

# SOME NOTES ON APPLYING THE HERFINDAHL-HIRSCHMAN INDEX

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ABSTRACT. The Herfindahl-Hirschman Index is one of the most commonly used indicators to detect anticompetitive behavior in industries. In fact, an increase in the value of the index is usually interpreted as an indicator of actions which may lessen competition or even create a monopoly. In this paper we show that this is not always the case. If all firms cooperate, then the index cannot be used since it does not depend on the cooperation level. We also show an example when competition even has a decreasing effect on the value of the index.

## 1. INTRODUCTION

The theory of oligopoly has a very large and diverse literature. Many different variants and extensions of the classical Cournot (1838) model were introduced and analyzed. A comprehensive summary of the most important results can be found in Okuguchi (1976), Okuguchi and Szidarovszky (1999) and Bischi et al. (2009). Almost all earlier studies examined the oligopolies with competing firms and very few works were devoted to the effect of cooperating groups of firms. Different kinds of cooperation were examined in Merlone (2007), his general model included different profit formulations in cross shareholding. Recently Matsumoto et al. (2009a,b) extended this model for partial cooperation and developed a nondifferentiable dynamics in which the firms stop cooperating when they suspect that their activity may draw the attention of the antitrust authorities. One of the commonly used measure is the Herfindahl-Hirschman Index, HHI, especially when examining horizontal mergers (Whinston, 2006), nevertheless this concentration index has been used by other authors to study cooperative behavior in other contexts (Porter and Zona, 1999). Theoretical and empirical evidence shows that higher HHI value indicates higher price-cost margin (Viscusi et al., 2005). Anticompetitive behavior can be much more diverse than horizontal mergers, see for instance Whinston (2006). Furthermore, Flath (1992) proved the anticompetitive effects of horizontal shareholding whereas it is shown in Matsumoto et al. (2009a) that shareholding interlocks are mathematically equivalent to partial cooperation. Therefore shareholding interlocks can also be interpreted as partial cooperation among the firms, so the HHI can be assumed to be a practical indicator of conspiracy. The literature has examined several drawbacks of the HHI index, for example Kwoka (1977) argues that this index embodies both size inequality and firm numbers with weights which are

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assumed a priori instead of being derived. Most of the other critiques are empirically grounded: for instance Borenstein et al. (1999) have shown that at least in the case of electricity markets the HHI is a poor measure of competitiveness. Furthermore, other authors (Foncel et al., 2008), (Liaukonyte, 2007) have questioned the use of this index even for analyzing mergers. In this paper we approach this topic from a theoretical perspective. We will examine some fundamental properties of the HHI index and show that it cannot be used in certain situations as it may lead to incorrect conclusions.

## 2. THE MATHEMATICAL MODEL

Consider an  $N$ -firm single-product oligopoly with homogeneous product and hyperbolic price function  $f(Q) = \frac{A}{Q}$ , where  $A > 0$  is a constant and  $Q$  is the total output of the industry. The cost function of firm  $k$  is assumed to be linear,  $c_k(x_k) = c_k x_k + d_k$ , where  $x_k$  denotes the output of firm  $k$ . The profit of firm  $k$  is the difference of its revenue and cost,

$$(2.1) \quad \varphi_k(x_1, \dots, x_N) = \frac{Ax_k}{x_k + Q_k} - (c_k x_k + d_k)$$

where  $Q_k = \sum_{l \neq k}^N x_l$  is the output of the rest of the industry.

It is next assumed that firms  $k = 1, 2, \dots, m$  partially cooperate with a common cooperation level  $\delta$  and firms  $k = m + 1, \dots, N$  have Cournot behavior. The payoff functions of these firms are given by (2.1), and the payoffs of the cooperating firms  $k = 1, 2, \dots, m$  are given as

$$(2.2) \quad \begin{aligned} \psi_k(x_1, \dots, x_N) &= \varphi_k(x_1, \dots, x_N) + \delta \sum_{\substack{l=1 \\ l \neq k}}^m \varphi_l(x_1, \dots, x_N) \\ &= \frac{Ax_k + \delta A \sum_{\substack{l=1 \\ l \neq k}}^m x_l}{x_k + Q_k} - (c_k x_k + d_k) - \delta \sum_{\substack{l=1 \\ l \neq k}}^m (c_l x_l + d_l). \end{aligned}$$

It is easy to see that the best responses of the competing firms are

$$(2.3) \quad x_k = \sqrt{\frac{AQ_k}{c_k}} - Q_k$$

and those of the cooperating firms are given as

$$(2.4) \quad x_k = \sqrt{\frac{A(Q_k - \delta q_k)}{c_k}} - Q_k$$

by assuming interior optimum, where  $q_k = \sum_{\substack{l=1 \\ l \neq k}}^m x_l$ . Equations (2.3) for  $k =$

$m + 1, \dots, N$  and equations (2.4) for  $k = 1, \dots, m$  provide  $N$  equations for the  $N$  unknown  $x_k$  output levels. A simple but lengthy calculation shows that the industry output at the equilibrium is

$$(2.5) \quad Q = \frac{A[(1 + \delta(m-1))(N - m - 1) + m]}{[1 + \delta(m-1)] \sum_{k=m+1}^N c_k + \sum_{k=1}^m c_k},$$

and the total output of the cooperating firms is

$$(2.6) \quad q = \frac{A[(1 + \delta(m-1))(N-m-1) + m] \left[ m \sum_{k=m+1}^N c_k - (N-m-1) \sum_{k=1}^m c_k \right]}{\left[ (1 + \delta(m-1)) \sum_{k=m+1}^N c_k + \sum_{k=1}^m c_k \right]^2}.$$

For the sake of simplicity we will consider the semi-symmetric case when  $c_k = c_x$  for  $k = 1, \dots, m$  and  $c_k = c_y$  for  $k = m+1, \dots, N$ , that is, the members of both cooperating and competing groups have identical marginal costs. By introducing the notation  $c = c_x/c_y$  as the cost ratio it is easy to verify that at the equilibrium point the common output of the cooperating firms is

$$(2.7) \quad x^C = \frac{q}{m} = \frac{A[N-m - (N-m-1)c][N-1 + \delta(m-1)(N-m-1)]}{[(1 + \delta(m-1))(N-m) + cm]^2 c_y}$$

and the common output of the competing firms has the form

$$(2.8) \quad x^N = \frac{Q - q}{N - m} = \frac{A[cm - (1 - \delta)(m-1)][N-1 + \delta(m-1)(N-m-1)]}{[(1 + \delta(m-1))(N-m) + cm]^2 c_y}.$$

In order to have a meaningful model we have to assume that  $N > 2$ ,  $m \geq 2$  and  $N \geq m+1$ ; to guarantee the positivity of the equilibrium outputs in both the partially cooperative and the noncooperative cases we also have to assume that

$$(2.9) \quad \frac{m-1}{m} < c < \frac{N-m}{N-m-1}$$

if  $N \neq m+1$ , otherwise the upper bound can be ignored.

The Herfindahl-Hirschman Index is the sum of squared market shares of the firms in the industry<sup>1</sup>; in our case we have

$$(2.10) \quad \begin{aligned} \text{HHI}^C &= \sum_{k=1}^N \left( \frac{x_k}{Q} \right)^2 = \frac{1}{Q^2} \left[ m(x^C)^2 + (N-m)(x^N)^2 \right] \\ &= \frac{m[N-m - (N-m-1)c]^2 + (N-m)[cm - (1-\delta)(m-1)]^2}{[(1 + \delta(m-1))(N-m) + cm]^2}. \end{aligned}$$

If all firms have Cournot behavior, then  $\delta = 0$ , so

$$(2.11) \quad \text{HHI}^N = \frac{m[N-m - (N-m-1)c]^2 + (N-m)[cm - (m-1)]^2}{[(N-m) + cm]^2}.$$

It is well known that in any nonsymmetric  $N$ -firm industry HHI has the unit upper bound which occurs in the case of a single monopolist, and the lower bound of HHI is  $1/N$  which occurs, for example, in the noncooperative symmetric case ( $\text{HHI}^N$  with  $c = 1$ ).

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<sup>1</sup>We recall that in the empirical applications the convention is to multiply the resulting sum of squared market shares by 10,000.

## 3. THREE CASES OF THE HHI INDEX

In this section three special cases will be shown in order to illustrate the applicability of this index.

- (1) Consider first the case of  $c = 1$ , when the firms have identical marginal costs. Then

$$(3.1) \quad \text{HHI}^C = \frac{m + (N - m) [1 + \delta(m - 1)]^2}{[(1 + \delta(m - 1))(N - m) + m]^2} \quad \text{and} \quad \text{HHI}^N = \frac{1}{N}$$

and simple calculation shows that  $\text{HHI}^N < \text{HHI}^C$ . Therefore in this case the increase of the index indicates cooperation.

- (2) Consider again the previous case and assume that all the firms cooperate, that is,  $m = N$ . In this case  $\text{HHI}^C = 1/N$  regardless of the value of  $\delta$ . That is, the cooperation of all firms cannot be detected.

- (3) Now we return to the semi-symmetric case and examine the difference between  $\text{HHI}^C$  and  $\text{HHI}^N$ . Simple but lengthy calculation shows that with  $\delta > 0$  the difference  $\text{HHI}^C - \text{HHI}^N$  has the same sign as

$$(3.2) \quad (c - 1)(N - 1)[N + (c - 1)m] + (m - 1)\delta[(N - m)(N - 2)(c - 1) + cN].$$

If  $c \geq 1$ , then this expression is positive, so  $\text{HHI}^C > \text{HHI}^N$  and partial cooperation is detected by the authorities.

Notice that  $c \geq 1$  if and only if  $c_x \geq c_y$ , that is, the cooperating firms are less efficient than the competing firms. Assume next that  $c < 1$ , and introduce the notation  $\gamma = 1 - c$ . From condition (2.10) we know that  $\gamma \in (0, 1/m)$ . Expression (3.2) will be denoted by  $g(\delta)$  and can be simplified as

$$(3.3) \quad -2\gamma(N - 1)(N - \gamma m) + \delta(m - 1)[-(N - m)(N - 2)\gamma + N - N\gamma].$$

Notice that this function is linear in  $\delta$ , and  $g(\delta)$  converges to a negative limit as  $\delta \rightarrow 0$ . So if the cooperative firms select a sufficiently small cooperation level  $\delta$ , then their cooperation is undetectable. If  $g(1) \leq 0$ , then no partial cooperation can be detected, since  $g(\delta) \leq 0$  for all  $0 \leq \delta \leq 1$ . If  $g(1) > 0$ , then there is a threshold

$$(3.4) \quad \delta^* = \frac{2\gamma(N - 1)(N - \gamma m)}{(m - 1)[-(N - m)(N - 2)\gamma + N - N\gamma]}$$

such that partial cooperation with  $\delta < \delta^*$  is undetectable and for  $\delta > \delta^*$  it can be detected by the authorities. Notice that condition  $g(1) > 0$  is equivalent to relation

$$(3.5) \quad 2\gamma^2 m(N - 1) - \gamma[2N(N - 1) + (m - 1)(N - m)(N - 2) + (m - 1)N] + N(m - 1) > 0.$$

The left hand side converges to  $N(m - 1)$  as  $\gamma \rightarrow 0$ , so with sufficiently small value of  $\gamma$  this relation necessarily holds. Small value of  $\gamma$  means that the cost ratio is sufficiently close to unit.

## 4. CONCLUSION

One of the most frequently used indicator of anticompetitive behavior among firms in an industry is the Herfindahl-Hirschman Index. There are however cases when it may be inappropriate. Two such cases were shown in the case of  $N$ -firm

oligopolies with partially cooperative firms. The cooperation of all firms cannot be detected since the value of the HHI index does not depend on the cooperation level in the case of symmetric oligopolies, that is, the index always shows no cooperation. In the semi-symmetric case there is the possibility that cooperation has a decreasing effect on the index.

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