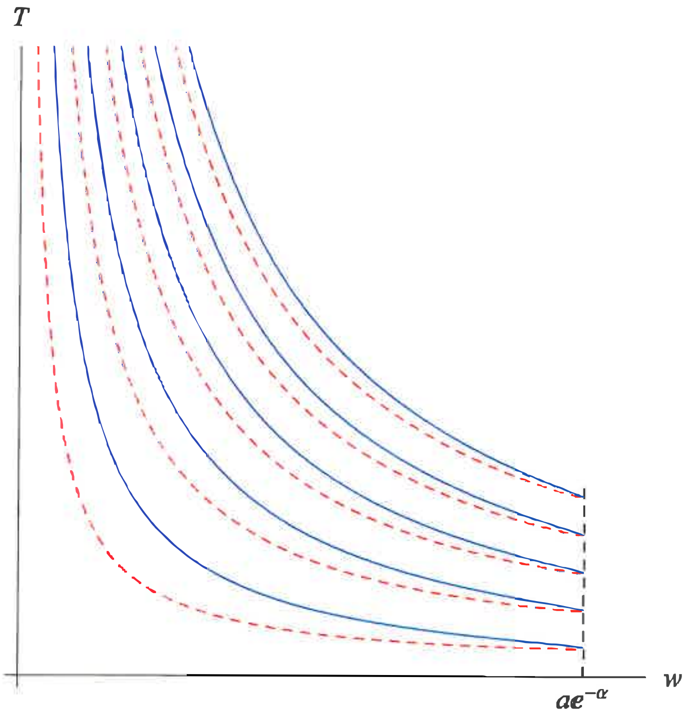

FIGURE 2. Shapes of $\bar{T}_1^{(n)}(w)$ and $\bar{T}_2^{(k)}(w)$ for $1 \leqq n \leqq 5$ and $0 \leqq k \leqq 4$

By simple differentiation,

$$\begin{aligned}\frac{\partial \bar{T}_2^{(k)}}{\partial w} &= \frac{1}{\alpha w^2}\left(\frac{1}{\sqrt{1-\frac{w^2}{a^2}e^{2\alpha}}}\frac{e^{\alpha}}{a}w - \left(\sin^{-1}\left(\frac{w}{a}e^{\alpha}\right) + (2k+1)\pi\right)\right) \\ &= \frac{1}{\alpha w^2}\left(\frac{x}{\sqrt{1-x^2}} - \sin^{-1}(x) - (2k+1)\pi\right)\end{aligned} \tag{20}$$

with $x = \frac{e^{\alpha} w}{a} \leqq 1$. For all $k \geqq 0$ there is a unique value w_k of w such that this derivative is zero, $w_0 < w_1 < w_2 < \ldots$ and $\bar{T}_2^{(k)}$ decreases as $w < w_k$ and increases when $w > w_k$. This fact is a consequence of the observation that the expression inside the parenthesis is strictly increases with negative value at $x = 0$ and infinite limit at $x = 1$.

It is also clear that $\bar{T}_1^{(n)}(w) > \bar{T}_2^{(k)}(w)$ for $n \geqq k+1$ in interval $[0, ae^{-\alpha})$, and $\bar{T}_1^{(n)}(w) < \bar{T}_2^{(k)}(w)$ for $n \leqq k$.

4. Stability Switching

The possible (w, T) stability switchings are obtained by the intercepts of $T_1(w)$ and $\bar{T}_1^{(n)}(w)$ and the intercepts of $T_2(w)$ and $\bar{T}_2^{(k)}(w)$. It is also clear that with small values of w, both T_1 and T_2 are larger than $\bar{T}_1^{(n)}$ and $\bar{T}_2^{(k)}$, furthermore at $w = ae^{-\alpha}$, $\bar{T}_1^{(n)}$ and $\bar{T}_2^{(k)}$ are larger than both T_1 and T_2 if n and k are sufficiently large. Therefore there are infinitely many solutions (w, T).

Next we check the directions of stability switching. Let α be fixed again, and let T be the bifurcation parameter. Implicitly differentiating (7) and using the fact that $\Delta = \alpha T$, we have

$$\lambda^2 + 2T\lambda\dot{\lambda} + \dot{\lambda} - ae^{-\alpha}e^{-\lambda\alpha T}(-\dot{\lambda}\alpha T - \lambda\alpha) = 0,$$

implying that

$$\dot{\lambda} = -\frac{\lambda^2 + a\alpha\lambda e^{-\alpha-\lambda\alpha T}}{2T\lambda + 1 + a\alpha T e^{-\alpha-\lambda\alpha T}}.$$

By using equation (7) again, we obtain

$$\dot{\lambda} = -\frac{\lambda^2 + a\alpha\lambda\frac{T\lambda^2+\lambda+a}{a}}{2T\lambda + 1 + a\alpha T\frac{T\lambda^2+\lambda+a}{a}} = -\frac{\lambda^2 + \alpha\lambda(T\lambda^2+\lambda+a)}{2T\lambda + 1 + \alpha T(T\lambda^2+\lambda+a)}.$$

When $\lambda = iw$, we find that

$$\dot{\lambda} = \frac{\alpha T w^3 i + w^2(1+\alpha) - a\alpha w i}{-w^2(\alpha T^2) + iw(2T+\alpha T) + (1+\alpha a T)} = \frac{w^2(1+\alpha) + i(\alpha T w^3 - a\alpha w)}{(-w^2T^2\alpha + 1 + \alpha a T) + iw(2T + \alpha T)}. \tag{21}$$

The sign of $\operatorname{Re}\dot{\lambda}$ is the same as the sign of its numerator after multiplying both numerator and denominator by the complex conjugate of the denominator, so that the numerator becomes

$$\begin{aligned} &w^2(1+\alpha)(-w^2T^2\alpha + 1 + \alpha a T) + w(2T+\alpha T)(\alpha T w^3 - a\alpha w) \\ &= w^2(\alpha T^2 w^2 - \alpha a T + \alpha + 1). \end{aligned} \tag{22}$$

Therefore stability is lost if

$$w^2 > \frac{\alpha a T - \alpha - 1}{\alpha T^2} := g(T) \tag{23}$$

and maybe regained if $w^2 < g(T)$. From (10),

$$w^2 = \frac{2aT - 1 \pm \sqrt{D}}{2T^2} \tag{24}$$

with

$$D = 4a^2T^2e^{-2\alpha} + 1 - 4aT.$$

The larger solution is the entire $T_2(w)$ and $T_1(w)$ between w^* and $ae^{-\alpha}$, the smaller solution is $T_1(w)$ between 0 and w^*.

We will next show that the larger solution of (24) is greater than $g(T)$, that is, stability is lost at the critical values located there.

We need to show that

$$\frac{2aT - 1 + \sqrt{D}}{2T^2} > \frac{\alpha a T - \alpha - 1}{\alpha T^2},$$

which can be rewritten as

$$2a\alpha T - \alpha + \alpha\sqrt{D} > 2\alpha a T - 2\alpha - 2$$

or

$$\alpha + 2 + \alpha\sqrt{D} > 0$$

which is clearly true.

The smaller root is also above $g(T)$ when

$$\frac{2aT - 1 - \sqrt{D}}{2T^2} > \frac{\alpha a T - \alpha - 1}{\alpha T^2}$$

or

$$\alpha + 2 > \alpha\sqrt{D}$$

which can be rewritten as

$$0 > a^2\alpha^2T^2e^{-2\alpha} - a\alpha^2T - \alpha - 1.$$

The right hand side has two roots for T, the only positive one being

$$T^* = \frac{a\alpha^2 + \sqrt{a^2\alpha^4 + 4(\alpha+1)a^2\alpha^2e^{-2\alpha}}}{2a^2\alpha^2e^{-2\alpha}} > \frac{1}{a}e^{2\alpha}$$

and stability is lost if $T < T^*$ and stability might be regained for $T > T^*$. Delay is harmless if $T < \frac{1+\sqrt{1-e^{-2\alpha}}}{2ae^{-2\alpha}}$ since in this region there is no possible stability switching by (16). Potential points of stability switching are the intercepts of $T_1(w)$ and $\bar{T}_1^{(n)}(w)$ and those of $T_2(w)$ and $\bar{T}_2^{(k)}(w)$.

If $T > T^*$ at the intercept of $T_1(w)$ and $\bar{T}_1^{(n)}(w)$, then stability might be regained, if only one eigenvalue had positive real part before. If $T < T^*$, then stability is lost, which is the case at the intercepts of $T_2(w)$ and $\bar{T}_2^{(k)}(w)$ for all k. Since at each intercept the sign of the real part of only one eigenvalue changes sign, we have to order the intercepts in increasing order in T, and have to divide them into two classes. In class 1, stability is lost, and in class 2, stability might be regained. Since at $T = 0$ the system is stable, the smallest intercept has to be in class 1. At each value of T, we have to count the number $N_1(T)$ of class 1 intercepts below T, and the number $N_2(T)$ of class 2 intercepts below T. If $N_1(T) \leqq N_2(T)$, then the system is asymptotically stable at T, otherwise it is unstable. The locations of the intercepts depend on the actual parameter values, which can be examined by computer studies.

5. Examples

The parameter value $a = 4$ was selected in our computer study. Three cases were considered, $\alpha = 1$, $\alpha = 2$ and $\alpha = \frac{1}{2}$. In the first case $T = \Delta$,

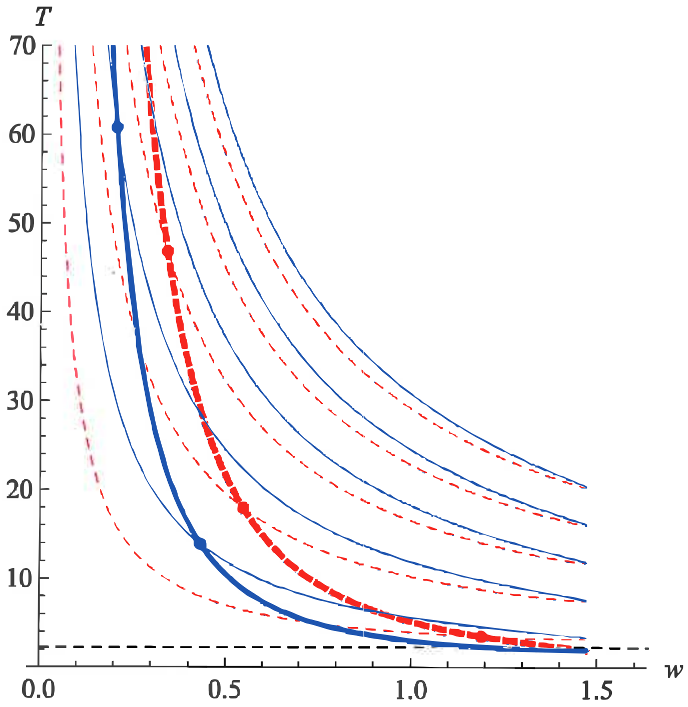

FIGURE 3. The case of $\alpha = 1$

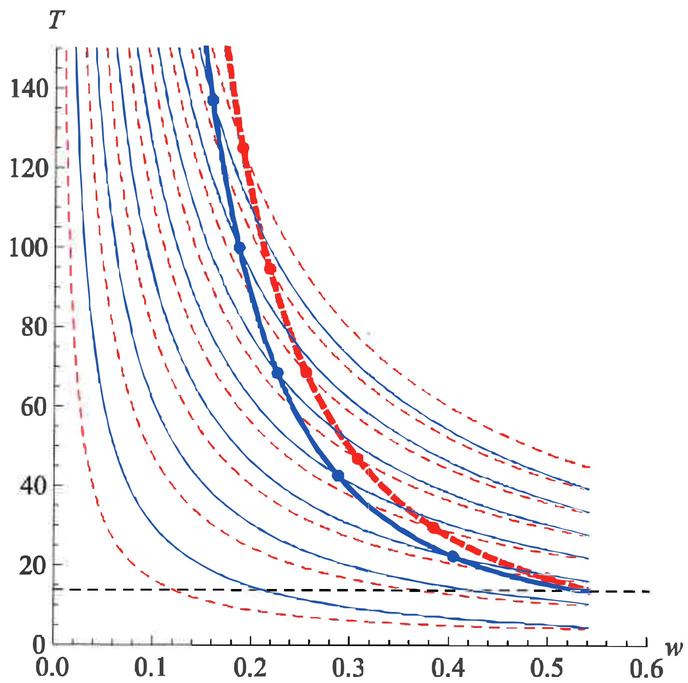

FIGURE 4. The case of $\alpha = 2$

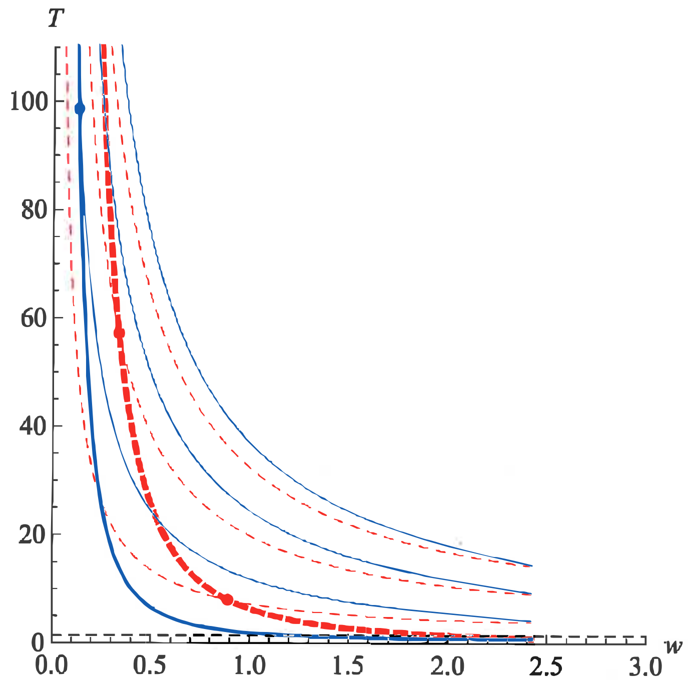

FIGURE 5. The case of $\alpha = \frac{1}{2}$

when the data with average delay are at the left endpoint of the interval $(t - \Delta, t)$. In the second case it is in the middle of the interval and in the third case it is below the endpoint. Figures 3, 4 and 5 show these cases. The thick red line is $T_2(w)$, the thick blue line shows $T_1(w)$. The horizontal dotted line is at $T = T^*$. The intercepts of $T_1(w)$ and $\bar{T}_1^{(n)}(w)$ are the blue dots and the intercepts of $T_2(w)$ and $\bar{T}_2^{(k)}(w)$ are the red dots. At each red dot stability is lost, since one eigenvalue changes the sign of its real part from negative to positive. This is the case with blue dots under the horizontal dotted line. However at the blue dots above the horizontal dotted line stability might be regained, since one eigenvalue changes the sign of its real part from positive to negative.

In each figure we have to consider the red and blue dots in increasing order with respect to T. If $\alpha = 1$, then stability is lost at the first red dot which is then regained, but it is lost twice afterwards. The next dot is blue, where only one eigenvalue changes the sign of its real part. However stability is not regained, since there is another eigenvalue with positive real part already. If $\alpha = 2$, then stability is regained three times, and if $\alpha = \frac{1}{2}$, then the first two dots are red, so there are two eigenvalues with positive real parts afterwards, so the next blue dot does not lead to a regain of stability. With a fixed value of $\alpha = \Delta/T$, the increase of the bifurcation parameter T implies an increase of the length Δ of the time window. Therefore large values of T do not have much practical importance.

6. Conclusions

A monopoly with continuously distributed delay has been examined when the time window of past data was bounded from below and had a fixed length. The characteristic polynomial of the resulting dynamic system is a mixture of a quadratic polynomial and an exponential term. The potential points of stability switching were then determined and a simple rule was developed to check if stability is lost or regained at these points, which are the intercepts of irrational and inverse trigonometric functions. A computer example illustrated the theoretical findings. Both the dynamic model and the stability conditions are different from those reported in earlier studies on continuously distributed lags. In our case several points of stability switching are possible, while in the earlier studies the number of points of stability switching were at most two.

In future research we will extend this methodology to duopolies and n-f

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