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A two-sector Keynesian model of business cycles

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Abstract

In this paper, we examine the effect of sectoral interactions on business cycles in a simple Keynesian model. As a first step for introducing viewpoints of multiple sectors in the context of business cycles, we consider a dual economy in which there are only two kinds of goods: the consumption good and the investment good. By examining a two-sector Keynesian model, we intend to take a look at some phenomena induced by interactions between the consumption good sector and the investment good sector, which cannot be observed in one-sector models. We then find that the stability of equilibrium and the possibility of emergence of a periodic orbit depend upon whether the Keynesian stability condition holds or not and that the consumption good sector lags behind the investment good sector along the periodic orbit (business cycles). Also, we supplement the analysis by performing numerical simulations.

Keywords: Business cycles; Keynesian economics; Quantity adjustment; Two-sector analysis

JEL classification: E12; E32; E37

1 Introduction

It is no exaggeration to say that one of the main goals of macroeconomic studies is to explain the mechanism of business cycles. Soon after the basis of macroeconomics was established by Keynes' *General Theory*, a lot of theories of business cycles were proposed from the late 1930s to the 1950s. For example, Kalecki (1935, 1937) and Kaldor (1940) put forward models of business cycles by synthesizing the Keynesian multiplier theory and the profit principle of investment, while Harrod (1936), Samuelson (1939), Metzler (1941), Hicks (1950) and Goodwin (1951) initiated the so-called multiplier-accelerator model of business cycles by combining the multiplier theory and the acceleration principle of investment.[1] [2] These classic models of business cycles can be characterized by the following

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[1]Precisely speaking, Kalecki (1935) and Harrod (1936) preceded Keynes (1936), but they can be included in the classic works of the Keynesian theory of business cycles because their models were based upon the Keynesian principle of effective demand.

[2]The profit principle of investment states that investment demand is determined by the level of income (or of profit) and by the volume of capital stock, while the acceleration principle means that investment demand is determined by *changes* in the level of income. These two principles are often confused with each other, but they are different.

"Keynesian" features: (i) quantity or income adjustment governed by the principle of effective demand prevails:[3] (ii) variations in investment are the main source of business cycles.[4] [5] In their models, the mechanism of business cycles is explained as follows: investment is linked to aggregate income and capital stock and aggregate income and capital stock are varied through the multiplier process and capital formation induced by investment.

It is true that a lot of models of business cycles, including the aforementioned ones, can describe some aspects of actual business cycles, but a certain important viewpoint is missing in them: the role of sectoral interactions in business cycles. This aspect is lacking in one-sector or one-commodity models of business cycles, but it should not be ignored in discussing actual business cycles. It goes without saying that propagations of shocks from one industry to another do enhance economic fluctuations in reality. Unfortunately, there have been only a few attempts made to study the role of sectoral interactions in business cycles from theoretical points of view.[6] [7] To understand the mechanism of actual business cycles in depth, it is necessary to present a theoretical foundation for multi-sector analysis of business cycles.

The purpose of this paper is to examine the effect of sectoral interactions on business cycles in a simple Keynesian model. As a first step for introducing viewpoints of multiple sectors in the context of business cycles, we consider a dual economy in which there are only two kinds of goods: the consumption good and the investment good. By examining a two-sector Keynesian model, we intend to take a look at some phenomena induced by interactions between the consumption good sector and the investment good sector, which cannot be observed in one-sector models.

This paper is organized as follows. In Section 2, we set up a two-sector model by decomposing the economy into the consumption-good sector and the investment-good sector. This two-sector model describes the dynamics of the quantities of output and stocks of capital of the two sectors. In Section 3, we study the characteristics of the two-sector model. In particular, we see that the stability of equilibrium and the possibility of emergence of a periodic orbit depend upon whether the Keynesian stability condition holds or not and that the consumption good sector lags behind the investment good sector along the periodic orbit (business cycles). In Section 4, we perform numerical simulations to confirm the validity of our analysis. In Section 5, we summarize our analysis and conclude this paper.

[3]We mean by the term "quantity adjustment" that firms adjust the quantity of their products so that it would be equal to the demand for their products. This is what the principle of effective demand (Keynes, 1936, chap. 3) implies. The opposite concept is "price adjustment," which means that firms respond to excess demand or supply by raising or lowering the price of their products.

[4]Keynes (1936, chap. 22) attributed the main cause of business cycles to violent fluctuations of the marginal efficiency of investment (i.e., profitability of investment).

[5]Precisely speaking, Metzler (1941) emphasized variations in inventory investment rather than those in fixed investment as a cause of economic fluctuations.

[6]In the context of international trade, Lorenz (1987a, 1987b), Asada et al. (2001), Asada et al. (2003), Asada (2004) and Chiarella and Flaschel (2000) presented multi-country open-economy Keynesian models of business cycles. However, their concern was with interactions between countries, not with those between domestic sectors.

[7]In the field of economic growth, Okishio (1967, 1968) extended the Harrod-Domar model (e.g., Harrod 1939; Domar 1946) to two-sector ones, while Dutt (1987), Lavoie and Ramírez-Gastón (1997), Franke (2000) and Nishi (2014) extended the Kaleckian model (e.g., Kalecki 1971; Asimakopulos 1975; Rowthorn 1981) to two-sector ones, but they only focused on the stability or instability of economic growth and did not examine the possibility of business cycles. In the neoclassical two-sector model of economic growth (e.g., Shinkai 1960; Uzawa 1961-1962, 1963), on the other hand, Inada (1963) and Stiglitz (1967) examined the possibility of emergence of cyclical fluctuations, but their studies are based upon the full-employment assumption, and so cycles in their models cannot correspond to actual business cycles.

2 The model

In this section, we set up a two-sector Keynesian model of business cycles.

We suppose that there are only two kinds of goods that differ in property and name each of them the *consumption good* and the *investment good*, respectively.[8] The consumption good is supposed to be used for consumption alone and not for production, while the investment good is assumed to be used for production alone and not for consumption.[9] We also suppose for simplicity that the prices of the consumption good and of the investment good are fixed.[10]

First, we consider the demand for the consumption good. We assume that the demand for the consumption good C is dependent upon both the income of the consumption good and that of the investment good.[11] Specifically, C is assumed to be represented by[12]

$$C = C(y_c, y_i), \tag{1}$$

where C is twice continuously differentiable with

$$0 < C_{y_c} \equiv \frac{\partial C}{\partial y_c} < 1,\ 0 < C_{y_i} \equiv \frac{\partial C}{\partial y_i} < 1. \tag{2}$$

In (1), y_c and y_i stand for the income or output of the consumption good and that of the investment good, respectively. Condition (2) means that the marginal propensity to consume with respect to each sector's income C_{y_c} or C_{y_i} lies between 0 and 1.

Second, the output (supply) of the consumption good y_c is assumed to be varied in response to the existing

[8]Traditionally, the consumption good and the investment good have been interpreted as the output in the agricultural sector and that in the manufacturing sector, respectively, but in a modern way of thinking, the consumption good may be regarded as the output in the service sector.

[9]It is possible to extend our analysis to a more general situation in which the investment good is also enjoyed for consumption. For simplicity, however, we make the above assumption about the investment good.

[10]It is quantity or income adjustment governed by Keynes' (1936, chap. 3) principle of effective demand that is the most distinguished feature of the Keynesian theory. This point was stressed by Leijonhufvud (1968) as follows:

> In [neoclassical] general equilibrium flow models, *prices* are the only endogenous variables which enter as arguments into the demand and supply functions of individual households. Tastes and initial resource endowments are parametric. In "Keynesian" flow models the corresponding arguments are real income and the interest rate. Of these, real income is a measure of *quantity*, not of prices. On a highly abstract level, the fundamental distinction between general equilibrium and Keynesian models lies in the appearance of this quantity variable in the excess demand relation to the latter. The difference is due to the assumptions made about the adjustment behavior of the two systems. In the short run, the "[Neo]Classical" system adjusts to changes in money expenditures by means of price-level movements; the Keynesian adjusts primarily by way of real income movements. (p. 51)

In the Keynesian theory, quantity or income, rather than prices, plays the primary role in dynamic adjustment (see also Tobin (1993) or Yoshikawa (1984)). To focus on the importance of quantity adjustment, price adjustment is ignored in our analysis.

[11]Since the prices of the consumption good and of the investment good are assumed to be fixed, we can use the terms "income" and "output" interchangeably.

[12]If consumption demand can be assumed to depends upon aggregate income, the consumption function may be expressed as

$$C = C(y_c + y_i),$$

with

$$0 < C_y \equiv \frac{\partial C}{\partial (y_c + y_i)} < 1.$$

Equation (1) is, of course, a more general formulation than this.

excess demand or supply. Specifically, y_c is assumed to obey the following process:

$$\dot{y}_c = \alpha_c[C(y_c, y_i) - y_c], \tag{3}$$

where α_c is a positive parameter which stands for the speed of adjustment. It is assumed in (3) that the adjustment of y_c is proportional to the existing excess demand or supply, represented by $C(y_c, y_i) - y_c$. Equation (3) expresses the Keynesian quantity adjustment process in the consumption good sector.[13]

Third, we take a look at the demand side of the investment good. For each sector, the consumption good sector or the investment good sector, the demand for the investment good (including the replacement demand for the investment good) is assumed to be dependent upon the output of the sector and upon the existing stock of the investment good (i.e., the existing stock of capital) of the sector. Specifically, we assume that the gross investment functions of the consumption good sector I_c and of the investment good sector I_i are represented by

$$I_c = I_c(y_c, k_c), \tag{4}$$

$$I_i = I_i(y_i, k_i), \tag{5}$$

where I_c and I_i are twice continuously differentiable with

$$0 < I_{y_c} \equiv \frac{\partial I_c}{\partial y_c} < 1,\ I_{k_c} \equiv \frac{\partial I_c}{\partial k_c} < 0, \tag{6}$$

$$0 < I_{y_i} \equiv \frac{\partial I_i}{\partial y_i} < 1,\ I_{k_i} \equiv \frac{\partial I_i}{\partial k_i} < 0. \tag{7}$$

In (4) and (5), k_c and k_i denote the existing stocks of the investment good of the consumption good sector and of the investment good sector, respectively. Equations (4) and (5) with (6) and (7) are consistent with the "profit principle" of investment.[14] It is also assumed in (6) and (7) that the marginal propensity to invest with respect to each sector's income I_{y_c} or I_{y_i} lies between 0 and 1.[15] In this paper, we assume for simplicity that the rate of interest r is fixed by the monetary authority. By so doing, we can omit the dependence of I_c or of I_i on r.[16]

Fourth, we turn to the output (supply) of the investment good y_i. As in the consumption good sector, y_i is

[13] It is implicitly assumed that the existing excess demand or supply is absorbed as unintended inventory investment.

[14] Strictly speaking, the profit principle argues that the more the rate of profit is, the more demand for investment is, but, under the assumption of fixed price and wage, the (gross) rate of profit for each sector can be shown to be increasing in the level of output and decreasing in the stock of capital. The profit principle in this form was used first by Kalecki (1937) and Kaldor (1940). For discussions on microeconomic foundation of this principle in the context of fixed investment, see Murakami (2016a), and, for that in the context of R&D investment, see Murakami (2017b).

[15] The usual profit principle does not require that the marginal propensity to invest be less than unity, but this assumption helps to simplify our analysis, especially the stability analysis in Section 3.

[16] We may include the rate of interest r in the investment functions I_c and I_i, but if r can be represented as a function of y_c and y_i (as the Keynesian liquidity preference theory implies) and the interest elasticity of liquidity preference is sufficiently large (as in the Keynesian liquidity trap), our argument shall not dramatically be changed.

assumed to be changed in response to the existing excess demand or supply in the following way:

$$\dot{y}_i = \alpha_i[I_c(y_c, k_c) + I_i(y_i, k_i) - y_i], \tag{8}$$

where α_i is a positive parameter which represents the speed of adjustment. Equation (8) is, of course, the Keynesian quantity adjustment process in the investment good sector.

Fifth, we describe the capital formation process of each sector. The demand for the investment good of each sector I_c or I_i is assumed to be realized as the gross increment of the stock of the investment good in the sector k_c or k_i, respectively, in the following way:

$$\dot{k}_c = I_c(y_c, k_c) - \delta k_c, \tag{9}$$

$$\dot{k}_i = I_i(y_i, k_i) - \delta k_i, \tag{10}$$

where δ is a positive constant which stands for the rate of capital depreciation. In (9) and (10), it is assumed that all the existing excess demand (resp. supply) is absorbed as unintended inventory decumulation (resp. accumulation).[17]

Thus, our two-sector model can be completed as follows:

$$\dot{y}_c = \alpha_c[C(y_c, y_i) - y_c], \tag{3}$$

$$\dot{y}_i = \alpha_i[I_c(y_c, k_c) + I_i(y_i, k_i) - y_i], \tag{8}$$

$$\dot{k}_c = I_c(y_c, k_c) - \delta k_c, \tag{9}$$

$$\dot{k}_i = I_i(y_i, k_i) - \delta k_i. \tag{10}$$

In what follows, the system of equations (3), (8), (9) and (10) is simply called "System." The structure of System is similar to that of Kaldor's (1940) model,[18] but the notable difference from his is that our System consists of two sectors.

Since System deals with medium-run economic fluctuations but not with long-run economic growth, the levels of income and capital stocks of both sectors, y_c, y_i, k_c and k_i, can be assumed to be bounded.[19] For this reason,

[17] More generally, the demand for the investment good of the investment good sector may not be realized as net capital accumulation due to the existence of excess demand or supply. To describe this more general situation, it may be better to formalize the capital formation process in the investment good sector as follows:

$$\dot{k}_i = I_i(y_i, k_i) - \delta k_i - \theta[I_c(y_c, k_c) + I_i(y_i, k_i) - y_i],$$

where $\theta \in [0, 1]$. This kind of formalization can be found in, for instance, Stein (1969) or Fischer (1971). For the sake of simplicity in analysis, however, equation (10) is adopted in our analysis.

[18] For extensions of Kaldor's (1940) model of business cycles (in continuous-time formalizations), see, for example, Chang and Smyth (1971), Varian (1979), Semmler (1986, 1987), Asada (1987, 1995, 2004), Lorenz (1987), Skott (1989), Chiarella and Flaschel (2000), Asada et al. (2003), Chiarella et al. (2013) or Murakami (2014, 2015, 2016b).

[19] In reality, the levels of income and capital stocks of both sectors grow with some trend due to population growth and technical progress, which implies that they cannot be considered to be bounded. If our variables y_c, y_i, k_c and k_i can be viewed as cyclical components (or detrended versions) of their actual time series, however, we can, without so much difficulty, extend our analysis even in the context of long-run economic growth.

we can restrict the domain of the variables in System as follows:

$$D \equiv \{(y_c, y_i, k_c, k_i) \in \mathbb{R}^4_{++}: \ y_c \in [\underline{y}_c, \overline{y}_c], \ y_i \in [\underline{y}_i, \overline{y}_i], \ k_c \in [\underline{k}_c, \overline{k}_c], \ k_i \in [\underline{k}_i, \overline{k}_i]\},$$

where $\underline{x}_j$ and $\overline{x}_j$ are positive constants with $\underline{x}_j < \overline{x}_j$ for $x = y, k$ and for $j = c, i$. For $x = y, k$ and for $j = c, i$, we may regard $\overline{x}_j$ and $\underline{x}_j$ as the maximum and minimum values of x_j in the medium-run context. In this sense, the compact domain D can be viewed as the set of combinations of y_c, y_i, k_c and k_i economically feasible in the medium run.[20]

For convenience, we summarize the assumptions made thus far as follows.

Assumption 1. *The real valued functions C, I_c and I_i are twice continuously differentiable everywhere on D and the following conditions are satisfied everywhere on D:*[21]

$$0 < C_{y_c} < 1, \ 0 < C_{y_i} < 1, \tag{2}$$

$$0 < I_{y_c} < 1, \ I_{k_c} < 0, \tag{6}$$

$$0 < I_{y_i} < 1, \ I_{k_i} < 0. \tag{7}$$

3 Analysis

In this section, we analyze System, which consists of (3), (8), (9) and (10). First, we verify the existence and uniqueness of equilibrium in System. Second, we examine the stability of this equilibrium. Third, we explore the possibility of persistent business cycles being generated.

3.1 Existence and uniqueness of equilibrium

To begin, we define an equilibrium point of System. An equilibrium point of System is defined as a point at which we have $\dot{y}_c = \dot{y}_i = \dot{k}_c = \dot{k}_i = 0$. We can easily see that an equilibrium point of System, $(y_c, y_i, k_c, k_i) \in D$, is given as a solution of the following simultaneous equations:

$$y_c = C(y_c, y_i), \tag{11}$$

$$y_i = I_c(y_c, k_c) + I_i(y_i, k_i), \tag{12}$$

$$\delta k_c = I_c(y_c, k_c), \tag{13}$$

$$\delta k_i = I_i(y_i, k_i). \tag{14}$$

[20] For more detailed economic interpretations of a related compact domain, see Murakami (2014).

[21] The assumption of twice-continuous differentiability is made for the application of the Hopf bifurcation theorem in our analysis. For details on this theorem, see Marsden and McCracken (1976).

In order to establish the existence and uniqueness of an equilibrium point of System, we make the following assumptions.

Assumption 2. *The following conditions are satisfied:*

$$C(\overline{y}_c,\overline{y}_i)-\overline{y}_c<0<C(\underline{y}_c,\underline{y}_i)-\underline{y}_c, \tag{15}$$

$$I_c(\overline{y}_c,\underline{k}_c)+I_i(\overline{y}_i,\underline{k}_i)-\overline{y}_i<0<I_c(\underline{y}_c,\overline{k}_c)+I_i(\underline{y}_i,\overline{k}_i)-\underline{y}_i, \tag{16}$$

$$I_c(\overline{y}_c,\overline{k}_c)-\delta\overline{k}_c<0<I_c(\underline{y}_c,\underline{k}_c)-\delta\underline{k}_c, \tag{17}$$

$$I_i(\overline{y}_i,\overline{k}_i)-\delta\overline{k}_i<0<I_i(\underline{y}_i,\underline{k}_i)-\delta\underline{k}_i. \tag{18}$$

Assumption 3. *The following conditions is satisfied everywhere on D:*

$$(1-C_{y_c})I_{k_c}I_{k_i}>[(1-C_{y_c})(1-I_{y_i})I_{k_c}+(1-C_{y_c}-C_{y_i}I_{y_c})I_{k_i}]\delta-[(1-C_{y_c})(1-I_{y_i})-C_{y_i}I_{y_c}]\delta^2. \tag{19}$$

Regarding Assumption 2, conditions (15)-(18) may seem technical, but it is possible to provide economic interpretations for them under Assumption 1. Indeed, Assumption 1 and conditions (15)-(18) imply that for every $(y_c,y_i,k_c,k_i)\in D$

$$C(\overline{y}_c,y_i)-\overline{y}_c\leq C(\overline{y}_c,\overline{y}_i)-\overline{y}_c<0<C(\underline{y}_c,\underline{y}_i)-\underline{y}_c\leq C(\underline{y}_c,y_i)-\underline{y}_c, \tag{20}$$

$$I_c(y_c,k_c)+I_i(\overline{y}_i,k_i)-\overline{y}_i\leq I_c(\overline{y}_c,\underline{k}_c)+I_i(\overline{y}_i,\underline{k}_i)-\overline{y}_i<0<I_c(\underline{y}_c,\overline{k}_c)+I_i(\underline{y}_i,\overline{k}_i)-\underline{y}_i\leq I_c(y_c,k_c)+I_i(\underline{y}_i,k_i)-\underline{y}_i, \tag{21}$$

$$I_c(y_c,\overline{k}_c)-\delta\overline{k}_c\leq I_c(\overline{y}_c,\overline{k}_c)-\delta\overline{k}_c<0<I_c(\underline{y}_c,\underline{k}_c)-\delta\underline{k}_c\leq I_c(y_c,\underline{k}_c)-\delta\underline{k}_c, \tag{22}$$

$$I_i(y_i,\overline{k}_i)-\delta\overline{k}_i\leq I_i(\overline{y}_i,\overline{k}_i)-\delta\overline{k}_i<0<I_i(\underline{y}_i,\underline{k}_i)-\delta\underline{k}_i\leq I_i(y_i,\underline{k}_i)-\delta\underline{k}_i. \tag{23}$$

Condition (20) or (21) means that, when income of each sector y_c or y_i is sufficiently large (resp. sufficiently small) or at the maximum $\overline{y}_c$ or $\overline{y}_i$ (resp. at the minimum $\underline{y}_c$ or $\underline{y}_i$), this sector is in excess supply (resp. in excess demand); condition (22) or (23) means that, when capital stock of each sector k_c or k_i is sufficiently large (resp. sufficiently small) or at the maximum $\overline{k}_c$ or $\overline{k}_i$ (resp. at the minimum $\underline{k}_c$ or $\underline{k}_i$), net investment in this sector is negative (resp. positive). Conditions (20) and (21) may be consistent with the reality in that the propensity to consume and that to invest tend to decline as the level of income rises (cf. Keynes 1936, chap. 10), while conditions (22) and (23) can be seen as realistic because net investment demand may be considered to be positive (resp. negative) when the volume of capital stock is sufficiently small (resp. sufficiently large). Therefore, Assumption 2 can be justified from economic viewpoints. Moreover, mathematically, conditions (20)-(21) imply that the compact domain D is positively invariant with respect to System.[22] This property is helpful for verifying the existence of an equilibrium

[22]A (nonempty) closed domain is said to be positively invariant if every solution of the dynamical system under consideration which starts from this domain at $t=0$ will remain in the same domain for all $t\geq 0$.

point of System.[23]

As for Assumption 3, condition (19) is satisfied under Assumption 1 if the constant rate of capital depreciation δ is sufficiently small. Since δ is annually around 0.09 in reality (in Japan), condition (19) is highly likely to hold. In this respect, Assumption 3 is realistic.

Now we are ready to establish the existence and uniqueness of a solution path and of an equilibrium point of System.

Proposition 1. *Let Assumptions 1-3 hold.*

(i) For every initial condition $(y_c(0), y_i(0), k_c(0), k_i(0)) \in D$, *there exists a unique solution path of System,* $(y_c(t), y_i(t), k_c(t), k_i(t)) \in D$, *for all* $t \geq 0$.

(ii) There exists a unique equilibrium point of System, $(y_c^*, y_i^*, k_c^*, k_i^*) \in D^\circ$, *where* D° *denotes the interior of* D.

Proof. See Appendix A. □

3.2 Stability and instability

Next, we study the stability of System. For this purpose, we have a look at the Jacobian matrix, denoted by J^*, evaluated at the unique equilibrium:

$$J^* = \begin{pmatrix} \alpha_c(C_{y_c}^* - 1) & \alpha_c C_{y_i}^* & 0 & 0 \\ \alpha_i I_{y_c}^* & \alpha_i(I_{y_i}^* - 1) & I_{k_c}^* & I_{k_i}^* \\ I_{y_c}^* & 0 & I_{k_c}^* - \delta & 0 \\ 0 & I_{y_i}^* & 0 & I_{k_i}^* - \delta \end{pmatrix},$$

where $*$ signifies the value evaluated at the unique equilibrium point of System. The characteristic equation associated with J^* is given by

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0, \tag{24}$$

[23] Assumption 2 may be viewed as "boundary conditions" for existence of equilibrium in our two-sector Keynesian model. For boundary conditions in related one-sector Keynesian models, see, for instance, Murakami (2014, 2017a).

where

$$a_1 = (1 - C^*_{y_c})\alpha_c + (1 - I^*_{y_i})\alpha_i + 2\delta - I^*_{k_c} - I^*_{k_i} > 0, \tag{25}$$

$$\begin{aligned} a_2 &= [(1 - C^*_{y_c})(1 - I^*_{y_i}) - C^*_{y_i} I^*_{y_c}]\alpha_c\alpha_i + (1 - C^*_{y_c})(2\delta - I^*_{k_c} - I^*_{k_i})\alpha_c \\ &\quad + [(1 - I^*_{y_i})(2\delta - I^*_{k_c}) - I^*_{k_i}]\alpha_i + (\delta - I^*_{k_c})(\delta - I^*_{k_i}), \end{aligned} \tag{26}$$

$$\begin{aligned} a_3 &= \{2[(1 - C^*_{y_c})(1 - I^*_{y_i}) - C^*_{y_i} I^*_{y_c}]\delta - (1 - C^*_{y_c})(1 - I^*_{y_i})I^*_{k_c} - (1 - C^*_{y_c} - C^*_{y_i} I^*_{y_c})I^*_{k_i}\}\alpha_c\alpha_i \\ &\quad + (1 - C^*_{y_c})(\delta - I^*_{k_c})(\delta - I^*_{k_i})\alpha_c + [(1 - I^*_{y_i})\delta - I^*_{k_c}](\delta - I^*_{k_i})\alpha_i, \end{aligned} \tag{27}$$

$$\begin{aligned} a_4 &= \{(1 - C^*_{y_c})I^*_{k_c} I^*_{k_i} \\ &\quad - [(1 - C^*_{y_c})(1 - I^*_{y_i})I^*_{k_c} + (1 - C^*_{y_c} - C^*_{y_i} I^*_{y_c})I^*_{k_i}]\delta + [(1 - C^*_{y_c})(1 - I^*_{y_i}) - C^*_{y_i} I^*_{y_c}]\delta^2\}\alpha_c\alpha_i > 0. \end{aligned} \tag{28}$$

The inequalities follow from Assumptions 1 and 3.

For the unique equilibrium of System to possess the local asymptotic stability, it is necessary and sufficient that all the roots of the characteristic equation (24) have negative real parts. According to the Routh-Hurwitz criterion, the following set of conditions is necessary and sufficient:

$$a_1 > 0, \; a_3 > 0, \; a_4 > 0, \; (a_1 a_2 - a_3)a_3 > a_1^2 a_4,$$

which are, under Assumption 1, equivalent to[24]

$$\begin{aligned} &\{[(1 - C^*_{y_c})(1 - I^*_{y_i}) - C^*_{y_i} I^*_{y_c}]\delta - (1 - C^*_{y_c})(1 - I^*_{y_i})I^*_{k_c} - (1 - C^*_{y_c} - C^*_{y_i} I^*_{y_c})I^*_{k_i}\}\alpha_c\alpha_i \\ &+(1 - C^*_{y_c})(\delta - I^*_{k_c})(\delta - I^*_{k_i})\alpha_c + [(1 - I^*_{y_i})\delta - I^*_{k_c}](\delta - I^*_{k_i})\alpha_i > 0, \end{aligned} \tag{29}$$

$$\begin{aligned} &f(\alpha_c, \alpha_i)(\equiv (a_1 a_2 - a_3)a_3 - a_1^2 a_4) \\ &= [(1 - C^*_{y_c})(1 - I^*_{y_i}) - C^*_{y_i} I^*_{y_c}]\{[(1 - C^*_{y_c})(1 - I^*_{y_i}) - C^*_{y_i} I^*_{y_c}]\delta - (1 - C^*_{y_c})(1 - I^*_{y_i})I^*_{k_c} - (1 - C^*_{y_c} - C^*_{y_i} I^*_{y_c})I^*_{k_i}\} \\ &\times [(1 - C^*_{y_c})\alpha_c + (1 - I^*_{y_i})\alpha_i]\alpha_c^2\alpha_i^2 \\ &+ (1 - C^*_{y_c})^3(\delta - I^*_{k_c})(\delta - I^*_{k_i})(2\delta - I^*_{k_c} - I^*_{k_I})\alpha_c^3 \\ &+ (1 - I^*_{y_i})[(1 - I^*_{y_i})\delta - I^*_{k_c}][(1 - I^*_{y_i})(2\delta - I^*_{k_c}) - I^*_{k_i}](\delta - I^*_{k_i})\alpha_i^3 \\ &+ b_1\alpha_c^3\alpha_i + b_2\alpha_c\alpha_i^3 + b_3\alpha_c^2\alpha_i^2 + b_4\alpha_c^2\alpha_i + b_5\alpha_c\alpha_i^2 + b_6\alpha_c^2 + b_7\alpha_i^2 + b_8\alpha_c\alpha_i \\ &+ (\delta - I^*_{k_c})(\delta - I^*_{k_i})^2(2\delta - I^*_{k_c} - I^*_{k_i})\{(1 - C^*_{y_c})(\delta - I^*_{k_c})\alpha_c + [(1 - I^*_{y_i})\delta - I^*_{k_c}]\alpha_i\} > 0. \end{aligned} \tag{30}$$

To discuss the stability of System, we impose the following realistic assumption.

[24] The exact values of b_l for $l = 1, ..., 8$ are omitted because they are irrelevant to our analysis.

Assumption 4. *The following condition is satisfied:*

$$2[(1 - C^*_{y_c})(1 - I^*_{y_i}) - C^*_{y_i} I^*_{y_c}]\delta > (1 - C^*_{y_c})(1 - I^*_{y_i}) I^*_{k_c} + (1 - C^*_{y_c} - C^*_{y_i} I^*_{y_c}) I^*_{k_i}. \tag{31}$$

Condition (31) is not so restrictive under Assumption 1 because the right hand side is likely to be negative due to $C_{yc},\ C_{y_i},\ I_{y_c},\ I_{y_i} \in (0,1)$ and $I_{k_c},\ I_{k_i} < 0$ and the rate of capital depreciation δ is sufficiently small (around 0.09) in reality. Therefore, Assumption 4 can safely be made from realistic viewpoints. Note that condition (29) is satisfied under Assumptions 1 and 4.

As regards the local asymptotic stability, we can state the following fact.

Proposition 2. *Let Assumptions 1-4 hold.*

(i) The unique equilibrium point of System is locally asymptotically stable if α_c and α_i are sufficiently small.

(ii) The unique equilibrium point of System is locally asymptotically stable if α_c is sufficiently large and α_i is sufficiently small.

(iii) The unique equilibrium point of System is locally asymptotically stable if α_c is sufficiently small and α_i is sufficiently large.

(iv) Assume that the following condition is satisfied:

$$(1 - C^*_{y_c})(1 - I^*_{y_i}) > C^*_{y_i} I^*_{y_c}. \tag{32}$$

Then, the unique equilibrium point of System is locally asymptotically stable if α_c and α_i are sufficiently large.

(v) Assume that the following condition is satisfied:

$$(1 - C^*_{y_c})(1 - I^*_{y_i}) < C^*_{y_i} I^*_{y_c}. \tag{33}$$

Then, the unique equilibrium point of System is locally asymptotically unstable if α_c and α_i are sufficiently large.

Proof. See Appendix B. □

Proposition 2 says that the (local) stability of System is gained when the quantity adjustments in both sectors are relatively slow or when the quantity adjustment in one of the sectors is comparatively fast while that in the other is comparatively slow. This proposition also states that when the quantity adjustments in both sectors are rapid enough, the (local asymptotic) stability of System rests upon whether condition (32) holds or not and the stability is lost if the reverse condition (33) is satisfied. Since the situation in which both α_c and α_i are large may be taken as the Keynesian case (cf. Leijonhufvud 1968; Tobin 1993; Yoshikawa 1984), we may state that conditions (32) and (33) are, respectively, the stability condition and the instability one in our Keynesian model.

We can find that our stability results are related to the well-established Keynesian stability condition. As is well known, the Keynesian stability condition, which states that the sum of the marginal propensity to consume and that to invest is less than unity, plays a vital role in stability in one-sector Keynesian models.[25] In our two-sector model, the Keynesian stability condition (at the equilibrium) may be written for each sector as follows:

$$C^*_{y_c} + I^*_{y_c} < 1, \tag{34}$$

$$C^*_{y_i} + I^*_{y_i} < 1. \tag{35}$$

It is seen that condition (32) holds under Assumption 1 and these conditions. Then, it follows from Proposition 2 (iv) that the local asymptotic stability of System obtains for α_c and α_i sufficiently large. In other words, the "two-sector" version of Keynesian stability condition, (34) and (35), can assure the stability in our two-sector System if the quantity adjustment is rapid enough in each sector. On the contrary, if the reverse inequalities hold both in (34) and in (35), condition (33) is satisfied and the instability arises for α_c and α_i sufficiently large in our two-sector System (cf. Proposition 2 (v)). Recently, some economists have cast doubt on the validity of the Keynesian stability condition and maintained that the reverse condition is realistic.[26] Thus, chances are not low that, in practice, the reverse inequalities may be fulfilled both in (34) and in (35) and that condition (33) holds. In this way, the (counterpart of) Keynesian stability condition also plays a key role in the stability in our two-sector Keynesian model.

3.3 Existence of a periodic orbit

Now we examine the phenomenon of the unique equilibrium of System turning from stable to unstable (or from unstable to stable), i.e., that of "stability switching" occurring to System, through changes in the parameters α_c and α_i.

It follows from the above argument that the stability switching occurs only if the following condition is satisfied:[27]

$$f(\alpha_c, \alpha_i) = 0. \tag{36}$$

Furthermore, according to Asada and Yoshida (2003, p. 527, Theorem 3), a non-constant periodic orbit is generated by a Hopf bifurcation if either of the following conditions as well as (36) holds:

$$f_{\alpha_c}(\alpha_c, \alpha_i) \neq 0, \tag{37}$$

$$f_{\alpha_i}(\alpha_c, \alpha_i) \neq 0. \tag{38}$$

[25] The term of "Keynesian stability condition" was coined by Marglin and Bhaduri (1990). To our best knowledge, Samuelson (1941, p. 117) was the first economist who found that this condition ensures the (local asymptotic) stability in the Keynesian system.

[26] For instance, Skott (2012) judged from his empirical study that the Keynesian stability condition is far from realistic.

[27] Due to (28), the characteristic equation (24) cannot possess any zero root.

To discuss the possibility of Hopf bifurcations, we make the following assumption.

Assumption 5. *The following condition is satisfied:*

$$(1 - C^*_{y_c})(1 - I^*_{y_i}) < C^*_{y_i} I^*_{y_c}. \tag{33}$$

According to Proposition 2 (v), condition (33) is a sufficient condition for the *instability* of System for α_c and α_i sufficiently large. As we have stated above, the validity of (33) may be verified on empirical grounds. In Section 4 and Appendix E, we shall see that condition (33) was empirically satisfied in Japan. In this respect, Assumption 5 can be justified.

Concerning the existence of a periodic orbit, we can establish the following fact.

Proposition 3. *Let Assumptions 1-5 hold.*

(i) Assume that α_i is fixed at some sufficiently large value. Then, there exists at least one positive value α^*_c such that the unique equilibrium of System is locally asymptotically stable (resp. unstable) for $\alpha_c < \alpha^*_c$ (resp. $\alpha_c > \alpha^*_c$) and that at least one (non-constant) periodic orbit is generated by a Hopf bifurcation for α_c sufficiently close to α^*_c.

(ii) Assume that α_c is fixed at some sufficiently large value. Then, there exists at least one positive value α^*_i such that the unique equilibrium of System is locally asymptotically stable (resp. unstable) for $\alpha_i < \alpha^*_i$ (resp. $\alpha_i > \alpha^*_i$) and that at least one (non-constant) periodic orbit is generated by a Hopf bifurcation for α_i sufficiently close to α^*_i.

Proof. See Appendix C. □

Proposition 3 states that there can be a periodic orbit generated by Hopf bifurcations through changes in α_c or in α_i. Note that it does not tell us about the stability of the periodic orbit. As is well known, there is a criterion to discern whether a Hopf periodic orbit is stable or not, i.e., whether a Hopf bifurcation is supercritical or subcritical (cf. Marsden and McCracken 1976, sect. 4). However, it is usually hard to derive economic implications from this criterion because it requires third order partial derivatives of the relevant functions, which usually do not have economic meanings, and so we do not analytically investigate the periodic stability.[28]

3.4 Characteristics of the Hopf periodic orbit

We investigate the properties of a periodic orbit generated by the aforementioned Hopf bifurcation, which are simply called a "Hopf periodic orbit" in what follows.

To begin, we calculate the period of a Hopf periodic orbit. When a Hopf bifurcation occurs to System, condition

[28]There have been some attempts made to derive the condition for stability of a Hopf periodic orbit in economics. For example, Lorenz (1993), Matsumoto (2009) and Dohtani (2010) provided periodic stability conditions in their systems of two-dimensional differential equations.

(36) is fulfilled and the characteristic equation (24) has a pair of purely imaginary root $\pm i\omega$ with

$$\begin{aligned}\omega &= \sqrt{\frac{a_3}{a_1}}\\ &= \sqrt{\frac{\{2[(1-C^*_{y_c})(1-I^*_{y_i})-C^*_{y_i}I^*_{y_c}]\delta-(1-C^*_{y_c})(1-I^*_{y_i})I^*_{k_c}-(1-C^*_{y_c}-C^*_{y_i}I^*_{y_c})I^*_{k_i}\}\alpha_c\alpha_i+(1-C^*_{y_c})(\delta-I^*_{k_c})(\delta-I^*_{k_i})\alpha_c+[(1-I^*_{y_i})\delta-I^*_{k_c}](\delta-I^*_{k_i})\alpha_i}{(1-C^*_{y_c})\alpha_c+(1-I^*_{y_i})\alpha_i+2\delta-I^*_{k_c}-I^*_{k_i}}}.\end{aligned}$$

According to the Hopf bifurcation theorem (cf. Marsden and McCracken 1976, sect. 5), the period of a Hopf periodic orbit can be approximated by

$$\begin{aligned}T &= \frac{2\pi}{\omega}\\ &= 2\pi\sqrt{\frac{(1-C^*_{y_c})\alpha_c+(1-I^*_{y_i})\alpha_i+2\delta-I^*_{k_c}-I^*_{k_i}}{\{2[(1-C^*_{y_c})(1-I^*_{y_i})-C^*_{y_i}I^*_{y_c}]\delta-(1-C^*_{y_c})(1-I^*_{y_i})I^*_{k_c}-(1-C^*_{y_c}-C^*_{y_i}I^*_{y_c})I^*_{k_i}\}\alpha_c\alpha_i+(1-C^*_{y_c})(\delta-I^*_{k_c})(\delta-I^*_{k_i})\alpha_c+[(1-I^*_{y_i})\delta-I^*_{k_c}](\delta-I^*_{k_i})\alpha_i}}.\end{aligned} \tag{39}$$

Next, we examine the movements of the variables y_c, y_i, k_c and k_i along Hopf periodic orbits. A simple way to look at what happens along Hopf periodic orbits is to draw phase diagrams. In figure 1, the phase diagram of System is illustrated on the y_c-y_i plane with $k_c = k_c^*$ and $k_i = k_i^*$ (it is drawn under Assumptions 1-5). By applying the conclusion of the Poincarè-Hopf theorem (cf. Milnor 1965, p. 35), we can find that the point (y_c^*, y_i^*) is, under Assumption 5, a saddle point on the y_c-y_i plane with $k_c = k_c^*$ and $k_i = k_i^*$ and that the loci of $\dot{y}_c = 0$ and of $\dot{y}_i = 0$ intersect at least two points other than (y_c^*, y_i^*) on this plane (note that, letting the other point be, say, (y_c^{**}, y_i^{**}), the point $(y_c^{**}, y_i^{**}, k_c^*, k_i^*)$ is not an equilibrium point of the four-dimensional System).[29] Since (y_c^*, y_i^*) is a saddle stationary point, however, we cannot directly gain from this simple method some useful information on the movements of y_c and of y_i along Hopf periodic orbits.

[29]For $k_c = k_c^*$ and $k_i = k_i^*$, the domain $D^* \equiv \{(y_c, y_i) \in \mathbb{R}^2_{++} :\ y_c \in [\underline{y}_c, \overline{y}_c],\ y_i \in [\underline{y}_i, \overline{y}_i]\}$ is positively invariant with respect to the subsystem of (3) and (8). By the same logic as used in the proof of Proposition 1 (cf. Appendix A), we can see from the Poincarè-Hopf theorem that the sum of the indices of all the stationary points of (3) and (8) is +1 on D^*. Since the index of (y_c^*, y_i^*) is -1 because it is a saddle point on the y_c-y_i plane, we can conclude that at least other two stationary points exist on D^*.

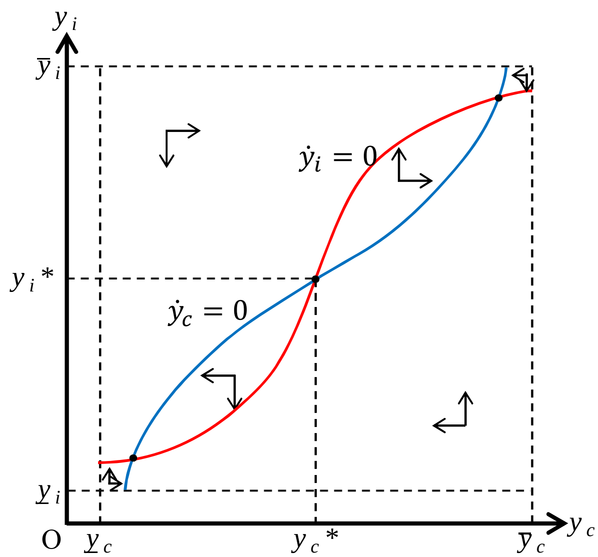

Figure 1: Phase diagram on y_c-y_i plane

Another way to get information on the properties of Hopf periodic orbits is to make use of the linearized version of the original System. The linearized version of System, which is linearized around the unique equilibrium point $(y_c^*, y_i^*, k_c^*, k_i^*)$, can be written in the following form:

$$\dot{y}_c = \alpha_c[(C^*_{y_c} - 1)(y_c - y_c^*) + C^*_{y_i}(y_i - y_i^*)], \tag{40}$$

$$\dot{y}_i = \alpha_i[I^*_{y_c}(y_c - y_c^*) + (I^*_{y_i} - 1)(y_i - y_i^*) + I^*_{k_c}(k_c - k_c^*) + I^*_{k_i}(y_i - y_i^*)], \tag{41}$$

$$\dot{k}_c = I^*_{y_c}(y_c - y_c^*) + (I^*_{k_c} - \delta)(k_c - k_c^*), \tag{42}$$

$$\dot{k}_i = I^*_{y_i}(y_i - y_i^*) + (I^*_{k_i} - \delta)(k_i - k_i^*). \tag{43}$$

A Hopf bifurcation is a *local* phenomenon in our analysis, a Hopf periodic orbit of System can be approximated by the solution path of the system of (40)-(43) with (36). In what follows, we denote the system of equations (40)-(43) with (36) by “System (L).”

A (purely) periodic solution of System (L) can be written in the following form:[30]

$$y_c = y_c^* + a_{y_c} \sin\Big(\frac{2\pi(t-\theta_{y_c})}{T}\Big), \tag{44}$$

$$y_i = y_i^* + a_{y_i} \sin\Big(\frac{2\pi(t-\theta_{y_i})}{T}\Big), \tag{45}$$

$$k_c = k_c^* + a_{k_c} \sin\Big(\frac{2\pi(t-\theta_{k_c})}{T}\Big), \tag{46}$$

$$k_i = k_i^* + a_{k_i} \sin\Big(\frac{2\pi(t-\theta_{k_i})}{T}\Big), \tag{47}$$

where T is the constant number defined by (39); a_{x_j} is a positive constant (amplitude) and θ_{x_j} is a real constant with $\theta_{x_j} \in [-T/2, T/2]$, for $x = y, k$ and $j = c, i$, where a_{x_j} and θ_{x_j} are dependent upon an initial condition.

To study the movements of variables along Hopf periodic orbits, we have a closer look at the phase differences (i.e., the differences in θ_{x_j}). By looking into the phase differences, we can gain helpful information about the lead and lag between the four variables y_c, y_i, k_c and k_i along Hopf periodic orbits. If we have $\theta_i < \theta_c$, for instance, we may say from figure 2 that y_i reaches its peak earlier than y_c does along the periodic orbit of System (L), i.e., that the movement of y_i precedes that of y_c (or that the latter lags behind the former). Then, we can approximately make a judgment on the lead-and-lag relationship between variables along Hopf periodic orbits of (the original) System.

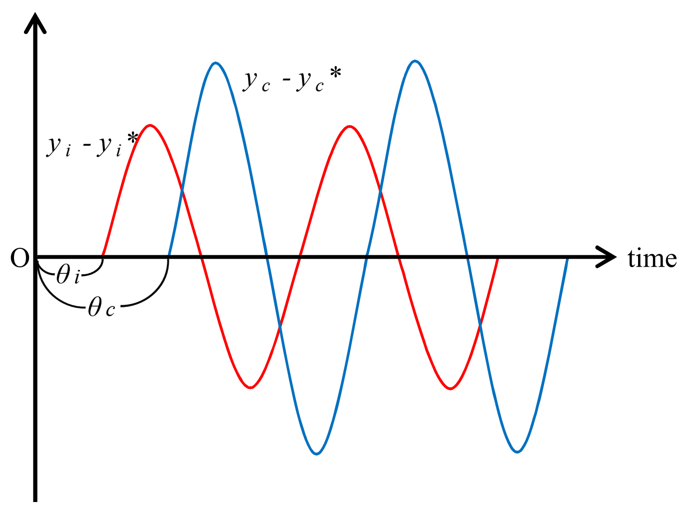

Figure 2: Phase difference

Concerning the phase differences, we can obtain the following proposition.

[30]The general solution of System (L), x_j, for $x = y, k$ and $j = c, i$, can be written in the form of

$$x_j = x_j^* + a_{x_j} \sin\Big(\frac{2\pi(t-\theta_{x_j})}{T}\Big) + b_{x_j} e^{\lambda_1 t} + c_{x_j} e^{\lambda_2 t},$$

where λ_1 and λ_2 are the negative roots of (24); a_{x_j}, b_{x_j} and c_{x_j} are constants that depend upon an initial condition. The pure periodic solution can be obtained if the initial condition satisfied $b_{x_j} = c_{x_j} = 0$. Note however that, due to $\lambda_1 < 0$ and $\lambda_2 < 0$, the above general solution converges to a periodic orbit with the passage of time.

Proposition 4. *Let Assumptions 1-5 hold. Let a (purely) periodic solution path of System (L) be represented by (44)-(47).*

(i) Assume that

$$\alpha_c(1 - C^*_{y_c}) + I^*_{k_i} - \delta \leq 0. \tag{48}$$

Then, the following condition holds:

$$\theta_{y_i} < \theta_{k_i} \leq \theta_{y_c} < \theta_{k_c}. \tag{49}$$

(ii) Assume that

$$\alpha_c(1 - C^*_{y_c}) + I^*_{k_i} - \delta > 0. \tag{50}$$

Then, either of the following conditions holds:

$$\theta_{y_i} < \theta_{y_c} < \theta_{k_c} \leq \theta_{k_i}, \tag{51}$$

$$\theta_{y_i} < \theta_{y_c} < \theta_{k_i} < \theta_{k_c}. \tag{52}$$

Proof. See Appendix D. □

We can find from Proposition 4 that the relationships $\theta_{y_i} < \theta_{y_c}$, $\theta_{y_c} < \theta_{k_c}$ and $\theta_{y_i} < \theta_{k_i}$ always hold but that which of θ_{y_c} and θ_{k_i} is greater is conditional.[31] It can thus be said that the ups and downs of y_i firstly occurs and those of y_c and k_i follow them (those of y_c are followed by those of k_c.) along (purely) periodic orbits of System (L). In particular, if condition (48) holds, the ups and downs of k_i precede those of y_c, while otherwise (if condition (50) is fulfilled), the reverse is true. Since Hopf periodic orbits of System can be approximated by periodic orbits of System (L), we can infer that, along Hopf periodic orbits of System, the movements of y_c and k_i lag behind that of y_i and which of y_c and k_i precedes the other depends upon which of conditions (48) and (50) holds. Moreover, the lead and lag between the four variables may imply that variations in the output of the investment good sector play a leading role in business cycles and that the consumption good sector passively responds to changes in the investment good sector. These lead and lag relationships reflect not only the Keynesian multiplier process but also the Keynesian view that what causes economic fluctuations is volatility of investment demand.

[31] It is also conditional which of θ_{k_c} and θ_{k_i} is larger. To avoid complexity in calculation, we do not dwell on the conditions upon which θ_{k_c} is larger than θ_{k_i} and which the reverse is true.

4 Numerical analysis

We proceed to perform some numerical simulations to check if the conclusions drawn from our analysis are valid. For this purpose, we need information about the consumption and investment functions.

4.1 Specification

To begin, we specify the investment functions of the consumption good and investment good sectors, I_c and I_i based upon the empirical data in Japan. We first estimate the gross investment functions of both sectors and then allow for depreciation of capital. To do so, we make the simplifying assumption that the shape of the gross capital accumulation function, which relates the rate of gross capital accumulation (i.e., the ratio of gross investment to capital stock) to the ratio of gross output to capital stock, is common for both sectors. Specifically, we assume that the gross capital accumulation functions are formalized as follows:

$$\frac{I_c}{k_c} = g\Big(\frac{y_c}{k_c}\Big), \tag{53}$$

$$\frac{I_i}{k_i} = g\Big(\frac{y_i}{k_i}\Big), \tag{54}$$

where g is the common gross capital accumulation function. By adopting this simplifying assumption, we give up formalizing the investment function for each sector. The reason for choosing this approach is that it is difficult to obtain separately information about the consumption good and investment good sectors. Indeed, to our best knowledge, there have been only a few attempts made to divide aggregate date into these two sectors (cf. Kuga 1967; Takahashi, et al. 2012). By making use of input-output tables, Kuga (1967) and Takahashi et al. (2012) estimated the volume of capital stock and the number of employed workers for each of the sectors, but because of the lack of reliable input-output tables, however, they were only able to obtain these data in every five years.[32] Since we have difficulty in getting input-output tables for every year and we want to estimate the investment functions for annual investment behaviors, we make the estimation of the investment function for each sector by assuming that both of these functions are common.

To formalize the gross capital accumulation function, we suppose that there are (only) two types of firms in terms of attitudes towards investment: "optimists" and "pessimists."[33] Optimists and pessimists are assumed to carry out the "optimistic" and "pessimistic" plans of investment, respectively, where the optimistic and pessimistic plans of investment are to spend $100\overline{g}$% and $100\underline{g}$%, respectively, of the (evaluated) value of capital stock on new (gross) investment with $\underline{g} < \overline{g}$. In other words, all firms are assumed to choose the binary options on investment plans, one of which realizes a higher rate of capital accumulation $\overline{g}$ and the other of which a lower one $\underline{g}$. Under

[32] In Japan, in particular, input-output tables are issued in every five years. What is worse, these tables may not be comparable with each other due to changes in details.

[33] This kind of assumption was made in Lux (1995) or Brock and Hommes (1997) to describe market sentiments. Also, Franke (2012, 2014) made use of Lux's (1995) framework to model the transition dynamics of "animal spirits" (cf. Keynes 1936, chap. 12) in the context of business cycles. Our model is the same in spirit as Franke's (2012, 2014), but, of course, the details are different.

this hypothetical setting, the rates of gross capital accumulation are the highest $\overline{g}$ (resp. the lowest $\underline{g}$), when all firms are optimists (resp. pessimists). Our binary-choice assumption may seem artificial but it can be justified in terms of aggregate analysis. Since the number of firms is almost infinite in reality, each firm's behavior cannot directly be observed. In this sense, aggregate data as macroeconomic or collective behavior should be interpreted in terms of "distributions."[34] The binary-choice treatment is one of the simplest ways to describe a macroeconomic phenomenon as a distribution.[35] Binary-choice settings may seem to describe only a few limited cases of phenomena due to the number of options, but this is not correct. Indeed, in our setting, if $\overline{g}$ and $\underline{g}$ are regarded as the maximum and minimum values of rate of gross capital accumulation, every number $g \in [\underline{g}, \overline{g}]$ can be realized as a rate of gross capital accumulation when the share of optimists among all firms is $(g - \underline{g})/(\overline{g} - \underline{g})$. We may thus argue that the above binary-option assumption is not inappropriate as a treatment in our analysis.

Now we proceed to give a specific shape to the gross capital accumulation function g. On the basis of the above argument, we set g in the following form:

$$g = p\overline{g} + (1 - p)\underline{g} \tag{55}$$

where p stands for the probability that a firm is an optimist or the share of optimists. Since all firms are assumed to choose the binary alternatives, optimists and pessimists, and the current economic condition, especially the current (gross) output-capital ratio y/k, can be considered to affect positively the share (probability) of optimists p, we can establish the following relationship:

$$\ln\Big(\frac{p}{1-p}\Big) = \beta_0 + \beta_u \frac{y}{k}.$$

where β_u is a positive constant. This is nothing but a logistic formulation of the probability p. Making use of (55), we obtain

$$\ln\Big(\frac{g - \underline{g}}{\overline{g} - g}\Big) = \beta_0 + \beta_u \frac{y}{k}. \tag{56}$$

By setting $\overline{g}$ and $\underline{g}$, we can estimate the parameters β_u and β_0 in (56). According to Appendix E, the rate of gross capital accumulation was annually around 0.3 at most (in Japan during 1970-2014). Therefore, we may set the maximum value of g or $\overline{g}$ as 0.3. Also, since we know from Appendix E that the estimated constant term is negative at the 1% confidence level in naive ordinary least squares relating g to y/k, we may conjecture that the minimum value of g or $\underline{g}$ is 0, which is required by the definition of the rate of gross capital accumulation, and so $\underline{g}$ is set as

[34] Aoki and Yoshikawa (2007, chaps. 3 and 4) and Yoshikawa (2015) emphasized distributions as collective behavior of heterogeneous agents or of the macro economy and introduced the concept of "stochastic macro-equilibrium" borrowing concepts from statistical physics.

[35] Income distributions among workers and capitalists are one of the classic and typical examples of distributions of binary options (classes). In the two-class income distributions, the share of wage income or that of profit income determines the macroeconomic distribution.

0. It thus follows from (56) that we have

$$\ln\Big(\frac{g}{0.3-g}\Big) = \beta_0 + \beta_u \frac{y}{k}. \tag{57}$$

From the results of regression analysis of (57) in Appendix E, we know that the estimated values of β_u and β_0 are $\hat{\beta}_u = 9.38$ and $\hat{\beta}_0 = -5.81$ and that they are significant at the 0.1% confidence level. Hence, we can set $\beta_u = 9.4$ and $\beta_0 = -5.8$. We can thus formalize the gross capital accumulation function g as

$$g = \frac{0.3}{1+\exp(5.8-9.4(y/k))}.$$

Utilizing (53) and (54), we can obtain the following investment functions of both sectors:

$$I_c = \frac{0.3k_c}{1+\exp(5.8-9.4(y_c/k_c))}, \tag{58}$$
$$I_i = \frac{0.3k_i}{1+\exp(5.8-9.4(y_i/k_i))}, \tag{59}$$

Note that the investment functions (58) and (59) have sigmoid shapes with respect to the level of output as in Kaldor's (1940) model of business cycles.

Next we specify the consumption function C on the basis of empirical data in Japan. For simplicity, we assume that C is expressed in the following form:

$$C = \alpha_0 + \alpha_y Y \tag{60}$$

where Y is the total (national) income, which is equal to the sum of y_c and y_i. Note that equation (60) satisfies Assumption 1 if we have $\alpha_y \in (0,1)$.

Since our analysis only deals with business cycles and not with economic growth, we should remove the trend component from raw data of national income and consumption expenditure. The standard way to do this is to make use of the Hodrick-Prescott filter suggested by Hodrick and Prescott (1997) or the band-pass filter proposed by Christiano and Fitzgerald (2003), which is more sophisticated than the Hodrick-Prescott one. Employing the band-pass filter, the resulting de-trended data will dramatically be changed if a different band of frequencies is chosen. So we adopt another simpler method. In our analysis, we get rid of the trend component from data by replacing aggregate income Y and consumption expenditure C with per-capita income and per-capita consumption both measured in the efficiency unit, respectively. Letting N and A be the number of population and the total factor productivity, respectively, we define per-capita income y and per-capita consumption measured in the efficiency unit as $y = Y/AN,\ c = C/AN$. According to the results of simple ordinary least squares relating c to y given in Appendix E, we can obtain the estimated values of the marginal propensity to consume $\hat{\alpha}_y = 0.532$ and of the fundamental consumption $\hat{\alpha}_0$ and these values are significant at the 0.1% confidence level. Thus, we may set $\alpha_y = 0.53$. Also, since the investment functions I_c and I_i, given in (53) and (54), are both homogeneous of degree one in the levels of output and capital stock and the consumption function is given in the form of (60), the equilibrium values of y_c, y_i, k_c and k_i are homogeneous of degree one in that of α_0. Hence, we may normalize α_0 as $\alpha_0 = 1$. Therefore, we may make the following formulation of C:

$$C = 1 + 0.53(y_c + y_i). \tag{61}$$

Finally, we determine the value of the rate of capital depreciation δ. Empirically, according to Appendix E, the average rate of capital depreciation, denoted by $\hat{\delta}$, was about 0.0873 in Japan (in the period of 1970-2014). Then, we may set $\delta = 0.087$.

By substituting (58), (59) and (61) in System, we can obtain the following system:

$$\dot{y}_c = \alpha_c[1 + 0.53(y_c + y_i) - y_c], \tag{62}$$

$$\dot{y}_i = \alpha_i\Big[\frac{0.3k_c}{1 + \exp(5.8 - 9.4(y_c/k_c))} + \frac{0.3k_i}{1 + \exp(5.8 - 9.4(y_i/k_i))} - y_i\Big], \tag{63}$$

$$\dot{k}_c = \frac{0.3k_c}{1 + \exp(5.8 - 9.4(y_c/k_c))} - 0.087k_c, \tag{64}$$

$$\dot{k}_i = \frac{0.3k_i}{1 + \exp(5.8 - 9.4(y_i/k_i))} - 0.087k_i. \tag{65}$$

In what follows, the system of equations (62)-(65) is redefined as "System."

In System, there is a unique equilibrium point $(y_c^*, y_i^*, k_c^*, k_i^*)$ represented as follows:

$$(y_c^*, y_i^*, k_c^*, k_i^*) = (2.748, 0.550, 5.266, 1.054) \tag{66}$$

It is not difficult to verify that all the conditions in Assumptions 1 and 3 are satisfied at least in a neighborhood of the unique equilibrium point given by (66) and that Assumptions 4 and 5 are fulfilled. Note that, since the existence and uniqueness of an equilibrium point is already established numerically, Assumption 2, which is made for the existence, is unnecessary in our numerical analysis.[36]

4.2 Existence of a Hopf periodic orbit

We discuss periodic orbits generated by Hopf bifurcations in System. To do so, we first explore for what values of α_c and α_i periodic orbits are generated by Hopf bifurcations. We have already known from Section 3 that Hopf bifurcations occur to System if we have (36). Under the specifications of (62)-(66), the curve of (36) can be illustrated in the following figure. Along the curve in figure 3, System undergoes Hopf bifurcations. That is, for

[36] It follows that we do not need to dwell upon the domain D of System.

every (α_c, α_i) on the curve, a Hopf bifurcation happens to System. In what follows, the curve is denoted by the "stability switching curve." Note that the unique equilibrium point is locally asymptotically stable (resp. unstable) if (α_c, α_i) is located on the left (resp. right) side of the stability switching curve.

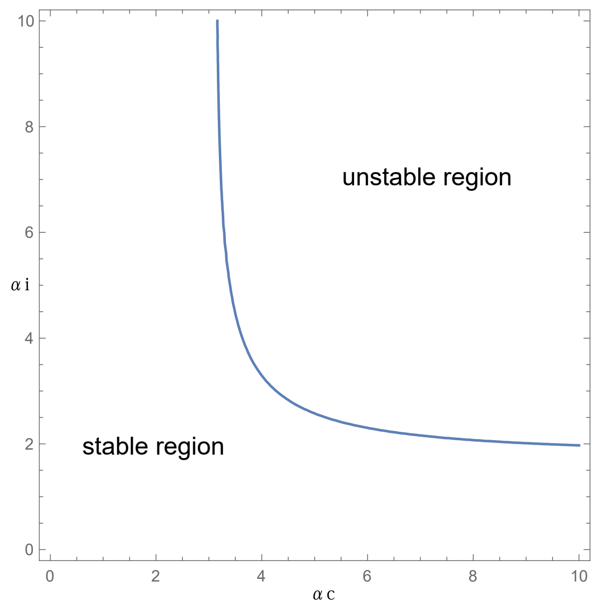

Figure 3: Stability switching curve

To see if a periodic orbit is actually generated by a Hopf bifurcation, we set the values of (α_c, α_i) sufficiently near the stability curve and perform a numerical simulation in of System. For instance, $(\alpha_c, \alpha_i) = (4, 3.292)$ is on the stability switching curve. Then, for our simulation, we set

$$(\alpha_c, \alpha_i) = (4, 3.293). \tag{67}$$

Finally, we set the initial condition for our simulation as follows:

$$(y_c(0), y_i(0), k_c(0), k_i(0)) = 2.748, 0.6, 5.266, 1.054) \tag{68}$$

We are now ready to conduct a numerical simulation of System. In figure 4 described is the solution path of System $(y_c(t), y_i(t), k_c(t), k_i(t))$ with (67) and (68). We can see from figure 4 that the solution path of System is periodic with (67). We can thus confirm the validity of Proposition 3.

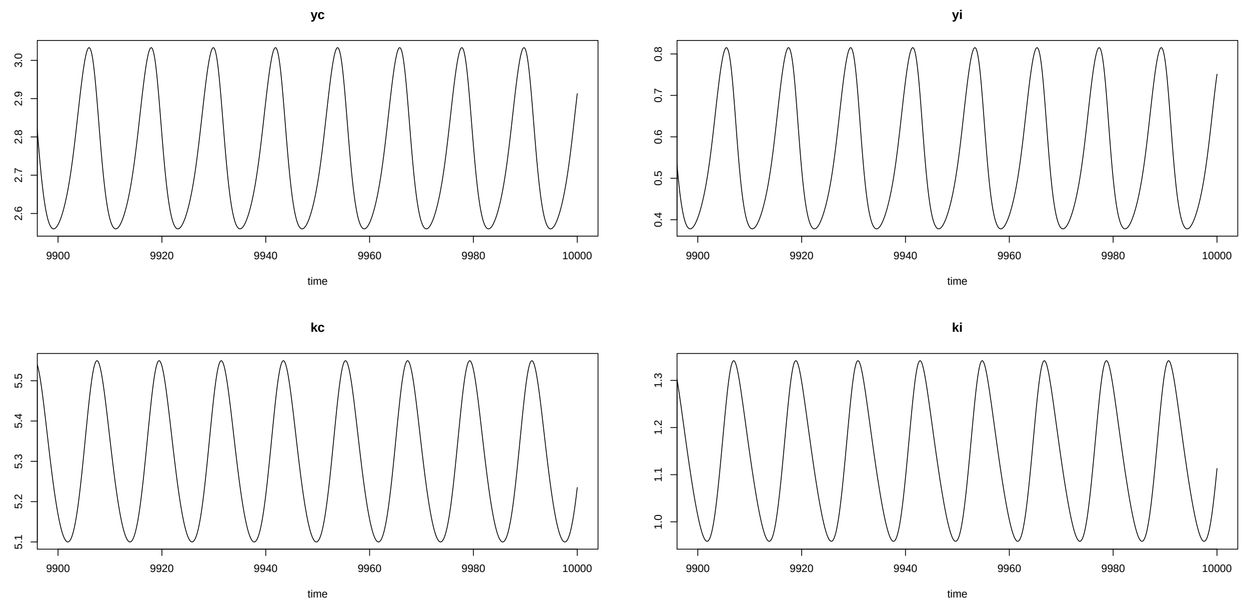

Figure 4: Periodic orbit

In Figure 5 depicted are the time paths of $y_c - y_c^*,\ y_i - y_i^*,\ k_c - k_c^*$ and $k_i - k_i^*$ of System with (67) and (68). This figure illustrates the lead and lag relationships between $y_c,\ y_i,\ k_c$ and k_i. We can find from figure 5 that the ups and downs of y_i come first and then those of $y_c,\ k_i$ and k_c follow in order. This consequence is consistent with Proposition 4.[37]

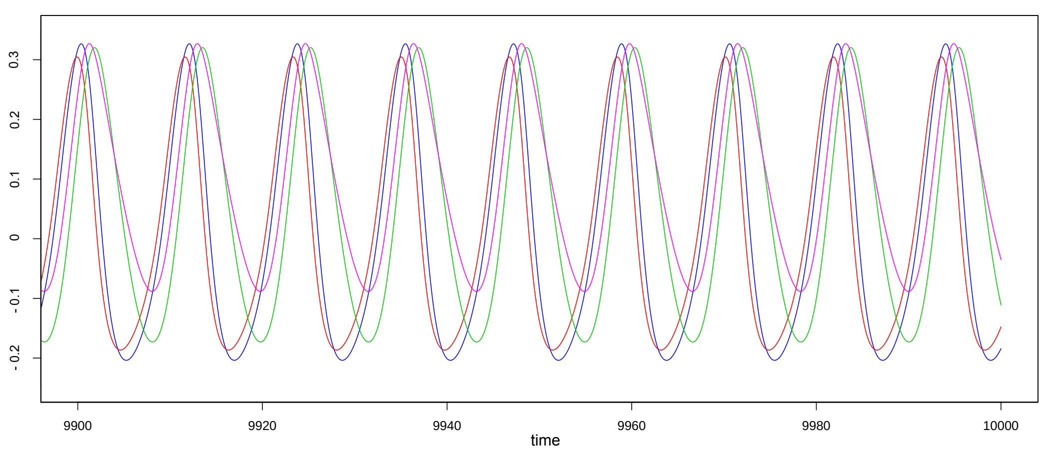

Figure 5: Lead and lag
(—: $y_c - y_c^*$, —: $y_i - y_i^*$, —: $k_c - k_c^*$, —: $k_i - k_i^*$)

[37]Since System with (67) satisfies (50), condition (51) holds.

5 Concluding remarks

We are in a position to summarize our analysis.

As a first step for multi-sectoral analysis of business cycles, we have put forward a two-sector model of business cycles by disaggregating the economy into two sectors: the consumption good sector and the investment good sector. We have examined the stability of equilibrium and the possibility of existence of a periodic orbit in our two-sector model. As a result, we have revealed that the counter part of the Keynesian stability condition plays a key role in the stability in our two-sector model and that a periodic orbit may arise by way of a Hopf bifurcation if the stability condition is not satisfied. We have also observed that the consumption good sector lags behind the investment good sector along the periodic orbit and that interactions between these two sectors do play a significant role in business cycles. Furthermore, we have numerically investigated the properties of the periodic orbit generated by Hopf bifurcations in our two-sector model. By conducting numerical simulations, we have verified that periodic orbits are actually generated by Hopf bifurcations in our two-sector model.

Both consumption and investment are components of aggregate expenditure. As Keynes (1936) argued, however, they differ on the demand side in that the former responds passively to aggregate income (through the "consumption function") while the latter has a decisive role in determination of aggregate income (through the "multiplier process") and is the main cause of economic fluctuations. As Harrod (1939) noticed, they also differ on the supply side in that the former is not utilized as a factor of production while the latter is durable and functions as a factor of production (the "duality of investment"). These differences cannot properly be described in one-sector models, but we believe that our two-sector model may shed a new light on them.

Our two-sector model is only a first step for developing a more general multi-sector model of business cycles. The dichotomy between consumption and investment in our model is artificial in that a lot of goods function both as a consumption good and as an investment good in reality. Also, our assumption of rigid prices is stringent because prices fluctuate over business cycles in reality. The removal of these unrealistic points may be left for future study.

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Appendices

A Proof of Proposition 1

Proof. (i)[38] Since System is (twice) continuously differentiable on D and D is a compact convex set, it follows from the mean value theorem that there exist a positive constant K such that for every $(y_c^0, y_i^0, k_c^0, k_i^0),\ (y_c^1, y_i^1, k_c^1, k_i^1) \in D$, we have

$$\begin{aligned}
&|\alpha_c[C(y_c^1, y_i^1) - y_c^1] - \alpha_c[C(y_c^0, y_i^0) - y_c^0]| + |\alpha_i[I_c(y_c^1, k_c^1) + I_i(y_i^1, k_i^1) - y_i^1] - \alpha_i[I_c(y_c^0, k_c^0) + I_i(y_i^0, k_i^0) - y_i^0]| \\
&+ |[I_c(y_c^1, k_c^1) - \delta k_c^1] - [I_c(y_c^0, k_c^0) - \delta k_c^0]| + |[I_i(y_i^1, k_i^1) - \delta k_i^1] - [I_i(y_i^0, k_i^0) - \delta k_i^0]| \\
&\leq K(|y_c^1 - y_c^0| + |y_i^1 - y_i^0| + |k_c^1 - k_c^0| + |k_i^1 - k_i^0|).
\end{aligned}$$

Hence, the Lipschitz condition (Coddington and Levinson 1955, chap. 1) is satisfied on D in System. Thus, for every initial condition $(y_c(0), y_i(0), k_c(0), k_i(0)) \in D$, there exists a nonnegative t_0 such that the solution path of System uniquely exists for $t \in [0, t_0]$. Such a constant t_0 is called the end point below.

Since conditions (20)-(23) (deduced from Assumptions 1 and 2) imply that D is positively invariant with respect to System, we can, by applying the argument on the continuation of solution paths of differential equations (cf. Coddington and Levinson 1955, chap. 1), conclude that, for every initial condition $(y_c(0), y_i(0), k_c(0), k_i(0)) \in D$, the end point t_0 can be extended to any positive number. $\square$

(ii) Since D is positively invariant with respect to System (by Assumptions 1 and 2), we can apply the Poincarè-Hopf theorem (cf. Milnor 1965, p. 35) to find that the sum of the indices of all equilibrium points of System is +1, which already implies the existence of an equilibrium point on D. Moreover, we know from Assumption 3 that the determinant of the Jacobian matrix J of System is positive everywhere on D, especially at every equilibrium point:

$$\det J = \alpha_c\alpha_i\{(1-C_{y_c})I_{k_c}I_{k_i} - [(1-C_{y_c})(1-I_{y_i})I_{k_c} + (1-C_{y_c} - C_{y_i}I_{y_c})I_{k_i}]\delta + [(1-C_{y_c})(1-I_{y_i}) - C_{y_i}I_{y_c}]\delta^2\} > 0.$$

By definition, we find that the index of each equilibrium point of System is +1. Hence, the number of equilibrium points has to be one.[39] Thus, the existence and uniqueness of an equilibrium point of System (on D) can be established.

It follows from (20)-(23) that $\overline{x}_j$ or $\underline{x}_j$ cannot be the equilibrium value of x_j for $x = y, k$ and $j = c, i$. Then, the unique equilibrium point $(y_c^*, y_i^*, k_c^*, k_i^*)$ must be on the interior D°. $\square$

[38] The method of proof is the same as used in Murakami (2014, 2017a).

[39] By the same method, Varian (1979) showed the existence and uniqueness of equilibrium of Kaldor's (1940) model of business cycles.

B Proof of Proposition 2

Proof. (i) If α_c and α_i are sufficiently small, the dominant term of f is given by[40]

$$(\delta - I^*_{k_c})(\delta - I^*_{k_i})^2(2\delta - I^*_{k_c} - I^*_{k_i})\{(1 - C^*_{y_c})(\delta - I^*_{k_c})\alpha_c + [(1 - I^*_{y_i})\delta - I^*_{k_c}]\alpha_i\} > 0.$$

The inequality follows from Assumption 1. Hence, condition (30) holds if α_c and α_i is sufficiently small.

(ii) If α_c is sufficiently large and α_i is sufficiently small, the dominant term of f is given by

$$(1 - C^*_{y_c})^3(\delta - I^*_{k_c})(\delta - I^*_{k_i})(2\delta - I^*_{k_c} - I^*_{k_I})\alpha_c^3 > 0,$$

where the inequality holds under Assumption 1. Then, the same logic as in (i) applies.

(iii) If α_c is sufficiently small and α_i is sufficiently large, the dominant term of f is

$$(1 - I^*_{y_i})[(1 - I^*_{y_i})\delta - I^*_{k_c}][(1 - I^*_{y_i})(2\delta - I^*_{k_c}) - I^*_{k_i}](\delta - I^*_{k_i})\alpha_i^3 > 0,$$

where the inequality holds under Assumption 1. The same logic as in (i) applies.

(iv) If α_c and α_i are sufficiently large, the dominant term of f is

$$\begin{aligned}[(1 - C^*_{y_c})(1 - I^*_{y_i}) - C^*_{y_i}I^*_{y_c}]\{[(1 - C^*_{y_c})(1 - I^*_{y_i}) - C^*_{y_i}I^*_{y_c}]\delta - (1 - C^*_{y_c})(1 - I^*_{y_i})I^*_{k_c} + (1 - C^*_{y_c} - C^*_{y_i}I^*_{y_c})I^*_{k_i}\}\\ \times[(1 - C^*_{y_c})\alpha_c + (1 - I^*_{y_i})\alpha_i]\alpha_c^2\alpha_i^2 > 0,\end{aligned}$$

where the inequality holds under Assumptions 1 and 4 and condition (32). Then, the same logic as in (i) applies.

(v) If α_c and α_i are sufficiently large, the dominant term of f is

$$\begin{aligned}[(1 - C^*_{y_c})(1 - I^*_{y_i}) - C^*_{y_i}I^*_{y_c}]\{[(1 - C^*_{y_c})(1 - I^*_{y_i}) - C^*_{y_i}I^*_{y_c}]\delta - (1 - C^*_{y_c})(1 - I^*_{y_i})I^*_{k_c} + (1 - C^*_{y_c} - C^*_{y_i}I^*_{y_c})I^*_{k_i}\}\\ \times[(1 - C^*_{y_c})\alpha_c + (1 - I^*_{y_i})\alpha_i]\alpha_c^2\alpha_i^2 < 0.\end{aligned}$$

The inequality holds under Assumptions 1 and 4 and condition (33). Thus, condition (30) is violated if both α_c and α_i are sufficiently large. □

C Proof of Proposition 3

Proof. (i) If α_i is sufficiently large, the dominant term of $f(0, \alpha_i)$ is

$$(1 - I^*_{y_i})[(1 - I^*_{y_i})\delta - I^*_{k_c}][(1 - I^*_{y_i})(2\delta - I^*_{k_c}) - I^*_{k_i}](\delta - I^*_{k_i})\alpha_i^3 > 0,$$

[40]We mean by a "dominant term" the one which is sufficiently large compared with the other terms in absolute values.

while we have

$$\lim_{\alpha_c \to \infty} f(\alpha_c, \alpha_i) = -\infty,$$

under Assumptions 1 and 5. Then, equation (36) has at least one positive root α_c^* for α_i sufficiently large. Since $f(\alpha_c, \alpha_i)$ turns from positive to negative at $\alpha_c = \alpha_c^*$ for α_i sufficiently large value, we have (37) at $\alpha_c = \alpha_c^*$. It follows from Asada and Yoshida's (2003) theorem that a periodic orbit is generated by a Hopf bifurcation.

(ii) Fix α_i at a sufficiently large value. Then, the dominant term of $f(\alpha_c, 0)$ is

$$(1 - C_{y_c}^*)^3 (\delta - I_{k_c}^*)(\delta - I_{k_i}^*)(2\delta - I_{k_c}^* - I_{k_I}^*)\alpha_c^3 > 0,$$

while we have

$$\lim_{\alpha_i \to \infty} f(\alpha_c, \alpha_i) = -\infty,$$

under Assumptions 1 and 5. Hence, the same logic as in (i) applies. □

D Proof of Proposition 4

Proof. (i) We substitute (44)-(47) in System (L) to obtain the following:

$$\frac{2\pi a_{y_c}}{T}\cos\Big(\frac{2\pi(t-\theta_{y_c})}{T}\Big) = \alpha_c\Big[(C_{y_c}^* - 1)a_{y_c}\sin\Big(\frac{2\pi(t-\theta_{y_c})}{T}\Big) + C_{y_i}^* a_{y_i}\sin\Big(\frac{2\pi(t-\theta_{y_i})}{T}\Big)\Big], \tag{69}$$

$$\begin{aligned}\frac{2\pi a_{y_i}}{T}\cos\Big(\frac{2\pi(t-\theta_{y_i})}{T}\Big) = \alpha_i\Big[&I_{y_c}^* a_{y_c}\sin\Big(\frac{2\pi(t-\theta_{y_c})}{T}\Big) + (I_{y_i}^* - 1)a_{y_i}\sin\Big(\frac{2\pi(t-\theta_{y_i})}{T}\Big) \\ &+ I_{k_c}^* a_{k_c}\sin\Big(\frac{2\pi(t-\theta_{k_c})}{T}\Big) + I_{k_i}^* a_{k_i}\sin\Big(\frac{2\pi(t-\theta_{k_i})}{T}\Big)\Big],\end{aligned} \tag{70}$$

$$\frac{2\pi a_{k_c}}{T}\cos\Big(\frac{2\pi(t-\theta_{k_c})}{T}\Big) = I_{y_c}^* a_{y_c}\sin\Big(\frac{2\pi(t-\theta_{y_c})}{T}\Big) + (I_{k_c}^* - \delta)a_{k_c}\sin\Big(\frac{2\pi(t-\theta_{k_c})}{T}\Big), \tag{71}$$

$$\frac{2\pi a_{k_i}}{T}\cos\Big(\frac{2\pi(t-\theta_{k_i})}{T}\Big) = I_{y_i}^* a_{y_i}\sin\Big(\frac{2\pi(t-\theta_{y_i})}{T}\Big) + (I_{k_i}^* - \delta)a_{k_i}\sin\Big(\frac{2\pi(t-\theta_{k_i})}{T}\Big). \tag{72}$$

Equations (69), (71) and (72) can be transformed into the following:

$$\begin{aligned} a_{y_c}\sqrt{\alpha_c^2(1-C_{y_c}^*)^2 + \Big(\frac{2\pi}{T}\Big)^2}\sin\Big(\frac{2\pi[t-\theta_{y_c} + (\theta_{y_c} - \theta_{y_i})]}{T}\Big) &= \alpha_c C_{y_i}^* a_{y_i}\sin\Big(\frac{2\pi(t-\theta_{y_i})}{T}\Big), \\ a_{k_c}\sqrt{(I_{k_c}^* - \delta)^2 + \Big(\frac{2\pi}{T}\Big)^2}\sin\Big(\frac{2\pi[t-\theta_{k_c} + (\theta_{k_c} - \theta_{y_c})]}{T}\Big) &= I_{y_c}^* a_{y_c}\sin\Big(\frac{2\pi(t-\theta_{y_c})}{T}\Big), \\ a_{k_i}\sqrt{(I_{k_i}^* - \delta)^2 + \Big(\frac{2\pi}{T}\Big)^2}\sin\Big(\frac{2\pi[t-\theta_{k_i} + (\theta_{k_i} - \theta_{y_i})]}{T}\Big) &= I_{y_i}^* a_{y_i}\sin\Big(\frac{2\pi(t-\theta_{y_i})}{T}\Big), \end{aligned}$$

where we can find that

$$
\begin{aligned}
&\sin\Big(\frac{2\pi(\theta_{y_c}-\theta_{y_i})}{T}\Big) = \frac{2\pi}{T}\Big/\sqrt{\alpha_c^2(1-C^*_{y_c})^2+\Big(\frac{2\pi}{T}\Big)^2} > 0,\\
&\cos\Big(\frac{2\pi(\theta_{y_c}-\theta_{y_i})}{T}\Big) = \alpha_c(1-C^*_{y_c})\Big/\sqrt{\alpha_c^2(1-C^*_{y_c})^2+\Big(\frac{2\pi}{T}\Big)^2} > 0,
\end{aligned} \tag{73}
$$

$$
\begin{aligned}
&\sin\Big(\frac{2\pi(\theta_{k_c}-\theta_{y_c})}{T}\Big) = \frac{2\pi}{T}\Big/\sqrt{(I^*_{k_c}-\delta)^2+\Big(\frac{2\pi}{T}\Big)^2} > 0,\\
&\cos\Big(\frac{2\pi(\theta_{k_c}-\theta_{y_c})}{T}\Big) = (\delta-I^*_{k_c})\Big/\sqrt{(I^*_{k_c}-\delta)^2+\Big(\frac{2\pi}{T}\Big)^2} > 0,
\end{aligned} \tag{74}
$$

$$
\begin{aligned}
&\sin\Big(\frac{2\pi(\theta_{k_i}-\theta_{y_i})}{T}\Big) = \frac{2\pi}{T}\Big/\sqrt{(I^*_{k_i}-\delta)^2+\Big(\frac{2\pi}{T}\Big)^2} > 0,\\
&\cos\Big(\frac{2\pi(\theta_{k_i}-\theta_{y_i})}{T}\Big) = (\delta-I^*_{k_i})\Big/\sqrt{(I^*_{k_i}-\delta)^2+\Big(\frac{2\pi}{T}\Big)^2} > 0.
\end{aligned} \tag{75}
$$

Since we have $\theta_{x'_{j'}} - \theta_{x_j} \in [-T, T]$, for $x, x' = y, k$ and $j, j' = c, i$, it follows from (73)-(75) that

$$0 < \theta_{y_c} - \theta_{y_i} < \frac{T}{4}, \tag{76}$$

$$0 < \theta_{k_c} - \theta_{y_c} < \frac{T}{4}, \tag{77}$$

$$0 < \theta_{k_i} - \theta_{y_i} < \frac{T}{4}. \tag{78}$$

Under Assumption 1 and (48), it is seen from (73) and (75) that

$$\sin\Big(\frac{2\pi(\theta_{k_i}-\theta_{y_i})}{T}\Big) \le \sin\Big(\frac{2\pi(\theta_{y_c}-\theta_{y_i})}{T}\Big).$$

Taking account of (76) and (78), this leads us to the following:

$$0 < \theta_{k_i} - \theta_{y_i} \le \theta_{y_c} - \theta_{y_i} < \frac{T}{4}.$$

Combined with (77), this implies (49). □

(ii) By the same argument as made in (i), the following holds under (50):

$$0 < \theta_{y_c} - \theta_{y_i} \le \theta_{k_i} - \theta_{y_i} < \frac{T}{4}.$$

Since we have (76)-(78) even under (50), condition (51) or (52) follows. □

E Data and estimations

The data on the levels of GDP (Y), consumption expenditure (C), investment expenditure (I), private capital stock (K) and private capital depreciation (D) are taken from Annual Report on National Accounts, issued by Cabinet

Office, Japan. Those on the number of population (N) and TFP (A) are taken from Population Estimates, issued by Statistics Bureau, Ministry of Internal Affairs and Communications, Japan and from Japan Main Productivity-indicators database, issued by Japan Productivity Center. Utilizing the data, we calculate the following variables:

$$y_t = \frac{Y_t}{A_t N_t},$$
$$c_t = \frac{C_t}{A_t N_t},$$
$$u_t = \frac{Y_t}{K_t},$$
$$g_t = \frac{I_t}{K_{t-1}}$$
$$\delta_t = \frac{D_t}{K_{t-1}}$$

The time series of these variables are presented in the following tables.

Table 1: Data on consumption

Year t	y_t	c_t	Year t	y_t	c_t
1955	52447.3506	34078.17756	1985	153864.6269	89699.92055
1956	55013.68838	36252.61448	1986	156125.3148	91544.53121
1957	57051.96345	37738.24008	1987	160426.0343	93905.94445
1958	58857.36951	38997.9724	1988	164549.1842	95469.94741
1959	61051.11748	40095.62564	1989	168585.833	97651.33145
1960	64366.22915	41542.98532	1990	171990.8208	98974.13324
1961	67253.57605	42877.80707	1991	176003.1668	100020.5226
1962	69996.92529	44211.67471	1992	177044.9549	101452.4419
1963	73190.66295	46248.96586	1993	175630.2417	101480.57
1964	76795.21284	48392.68646	1994	176118.6834	103152.4538
1965	80267.59664	50603.66094	1995	175845.3864	103596.6206
1966	85122.8436	53539.14072	1996	182791.9129	105245.92
1967	89969.47849	56223.1002	1997	186026.6593	105731.2342
1968	95430.56721	57869.02348	1998	181965.3922	105323.1219
1969	100619.3079	60123.4434	1999	182625.5191	102412.1456
1970	107199.8807	62343.77079	2000	182809.3908	101142.4088
1971	112666.7582	66164.99703	2001	180842.1034	100756.6888
1972	116869.9426	68892.10944	2002	178970.9385	100721.831
1973	122953.1865	72984.65676	2003	178867.2086	99765.28645
1974	123443.5707	74273.6232	2004	179031.4318	99019.39508
1975	124399.5791	75746.47524	2005	178190.8678	99257.60558
1976	129684.7461	78144.77558	2006	179831.7834	100116.0771
1977	133693.7278	80248.55244	2007	182361.6312	100462.7929
1978	137986.3474	82740.0092	2008	178401.1968	100210.9704
1979	143074.6262	86525.30545	2009	172986.7644	101109.3692
1980	146750.7696	87371.41093	2010	177937.5003	103898.083
1981	147392.5621	86463.48082	2011	176238.1682	105074.604
1982	149528.3098	88612.99813	2012	173891.8425	103461.576
1983	151563.7855	90633.92518	2013	173969.832	103845.6398
1984	154098.0531	90951.2639	2014	175641.2031	104349.2264

Table 2: Data on investment

Year t	u_{t-1}	g_t	Year t	u_{t-1}	g_t
1970	0.920540411	0.2981151	1993	0.586731409	0.124239961
1971	0.861842909	0.240154214	1994	0.580881778	0.117256516
1972	0.816810644	0.229252888	1995	0.582359019	0.11697812
1973	0.746888181	0.243385516	1996	0.583256859	0.125143077
1974	0.656871046	0.202108329	1997	0.588555194	0.124770106
1975	0.630956113	0.163178192	1998	0.575262762	0.107852181
1976	0.645727528	0.162766744	1999	0.570474009	0.101667537
1977	0.641003805	0.150631948	2000	0.521879901	0.095604815
1978	0.668038815	0.151325608	2001	0.525945192	0.091719391
1979	0.684368258	0.161263592	2002	0.528027288	0.086284506
1980	0.637740358	0.152370902	2003	0.528254957	0.088409658
1981	0.620540451	0.141145521	2004	0.524990784	0.089575079
1982	0.630148695	0.135851506	2005	0.525998606	0.092253872
1983	0.62946249	0.128638911	2006	0.518379046	0.094179211
1984	0.638086432	0.136205692	2007	0.509119624	0.093929706
1985	0.647524516	0.143041077	2008	0.503120972	0.089334967
1986	0.642018714	0.138712231	2009	0.482781297	0.072974815
1987	0.656702825	0.147648958	2010	0.473082549	0.07450905
1988	0.638795784	0.15672139	2011	0.490865069	0.077885821
1989	0.639862382	0.164925227	2012	0.483356602	0.080982492
1990	0.616123054	0.166810789	2013	0.49127967	0.083401591
1991	0.60327599	0.159280002	2014	0.485373411	0.084423299
1992	0.593849223	0.140180568			

Table 3: Data on capital depreciation

Year	δ_t	Year	δ_t
1970	0.127447471	1993	0.085978657
1971	0.113426883	1994	0.085382263
1972	0.115418444	1995	0.08625172
1973	0.110588994	1996	0.088598348
1974	0.093032028	1997	0.088046291
1975	0.079194407	1998	0.085682152
1976	0.079162266	1999	0.096150697
1977	0.077645084	2000	0.086042349
1978	0.078456826	2001	0.085210528
1979	0.080642677	2002	0.085517862
1980	0.077040678	2003	0.084997226
1981	0.076879552	2004	0.084576848
1982	0.077099521	2005	0.084427673
1983	0.077819367	2006	0.085566701
1984	0.08060643	2007	0.085775444
1985	0.08465134	2008	0.085740372
1986	0.084312335	2009	0.082681698
1987	0.088034464	2010	0.083173753
1988	0.088379331	2011	0.082248902
1989	0.092058328	2012	0.081851671
1990	0.090016885	2013	0.083653054
1991	0.089415745	2014	0.083354347
1992	0.087763779		

Based upon the argument in Section 4, we make use of the following equations for our estimations.

$$c_t = \alpha_0 + \alpha_y y_t + \varepsilon_c, \tag{79}$$

$$\ln\Big(\frac{g_t}{0.3 - g_t}\Big) = \beta_0 + \beta_u u_{t-1} + \varepsilon_g, \tag{80}$$

where ε_c and ε_g obey $N(0, \sigma_c^2)$ and $N(0, \sigma_g^2)$, respectively. In (79), the level of consumption is assumed to be dependent upon that of income in the year, while in (80), the level of investment is assumed to be dependent upon that of income in the previous year. This difference reflects the reality that the Robertsonian consumption lag (i.e., the time lag between earnings of income and consumption expenditure) is shorter than the investment decision lag (i.e., the time lag between earnings of income and investment expenditure).[41]

The results of estimations on (79) and (80) are given in the following tables.

Table 4: Estimations

	(79)		(80)
y_t	0.5318***	u_{t-1}	9.3798***
	(0.005691)		(0.6899)
Constant	7675***	Constant	−5.8092***
	(829.9)		(0.4212)
Observations	60	Observations	45
Adjusted R^2	0.9933	Adjusted R^2	0.8069

Standard errors in parentheses
*** $p < 0.001$

The average rate of capital depreciation can directly be calculated from table 3 as follows:

$$\hat{\delta} = 0.0873.$$

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