Discussion Paper No.244

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January 2015

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Abstract

A modified Cournot oligopoly is introduced and examined in which the firms can treat their wastes up to a certain amount, and if the amount of waste is even higher, then an outside facility is used with a given fixed cost and higher unit cost. The resulting payoff functions become discontinuous. The best response functions of firms are nonincreasing and also might be discontinuous. The best response functions can be modified to depend on the total indutry output, which also might be discontinuous as well as multiple valued. The existence of at least one equilibrium is proved and numerical examples show the possibility of a unique equilibrium as well as that of multiple equilibria.

Keywords: Cournot oligopoly, Discontinuous best responses, Nash equilibrium

*This paper has been prepared while the first author was a visiting professor at the Department of Economics of Chuo University, Tokyo, Japan. The authors highly appreciate the financial supports from the MEXT-Supported Program for the Strategic Research Foundation at Private Universities 2013-2017, the Japan Society for the Promotion of Science (Grant-in-Aid for Scientific Research (C) 24530202, 25380238 and 26380316). The usual disclaimers apply.

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1 Introduction

Oligopoly theory is one of the most frequently studied subjects in mathematical economics. It dates back to the pioneering work of Cournot (1838) and since then a large number of researchers devoted their efforts to this interesting area. The monograph of Okuguchi (1976) is considered as the most important summary of the earlier results which also includes some of his own fundamental contributions. The multi-product extension of the classical theory with some applications is presented in Okuguchi and Szidarovszky (1999). In recent years nonlinear models occupied the main research focus, a comprehensive summary of the more recent developments is presented in Bischi et al. (2010). Most earlier models assumed differentiable payoff functions, where the analytic treatment was straightforward. In Szidarovszky and Yakowitz (1982) only the continuity of the price and cost functions are assumed with the possibility of infinitely many equilibria where the industry output is unique. The uniqueness of the equilibrium is guaranteed if the price function is differentiable at this point. More recently, Zhao and Szidarovszky (2008) introduced production adjustment costs resulting in non-differentiable payoff functions, where the existence of usually infinitely many equilibria is proved. Similar situation is found in Burr et al. (2014) where the output adjustments are limited from both above and below. The payoff functions are non-differentiable and the best responses discontinuous in this case and there are again infinitely many equilibria except in special cases.

In this paper another variant of the Cournot model is introduced, where the payoff functions are discontinuous, therefore the best responses as functions of the output of the rest of the industry are also discontinuous. Furthermore the best responses as functions of the total industry output are not only discontinuous but might have multiple values.

This paper develops as follows. In Section 2, the mathematical model is introduced and the best responses are determined. In Section 3, the existence of at least one equilibrium is proved and numerical examples shown the possibility of both unique and multiple equilibria. Section 4 concludes the paper and further research directions are mentioned.

2 The Model and Best Responses

Consider an N-firm single-product oligopoly without product differentiation. For mathematical simplicity assume linear price and cost functions,

$$p(s) = A - Bs$$

and

$$C_k(x_k) = c_k x_k$$

where x_k is the output of firm k and $s = \sum_{k=1}^{N} x_k$ is the industry output. Assume that each firm produces some waste proportional to its production level, which can be cleaned or deposited by the firm until a certain amount, and if the waste amount is larger than a certain threshold, then it has to be shipped to be cleaned or deposited by a contractor with higher unit cost than that if the firm itself does the cleaning or depositing and in addition the firm also has to pay a certain fixed cost, which can be interpreted as the setup or transportation cost. So the payoff of firm k can be written as follows:

$$\varphi_k = x_k(A - Bs_k - Bx_k) - c_k x_k - \begin{cases} \alpha_k x_k & \text{if } x_k \le K_k, \\ a_k + \beta_k x_k & \text{if } x_k > K_k \end{cases} \tag{1}$$

where $s_k = \sum_{\ell \ne k}^{N} x_\ell$ is the output of the rest of the industry for firm k, α_k and β_k are the costs of cleaning or depositing the waste per unit production (since amount of waste is proportional to the output level), and a_k is the fixed cost and K_k is the maximum output level which generates the maximum waste amount that the firm can treat. There are several possibilities for the shape of φ_k, which are summarized in Figure 1, where L_k is the capacity limit of the firm.

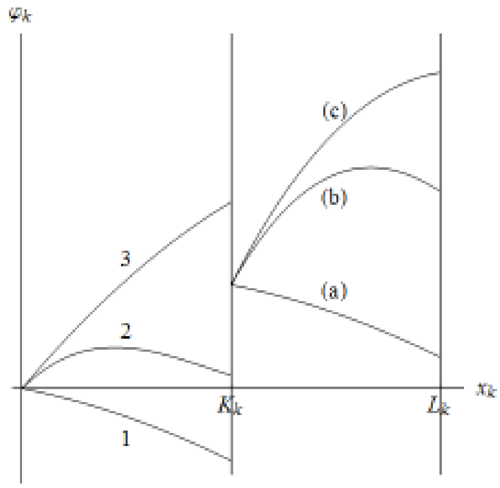

Figure 1. Possible shapes of the payoff functions

Case 1

It occurs, when

$$\frac{\partial \varphi_k}{\partial x_k} \le 0 \text{ at } x_k = 0,$$

which is the case when

$$A - Bs_k - c_k - \alpha_k \le 0$$

that is,

$$s_k \ge \frac{A - c_k - \alpha_k}{B}. \tag{2}$$

In this case the best response of the firm is

$$R_k(s_k) = 0. \tag{3}$$

Case 2

It occurs, when

$$\frac{\partial \varphi_k}{\partial x_k} > 0 \text{ at } x_k = 0 \text{ and } \frac{\partial \varphi_k}{\partial x_k} \leq 0 \text{ at } x_k = K_{k^-},$$

that is, when (2) is violated with strict inequality and

$$A - Bs_k - 2BK_k - c_k - \alpha_k \leq 0,$$

so

$$\frac{A - c_k - \alpha_k}{B} - 2K_k \leq s_k < \frac{A - c_k - \alpha_k}{B}, \tag{4}$$

and in this case the best response is the stationary point:

$$R_k(s_k) = \frac{A - c_k - \alpha_k}{2B} - \frac{s_k}{2}. \tag{5}$$

Case 3

This is the case when

$$\frac{\partial \varphi_k}{\partial x_k} > 0 \text{ at both } x_k = 0 \text{ and } x_k = K_{k^-},$$

so

$$s_k < \frac{A - c_k - \alpha_k}{B} - 2K_k. \tag{6}$$

There are three subcases depending on the signs of the derivative

$$\frac{\partial \varphi_k}{\partial x_k} \text{ at } x_k = K_{k^+} \text{ and } x_k = L_k.$$

In case 3(a),

$$\frac{\partial \varphi_k}{\partial x_k} \leq 0 \text{ at } x_k = K_{k^+}$$

which can be written as

$$s_k \geq \frac{A - c_k - \beta_k}{B} - 2K_k, \tag{7}$$

and clearly

$$R_k(s_k) = K_k \tag{8}$$

in this case.

Case 3(b) occurs when in addition to (6),

$$\frac{\partial \varphi_k}{\partial x_k} > 0 \text{ at } x_k = K_{k^+} \text{ and } \frac{\partial \varphi_k}{\partial x_k} < 0 \text{ at } x_k = L_k.$$

That is,

$$\frac{A - c_k - \beta_k}{B} - 2L_k < s_k < \frac{A - c_k - \beta_k}{B} - 2K_k. \qquad (9)$$

The stationary point between K_k and L_k is

$$x_k^* = \frac{A - c_k - \beta_k}{2B} - \frac{s_k}{2} \qquad (10)$$

and now the function values $\varphi_k(x_k^*)$ and $\varphi_k(K_k)$ have to be compared. Notice first that

$$\begin{aligned} \varphi_k(x_k^*) &= \left(\frac{A - c_k - \beta_k}{2B} - \frac{s_k}{2}\right)\left(A - Bs_k - \frac{A - c_k - \beta_k - Bs_k}{2} - c_k - \beta_k\right) - a_k \\ &= \frac{B}{4}\left(\frac{A - c_k - \beta_k}{B} - s_k\right)^2 - a_k. \end{aligned}$$

Since

$$\varphi_k(K_k) = K_k\left(A - Bs_k - BK_k - c_k - \alpha_k\right),$$

we have

$$\varphi_k(x_k^*) > \varphi_k(K_k)$$

if and only if

$$\left(\frac{A - c_k - \beta_k}{B} - s_k\right)^2 > \frac{4K_k}{B}\left(A - Bs_k - BK_k - c_k - \alpha_k\right) + \frac{4a_k}{B},$$

which is a quadratic inequality for s_k:

$$0 < s_k^2 + s_k\left(4K_k - 2\frac{A - c_k - \beta_k}{B}\right) + \left(-\frac{4AK_k}{B} + 4K_k^2 + \frac{4K_k(c_k + \alpha_k)}{B} + \left(\frac{A - c_k - \beta_k}{B}\right)^2 - \frac{4a_k}{B}\right).$$

The discriminant of the right hand side is

$$D = \frac{16}{B}\left(a_k + K_k(\beta_k - \alpha_k)\right),$$

so its two roots are

$$s_k^{\pm} = \frac{A - c_k - \beta_k}{B} - 2K_k \pm \sqrt{\frac{4}{B}\left(a_k + K_k(\beta_k - \alpha_k)\right)}. \qquad (11)$$

The root s_k^+ violates (9), so the only feasible root is s_k^-, if it satisfies the left hand side of (9),

$$\frac{A - c_k - \beta_k}{B} - 2K_k - \sqrt{\frac{4}{B}\left(a_k + K_k(\beta_k - \alpha_k)\right)} > \frac{A - c_k - \beta_k}{B} - 2L_k$$

which can be rewritten as

$$L_k - K_k > \sqrt{\frac{a_k + K_k(\beta_k - \alpha_k)}{B}}. \qquad (12)$$

In the case of $s_k < s_k^-$,

$$R_k(s_k) = x_k^*.$$

If $s_k > s_k^-$, then

$$R_k(s_k) = K_k$$

and if $s_k = s_k^-$, then

$$R_k(s_k) = \{K_k, x_k^*\}.$$

If s_k^- is not feasible, then

$$s_k^- \leq \frac{A - c_k - \beta_k}{B} - 2L_k,$$

so in the entire interval (9),

$$R_k(s_k) = K_k.$$

Notice that s_k^- satisfies the right hand side of (9).

In the case of $3(c)$,

$$\frac{\partial \varphi_k}{\partial x_k} \geq 0 \text{ at } x_k = L_k,$$

so

$$0 \leq s_k \leq \frac{A - c_k - \beta_k}{B} - 2L_k \tag{13}$$

and in this case both K_k and L_k might be best response, so we have to compare the values of $\varphi_k(K_k)$ and $\varphi_k(L_k)$. Notice first that

$$\varphi_k(K_k) > \varphi_k(L_k)$$

if and only if

$$K_k (A - Bs_k - BK_k - c_k - \alpha_k) > L_k (A - Bs_k - BL_k - c_k - \beta_k) - a_k$$

which can be written as

$$s_k > s_k^* = \frac{A - c_k - \beta_k}{B} - (L_k + K_k) - \frac{a_k + K_k(\beta_k - \alpha_k)}{B(L_k - K_k)}. \tag{14}$$

It is easy to see that

$$s_k^* \leq \frac{A - c_k - \beta_k}{B} - 2L_k$$

if and only if

$$L_k - K_k \leq \sqrt{\frac{a_k + K_k(\beta_k - \alpha_k)}{B}}. \tag{15}$$

It is easy to see that $s_k^* \leq s_k^-$, and equality holds if and only if

$$s_k^- = s_k^* = \frac{A - c_k - \beta_k}{B} - 2L_k.$$

In comparing (12) and (15), we can conclude the followings:

If s_k^- is interior in interval (9), then (12) holds with strict inequality, then (15) is violated, so

$$s_k^* > \frac{A - c_k - \beta_k}{B} - 2L_k$$

implying that $R_k(s_k) = L_k$ in interval (13).

If

$$s_k^- = \frac{A - c_k - \beta_k}{B} - 2L_k,$$

then (12) holds with equality, so (15) implies that $s_k^- = s_k^*$ at this point, so

$$R_k(s_k^-) = \{K_k; L_k\}$$

and

$$R_k(s_k) = L_k \text{ again in all other points of interval (13).}$$

If

$$s_k^- < \frac{A - c_k - \beta_k}{B} - 2L_k,$$

then (12) is violated with strict inequality, so (15) implies that

$$s_k^* < \frac{A - c_k - \beta_k}{B} - 2L_k.$$

If it is positive, then

$$R_k(s_k^*) = \{K_k; L_k\},$$

$$R_k(s_k) = \begin{cases} K_k & \text{for } s_k > s_k^* \\ L_k & \text{for } s_k < s_k^*. \end{cases}$$

If $s_k^* = 0$, then

$$R_k(s_k) = K_k \text{ for all positive } s_k \text{ values in interval (13),}$$

$$R_k(0) = \{K_k; L_k\}.$$

If $s_k^* < 0$, then

$$R_k(s_k) = K_k \text{ for all } s_k \text{ from interval (13).}$$

The possible shapes of $R_k(s_k)$ are shown in Figure 2, where 2(a) represents the case when neither s_k^* nor s_k^- is feasible, part 2(b) shows the case, when only s_k^- is feasible and 2(c) the case when only s_k^* is feasible which includes the extreme case of

$$s_k^* = s_k^- = \frac{A - c_k - \beta_k}{B} - 2L_k.$$

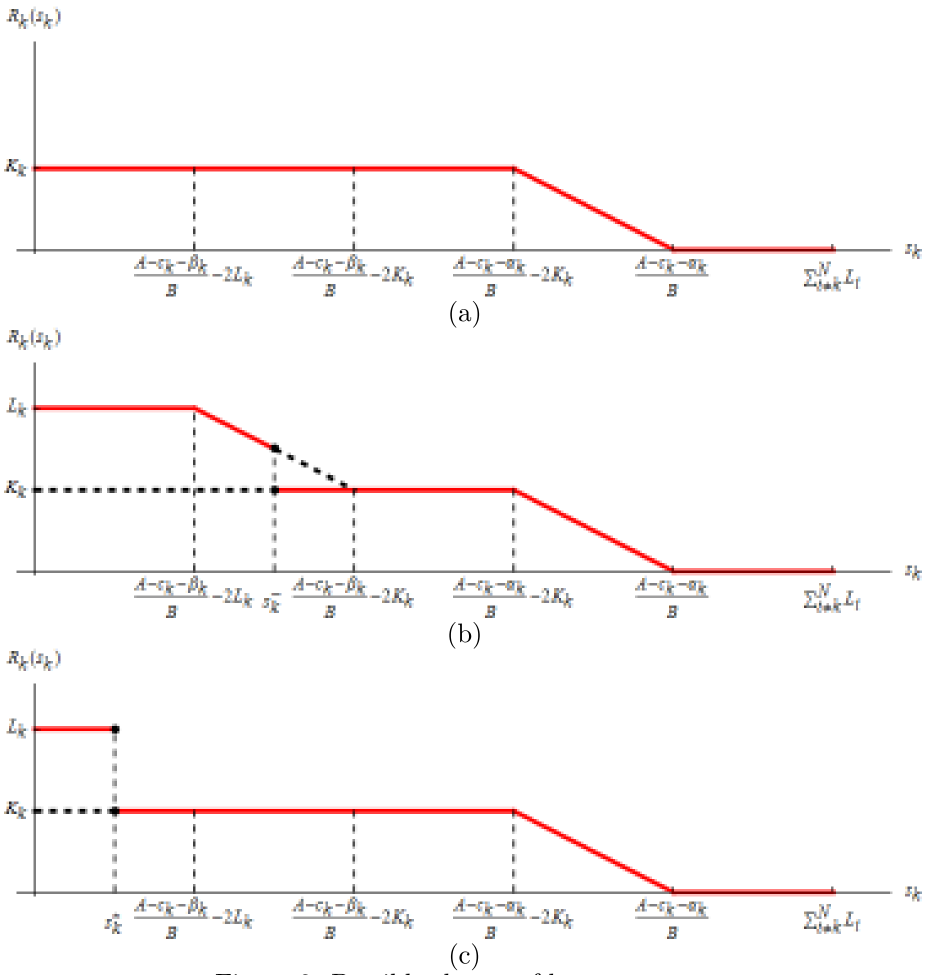

Figure 2. Possible shapes of best responses

In case (a) the best response function is continuous, however in cases (b) and (c), it is discontinuous by having jump at $s_k = s_k^-$ and $s_k = s_k^*$, respectively.

We can also rewrite the best responses as functions of the total industry output s. In case 1, $\bar{R}_k(s) = 0$ with

$$s \geq \frac{A - c_k - \alpha_k}{B}. \tag{16}$$

In case 2, from (5) we have

$$s = s_k + x_k = \frac{A - c_k - \alpha_k}{2B} + \frac{s_k}{2}$$

therefore the range of s is

$$\frac{A - c_k - \alpha_k}{B} - K_k \leq s < \frac{A - c_k - \alpha_k}{B} \tag{17}$$

and the best response satisfies equation

$$x_k = \frac{A - c_k - \alpha_k}{2B} - \frac{s - x_k}{2}$$

which implies that

$$x_k = \frac{A - c_k - \alpha_k}{B} - s. \tag{18}$$

In case 3(a), $\bar{R}_k(s) = K_k$ and since s_k satisfies (6) and (7),

$$\frac{A - c_k - \beta_k}{B} - K_k \leq s < \frac{A - c_k - \alpha_k}{B} - K_k. \tag{19}$$

In case 3(b) we have two subcases. If s_k^- is not feasible, then $\bar{R}_k(s) = K_k$ and from (9),

$$\frac{A - c_k - \beta_k}{B} - 2L_k + K_k < s < \frac{A - c_k - \beta_k}{B} - K_k.$$

If s_k^- is feasible, then in the case of $s_k > s_k^-$, $\bar{R}_k(s) = K_k$ with the domain

$$s_k^- + K_k < s < \frac{A - c_k - \beta_k}{B} - K_k. \tag{20}$$

If $s_k < s_k^-$, then $\bar{R}_k(s) = x_k^*$, so from (10),

$$s = x_k^* + s_k = \frac{A - c_k - \beta_k}{2B} + \frac{s_k}{2}$$

with domain

$$\frac{A - c_k - \beta_k}{B} - L_k < s < \frac{A - c_k - \beta_k}{2B} + \frac{s_k^-}{2} \tag{21}$$

and the best response is obtained from equation

$$s = \frac{A - c_k - \beta_k}{2B} + \frac{s - x_k}{2}$$

so

$$\bar{R}_k(s) = \frac{A - c_k - \beta_k}{B} - s. \tag{22}$$

In case 3(c) we also have two subcases. Assume first that s_k^* is not feasible, then $R_k(s_k) = K_k$ if s_k^- is not feasible as well, or $R_k(s_k) = L_k$ if s_k^- is feasible. In the first case,

$$K_k \leq s \leq \frac{A - c_k - \beta_k}{B} - 2L_k + K_k, \; \bar{R}_k(s) = K_k \tag{23}$$

and in the second case

$$L_k \leq s \leq \frac{A - c_k - \beta_k}{B} - L_k, \; \bar{R}_k(s) = L_k. \tag{24}$$

If s_k^* is feasible, then $\bar{R}_k(s_k^*) = \{K_k; L_k\}$, $R_k(s_k) = K_k$ for $s_k > s_k^*$ and $R_k(s_k) = L_k$ for $s_k < s_k^*$. The last two cases are as follows:

$$s_k^* + K_k \leq s \leq \frac{A - c_k - \beta_k}{B} - 2L_k + K_k, \;\; \bar{R}_k(s) = K_k \tag{25}$$

and

$$L_k \leq s \leq s_k^* + L_k, \;\; \bar{R}_k(s) = L_k. \tag{26}$$

Figure 3 shows the possible shapes of $\bar{R}_k(s)$, where part (a) gives the case when neither s_k^- nor s_k^* is feasible, part (b) shows the graph when only s_k^- is feasible and part (c) is the case when only s_k^* is feasible or the border line case of

$$s_k^- = s_k^* = \frac{A - c_k - \beta_k}{B} - 2L_k.$$

In particular cases one or more segments for smaller or larger values of s might be missing.

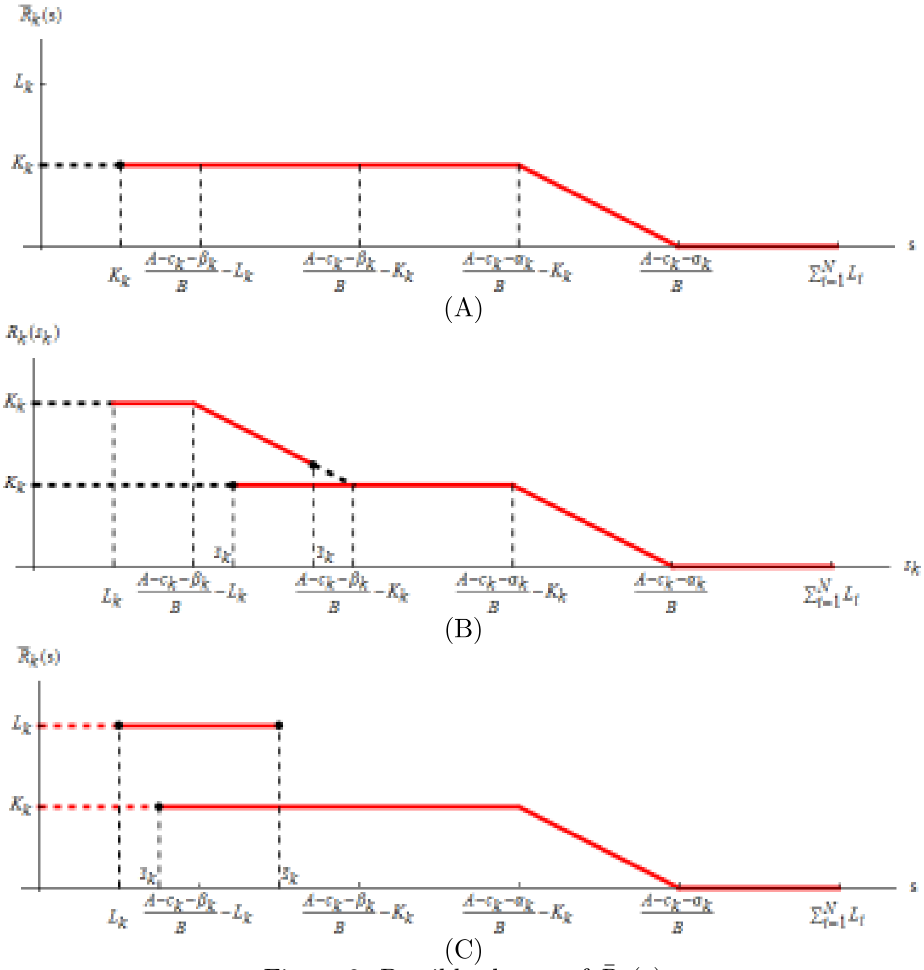

Figure 3. Possible shapes of $\bar{R}_k(s)$

In Figure 3(b)

$$\bar{s}_k = s_k^- + K_k \text{ and } \tilde{s}_k = \frac{A - c_k - \beta_k}{2B} + \frac{s_k^-}{2}.$$

Notice that from (9),

$$\tilde{s}_k = \frac{A - c_k - \beta_k}{2B} + \frac{s_k^-}{2} \leq \frac{A - c_k - \beta_k}{2B} + \frac{A - c_k - \beta_k}{2B} - K_k = \frac{A - c_k - \beta_k}{B} - K_k$$

and

$$\tilde{s}_k \geq \frac{A - c_k - \beta_k}{2B} + \frac{A - c_k - \beta_k}{2B} - L_k = \frac{A - c_k - \beta_k}{B} - L_k.$$

In addition,

$$\tilde{s}_k = \frac{A - c_k - \beta_k}{2B} + \frac{s_k^-}{2} > s_k^- + K_k = \bar{s}_k$$

since it can be written as

$$s_k^- < \frac{A - c_k - \beta_k}{B} - 2K_k,$$

which is true by (11).

Notice that the jumps in Figures 3(b) and 3(c) have 45° slopes, since in Figure 3(b), by (10),

$$x_k^*(\tilde{s}_k) - K_k = \left(\frac{A - c_k - \beta_k}{B} - \tilde{s}_k\right) - K_k = \frac{A - c_k - \beta_k}{2B} - \frac{s_k^-}{2} - K_k$$

and

$$\tilde{s}_k - \bar{s}_k = \left(\frac{A - c_k - \beta_k}{2B} + \frac{s_k^-}{2}\right) - \left(s_k^- + K_k\right)$$

are equal. Furthermore in Figure 3(c),

$$\bar{s}_k = s_k^* + K_k \text{ and } \tilde{s}_k = s_k^* + L_k,$$

and

$$\tilde{s}_k - \bar{s}_k = (s_k^* + L_k) - (s_k^* + K_k) = L_k - K_k.$$

3 Equilibrium Analysis

The equilibrium is the solution of equation

$$\sum_{k=1}^{N} \bar{R}_k(s) = s. \tag{27}$$

Consider the left hand side, which can be denoted by $H(s)$. With any value $s > 0$ an $\bar{R}_k(s)$ exists if $s \geq R_k(0)$. So $\left[R_k(0),\ \sum_{\ell=1}^{N} L_\ell\right]$ is the interval for

s such that $\bar{R}_k(s)$ is defined. So all $\bar{R}_k(s)$ values exist if s is greater than or equal to the largest value of the left end points of these intervals. Clearly, at the minimal s value $H(s) \geq s$, since for at least one k, that minimal value equals s. At $s = \sum_{\ell=1}^{N} L_\ell$, the value of $H(s)$ is below s, since for all k, $\bar{R}_k(s) \leq L_k$. The only way of having no equilibrium is when the 45 degree line skips through a jump created by at least one function $\bar{R}_k(s)$.

We will next show that $\sum_{k=1}^{N} \bar{R}_k(s)$ cannot have jumps with more than 45° slope. The simple structure of the best response function implies the following simple fact. Let s_A and s_B be two points of the domain of $H(s)$ such that $s_A < s_B$, and let $\bar{R}_B$ be the value of $\bar{R}(s_B)$ or one of the values. Then there is a value $\bar{R}_A$ of $\bar{R}(s_A)$ such that $\bar{R}_A \geq \bar{R}_B$. Clearly the same holds if we add up some or all of the $\bar{R}_k(s)$ functions. Let now $g_1(s)$ be sum of some $\bar{R}_k(s)$ functions and $g_2(s)$ the sum of all others. We will show that the slopes of the jumps in $g_1(s)$ $(g_2(s))$ cannot increase by adding $g_2(s)$ $(g_1(s))$, that is, the slopes of the jumps of $H(s)$ cannot acceed 45°. Consider now a jump of $g_1(s)$ with points, $(A,\ R_A)$ and (B, R_B) such that $A < B$ and $R_A < R_B$. Let R'_B be a value of $g_2(s)$ at $s = B$, so the corresponding point of $g_1(s)+g_2(s)$ is $(B,\ R_B + R'_B)$. However there is a point (A, R'_A) on $g_2(s)$ such that $R'_A \geq R'_B$, and the corresponding point on $g_1(s) + g_2(s)$ is $(A,\ R_A + R'_A)$, so the hight of the jump between these points becomes

$$(R_B + R'_B) - (R_A + R'_A) \leq R_B - R_A,$$

so the height of this new jump cannot be larger than that of the jump of $g_1(s)$ in $[A, B]$, so its slope cannot be larger either. Therefore adding the best responses $\bar{R}_k(s)$ the slope of their jumps cannot increase, so the 45° line must cross the curve of $H(s)$ implying the existence of at least one equilibrium.

If all firms can be described by Figure 3(a), then there is a unique equilibrium. The next two examples show cases of a unique equilibrium and multiple equilibria.

Example 1. Assume $N = 2,\ A = 12,\ B = 2,\ a_k = 1,\ c_k = 3,\ \alpha_k = 1,\ \beta_k = 7,\ K_k = 0.5$ and $L_k = 2$ for both firms. In this case

$$\sum_{k=1}^{2} L_k = 4,\ \frac{A - c_k - \alpha_k}{B} = 4,\ \frac{A - c_k - \alpha_k}{B} - 2K_k = 3,$$

$$\frac{A - c_k - \beta_k}{B} - 2K_k = 0,\ \frac{A - c_k - \beta_k}{B} - 2L_k = -3.$$

The two best response functions are shown in Figure 4, from which it is clear that $x_1 = x_2 = 0.5$ is the unique equilibrium.

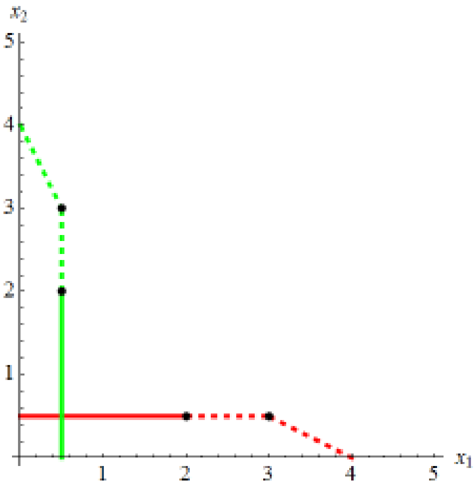

Figure 4. Equilibrium in Example 1

Example 2. Assume again $N = 2,\ A = 14,\ B = 2,\ a_k = \alpha_k = c_k = 1,\ K_k = 1,\ \beta_k = L_k = 2$. Then

$$\frac{A - c_k - \alpha_k}{B} = 6,\ \frac{A - c_k - \alpha_k}{B} - 2K_k = 4,$$

$$\frac{A - c_k - \beta_k}{B} - 2K_k = 3.5,\ \frac{A - c_k - \beta_k}{B} - 2L_k = 1.5,$$

and

$$s_k^- = s_k^* = 1.5.$$

So

$$R_k(s_k) = \begin{cases} 2 & \text{if } 0 \leq s_k < 1.5, \\ \{1; 2\} & \text{if } s_k = 1.5, \\ 1 & \text{if } s_k > 1.5. \end{cases}$$

Consider next the duopoly when both firms have the same parameters as given above. Their best responses are show in Figure 5, where notice that $\sum_{\ell \neq k} L_\ell = 2$ for both firms. It is clear that $x_1 = 1,\ x_2 = 2$ and $x_1 = 2,$

$x_2 = 1$ are the two equilibria.

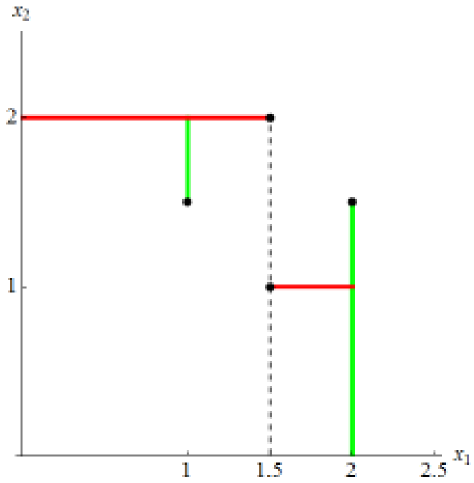

Figure 5. Equilibria in Example 2

Example 3. We can redo the previous examples by using the best responses as functions of the total industry output.

(a) In the case of Example 1,

$$\frac{A - c_k - \alpha_k}{B} = 4, \ \frac{A - c_k - \alpha_k}{B} - K_k = 3.5, \ \frac{A - c_k - \beta_k}{B} - K_k = 0.5, \ \frac{A - c_k - \beta_k}{B} - L_k < 0.$$

Function $\bar{R}_k(s)$ is shown in Figure 6. In a duopoly with the identical firms with the above parameters,

$$\sum_{k=1}^{2} \bar{R}_k(s) = 2\bar{R}_k(s),$$

and it is also shown with broken lines. It has a unique intersection with the 45° line at $s = 1$ and since $\bar{R}_k(1) = 0.5$, the unique equilibrium is $x_1 = x_2 = 0.5$.

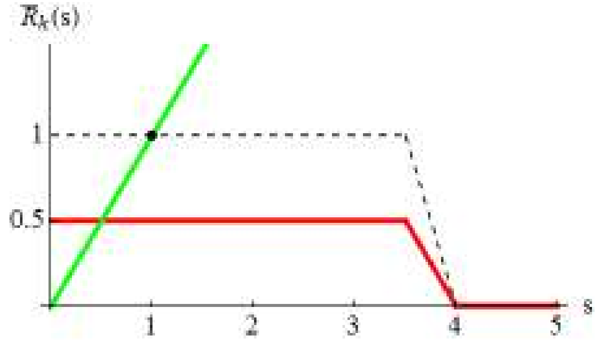

Figure 6. Equilibrium of Part (a) of Example 3

(b) In the case of Example 2, the best response $\bar{R}_k(s)$ is shown in Figure 7.

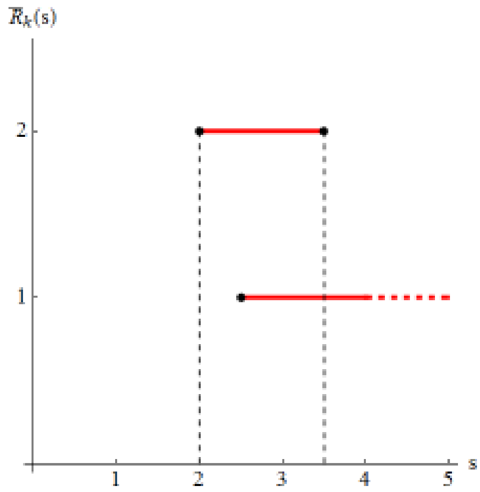

Figure 7. Best response $\bar{R}_k(s)$ in Part (b) of Example 3

If the same kind of firms form a duopoly, then

$$\bar{R}_1(s) + \bar{R}_2(s) = \begin{cases} 4 & \text{if } 2 \le s < 2.5, \\ \{2,3,4\} & \text{if } 2.5 \le s \le 3.5, \\ 2 & \text{if } s > 3.5. \end{cases}$$

Since $\bar{R}_k(s)$ is either 1 or 2 in the second case, the possible combinations for $\bar{R}_1(s)+\bar{R}_2(s)$ are $1+1$, $1+2$, $2+1$ and $2+2$. $H(s) = \bar{R}_1(s)+\bar{R}_2(s)$ is shown in Figure 8, from which it is clear that $s = 3$ is the unique solution. Notice that at $s = 3$, both $\bar{R}_k(3) = 1$ and $\bar{R}_2(3) = 2$ are feasible, so both

$x_1 = 1,\ x_2 = 2$ and $x_1 = 2,\ x_2 = 1$ are equilibria.

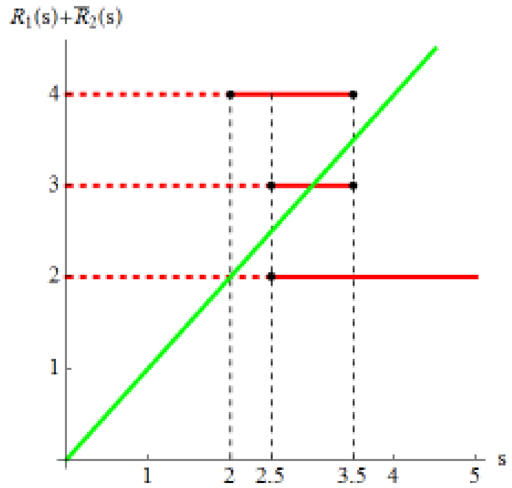

Figure 8. Equilibrium of Part (b) of Example 3

4 Conclusions

A modified N-firm single-product oligopoly without product differentiation was examined in which the payoff functions of the firms were discontinuous. The best responses of the firms were determined as functions of the outputs of the rest of the industry and then they were modified as functions of the total industry output. Based on these modified best responses the existence of at least one equilibrium was proved, and numerical examples showed the possibility of a unique as well as of multiple equilibria. The more general cases with nonlinear price and cost functions will be the subjects of our next research project.

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