Time Delayed Dynamic Model of Renewable Resource and Population

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Abstract

This paper reconsiders a "Ricardo-Malthus" dynamic model of population and renewable resource developed by Brander and Taylor [1998]. To this end, it sheds light on a *delay* in production, a key feature of long-run evolution in a pre-industrial economic society. It is clear that the delay has a considerable effect on long-run dynamics of natural resource and population This notwithstanding, many previous works relate to a case in which production is instantaneous. As such, the purpose of this paper is to investigate the effect caused by the delay in production. Our analysis shows that there is a critical value of the delay with which a delayed version of an otherwise stable system becomes unstable. In addition, it numerically shows that the critical value is negatively related to the size of the carrying capacity and the exogenous net birth rate.

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1 Introduction

This study reconstructs an economic dynamic model for a small island based on the Easter Island study by Brander and Taylor [1998] (BT henceforth). Introducing a delay in production into the BT model, it deals with analytical as well as numerical representations of evolutions of a small island economy, evolutions of which are characterized by a two-fold process of transformation; environmental degradation and population growth. Its main purpose is to investigate effects caused by the delay in production on an adjustment process in population and natural resource stock.

Archeological studies have suggested that many Pacific islands followed similar evolutionary patterns of natural resource and population dynamics, that is, rapid population growth, resource degradation, economic decline and then population collapse. BT reconsider archeological and anthropological evidences of Easter Island, as an example, from an economic point of view. In particular, BT present a general equilibrium model of renewable resource and population dynamics and seek to explain the rise and fall of Easter Island for 1400 years between the 4th century and the middle of the 18th century. They bring to light on economic conditions under which the small island economy can survive or perish. Their findings indicate that an economic model linking resource and population dynamics may explain not only the sources of past historical evolutions discovered in these small islands but also a possibility of sustainable growth of our world economy in which rapid-increasing population and rapid-degrading environment become serious problems. The analysis in the BT model has been extended in various directions. Dalton and Coats [2000] examine the impact of market institutions and different property-rights structures. Reuveny and Decker [2000] numerically consider how technological progress and population management reform affect the long-run dynamics of Easter Island. Matsumoto [2002] reconstructs the BT continuous model in discrete steps and shows that the modified model can generate various dynamics ranging from simple dynamics to complex dynamics involving chaos.

In the existing literature, however, not much has been revealed with respect to a "history" of Easter Island. It has been believed that a small group of Polynesians arrived the island around 400 A.D., deforestation occurred around 1000 A.D., most of statures were carved during 1000-1400 A.D., and so forth.[1] Based on this "conventional wisdom," BT as well as other researchers attempt to reproduce a dynamic pattern of natural resource and population. However, the "wisdom" is still one of possible hypotheses and

[1] See section 1 of Brander and Taylor [1998] for more details.

not fully confirmed yet. In particular, according to Intoh [2000], recent reconsiderations of archeological evidences on the island imply indeterminacy of arrival time of Polynesians. It can only be estimated that a small group settled between 410 and 1270 A.D. This new finding is inconsistent with the traditional wisdom. In other words, we may have a different history (as well as different evolution pattern) of Easter Island, even though the available historical evidences are the same. It is thus imperative to construct a model for small islands that can generate various patterns of dynamics in order to deal with such ambiguous characteristics of archeological evidence.

In this study, we extend the BT model to include a delay or lag in production with the following reason. Agricultural production may play an essential part of economic activity in a pre-industrial economic society. It is well-known that an important characteristic of such an agricultural production is the significant time lag between the time at which producers make their decisions to plant seeds in the fields and the time when they actually gather crops from the fields. It is thus natural to raise a question: *how the delayed production affects an evolutionary pattern of small island economy?*

This study is organized as follows. Section 2 constructs a simple economic dynamics model of an small island based on BT's Easter Island analysis. Section 3 analytically as well as numerically examines effects caused by a delay in production on evolutions of natural resource and population. Section 4 provides summary and concluding remarks.

2 Basic Model for Small Islands

Since our analysis is based on BT's dynamic model of renewable resource and population, we recapitulate the basic part of their model in this section. See BT's paper for more details.

The model describes the dynamics of an economy with two goods and three types of economic agents (two producers and one consumer). The harvest of the renewable resource is called agricultural good and some other good called manufactured good. The model functions as follows. At time t, the stock of natural resource $S(t)$ and the size of population $L(t)$ are given.[2] Producers determine their demands for labor and supplies of goods so as to maximize their profits. A manufactured producer supplies the manufactured good produced with constant returns to scale using only labor. Since, by choice of units, one unit of manufactured goods can be produced by one unit of labor, the total supply of manufactured good M^S is determined by the

[2] For a time being, t is suppressed for the notational simpliciy.

demand for labor L_M^D

$$M^S = L_M^D. \tag{1}$$

An agricultural producer supplies the agricultural good carried out according to the Shaefer harvesting production function,

$$H^S = \alpha S L_H^D, \tag{2}$$

where H^S is the harvest supplied, α is a positive constant indicating the harvesting efficiency and L_H^D is the labor demand in resource harvesting. A representative consumer is endowed with one unit of labor and is assumed to have a Cobb-Douglas utility function,

$$u(h, m) = h^\beta m^{1-\beta}, \tag{3}$$

where h and m are individual consumptions of the agricultural good and of manufactured good, and $\beta \in (0, 1)$ is a positive constant reflecting preference of the agricultural good. Each consumer supplies one unit of labor and demands both goods so as to maximize his utility subject to the budget constraint, $ph + m = w$ where p is the price of the agricultural good, w is the wage rate, and the price of the manufactured good is normalized to 1 as it is treated as a numeraire. The usual utility maximization procedure yields optimal demands for both goods, $h^d = \frac{w\beta}{p}$ and $m^d = (1 - \beta)L$ where d is attached to a variable indicating individual demand. The total number of population is L so that the total demands are

$$H^D = Lh^d \text{ and } M^D = Lm^d. \tag{4}$$

Prices are adjusted to establish temporary equilibrium in each of three markets; agricultural good market ($H^D = H^S$), manufactured good market ($M^D = M^S$) and labor market ($L_M^D + L_H^D = L$) in which labor force is assumed to be equal to the population. It can be verified that the fixed proportion of the total population is employed in the agricultural section, $L_H^D = \beta L$ and thus the resource harvest is $H^S = \alpha\beta S L$ at the temporary equilibrium state. After finishing transactions in each market, new values of natural resource and the size of population are determined at the next instant of time. With these new values, the process repeats until the stationary state is attained.

Dynamics of temporary equilibrium is described as follows. A change in the stock at time t is determined by the natural growth rate $G(S)$ minus the harvest rate,

$$\frac{dS(t)}{dt} = G(S(t)) - H^S(t). \tag{5}$$

For the analytical simplicity, the logistic functional form for G is assumed, $G(S) = r(1 - S/K)S$ where K is the maximum possible size for the resource stock, r is an intrinsic growth rate of natural resource, and both are positive constants. A change of population depends on a difference between an underlying birth rate and death rate. b denotes the exogenously-determined birth rate, and d the endogenously-determined death rate. It is assumed that the net rate, denoted as $c = b - d$, is negative. Following the formulation of Malthusian population dynamics, it is further assumed that the endogenously-determined birth rate depends on the economic activities, that is, per capita consumption of the agricultural good increases fertility and/or decreases mortality. Let $\phi \frac{H^S}{L}$ be a fertility function where ϕ is positive constant. Then the population growth rate is

$$\frac{1}{L(t)} \frac{dL(t)}{dt} = (c + \phi \frac{H^S(t)}{L(t)}), \tag{6}$$

where the first factor is the exogenous net birth rate, and the second is the endogenous birth rate. Substituting the logistic function into the natural growth rate and the optimal harvest into the fertility function yields the dynamics process of the natural resource and population,

$$\begin{cases} \dfrac{dS(t)}{dt} = \left(r(1 - \dfrac{S(t)}{K}) - \alpha\beta L(t) \right) S(t), \\ \\ \dfrac{dL(t)}{dt} = (c + \alpha\beta\phi S(t))\, L(t). \end{cases} \tag{7}$$

This is a two-dimensional dynamic system of differential equations and is a variant of the Lotka-Volterra perdator-prey model in which the human is the predator and the resource stock is the prey. The system has a steady state if $\frac{dS}{dt} = 0$ and $\frac{dL}{dt} = 0$ hold. We can solve the last simultaneous system for stocks of the natural resource and population size.[3] Coordinates of an interior solution are

$$S^e = -\frac{c}{\alpha\beta\phi} \text{ and } L^e = \frac{r}{\alpha\beta} \left(1 + \frac{c}{\alpha\beta\phi} \right). \tag{8}$$

BT find the following result about the stability of the interior steady point:

[3] The system can have a corner solution depending on values of parameters. However, the corner solution with $L = 0$ means an extinction of human, which is not interesting. Thus, in this study, we focus only on the interior solution.

Theorem 1 *(Proposition 4 (iii) of Brander and Taylor) An interior steady state* ($L^e > 0$ *and* $S^e > 0$) *is a spiral node with cyclical convergence if*

$$\frac{-cr}{\alpha\beta\phi K} + 4(-c - \alpha\beta\phi K) < 0$$

and an improper node allowing monotonic convergence if not.

Using the parameterization for the basic model for which BT provide a detailed justification,[4] we perform two simulations, which are reproductions of BT's numerical examples. Figure 1 illustrates time series for population size (i.e., a mountain-shaped curve) and resource stocks (i.e., a downward sloping concave-convex curve) when the intrinsic growth rate of renewable resource is low, $r = 0.04$. Figure 2 also illustrate time series for the same variables when the rate is roughly 9 times higher, $r = 0.35$. In these figures, one period represents one decade and the horizontal axis shows 140 periods. The initial period corresponds to the year 400 A.D., when the first indigenous people are said to have arrived on the island, and the last period corresponds to sometimes of the 18th century when European first arrived on the island, after which the substantial changes in environments make the dynamic model, (7), hold no longer.

Figure 1 shows oscillatory dynamics and appears to replicate what is known of Easter Island history. The island was settled by a Polynesian group about 400 A.D and was covered with great palm trees at this time. For the first 300 years, the populations size is small, and the resource has little degradation. However, soon after, the population size begins to increase rapidly, and the resource begins to decline correspondingly. The heyday of Easter Island is supposed to be between 1100 and 1300: the population reaches its maximum (about 10,000) and the statue carving is intensive. After reaching its peak of the population size, the island entered into a declining period and then disappeared from the history; the palm forest was entirely gone by 1400, carving ceased by 1500 and violent internecine conflict appeared.

[4]BT use the following parameters' values for their simulations: $L_0 = 40$ (initial human populatoin), $S_0 = 12,000$ (initial stock of the renewable resource), $K = 12,000$ (carrying capacity), $\alpha = 0.00001$ (harvesting efficienty), $\beta = 0.4$ (preference for agricultural good), $r = 0.04$ (intrinsic growth rate of the renewable resource), $\phi = 4$ (fertility rate), $c = -0.1$ (intrinsic net birth rate).

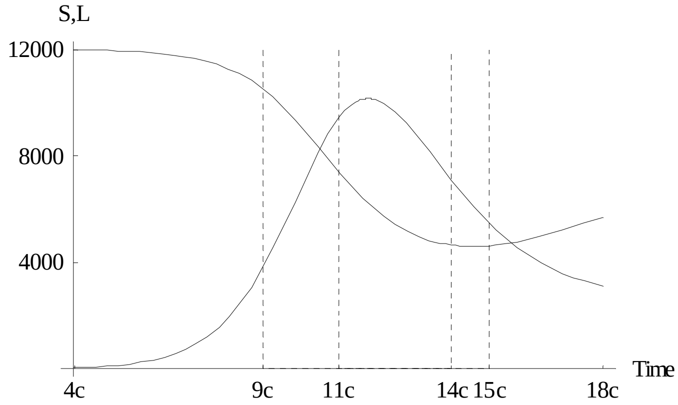


Figure 1.Slow Growth Rate of Natural Resource (r = 0.01)

According to Theorem 1, the model can generate monotonic behavior for some combinations of parameters values, which may explain monotonic evolutions observed on some Polynesian islands. In the second simulation we change the growth rate of the natural resource from the lower value (r = 0.04) to the higher value (r = 0.35) and use the same values of any other parameters as in the first simulation. As illustrated in Figure 2, the simulation shows a entirely different dynamics; a smooth adjustment converging to the stationary state. Comparing these two figures, we observe that an island with a slow-growing resource base will exhibit overshooting and collapse while an island with a rapid growing resource exhibits a near-monotonic adjustment of population and resource stocks towards steady state.

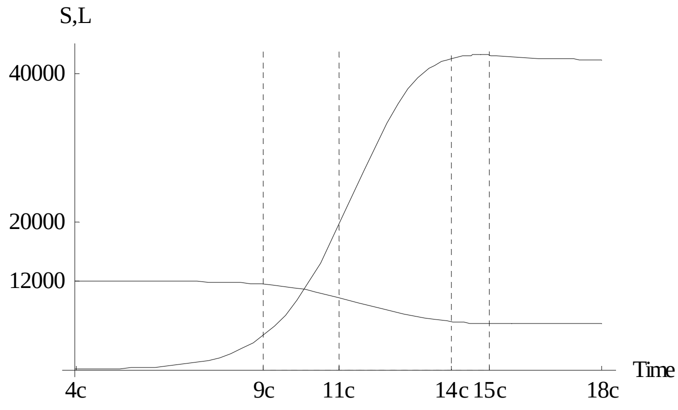


Figure 2. Rapid Growth Rate of Natural Resource (r = 0.35)

3 Application Model with Time Delay

In this section, we investigate the hypothetical role that a delay might has played in disturbing monotonic dynamics and disturbing further an oscillatory dynamics. In order to see this disturbing or destabilizing effect, we introduce a delay in production into the basic model, taking account of the fact that it takes some time for agricultural good from seeding to harvesting,

$$H(t) = \alpha\beta L(t)S(t-\tau). \tag{9}$$

τ is a delay in production. The current harvesting depends on the current amount of labor and the stock of natural resource at time $t-\tau$. Substituting (9) into the basic system (7) generates the dynamical system with the delay in production, τ,

$$\begin{cases} \dfrac{dS(t)}{dt} = rS(t)(1-\frac{S(t)}{K}) - \alpha\beta L(t)S(t-\tau) \\ \\ \dfrac{dL(t)}{dt} = L(t)(b-d+\phi\alpha\beta S(t-\tau)). \end{cases} \tag{10}$$

It can be verified that the delayed system has the same equilibrium point as the basic system. To investigate the stability of the delayed system, we first make a coordinate transformation such that a new system is centered at the equilibrium point (S^e, L^e) and then linearize the resultant system at the origin to derive its characteristic equation.

Let $\overline{S} = S - S^e$ and $\overline{L} = L - L^e$. Then the centered system is reduced to

$$\begin{cases} \dfrac{d\overline{S}(t)}{dt} = r(1-\frac{2S^e}{K})\overline{S}(t) - \alpha\beta S^e\overline{L}(t) - \frac{r}{K}\overline{S}(t)^2 \\ \qquad\qquad -\alpha\beta L^e\overline{S}(t-\tau) - \alpha\beta\overline{L}(t)\overline{S}(t-\tau) \\ \\ \dfrac{d\overline{L}(t)}{dt} = (L^e + \overline{L}(t))\phi\alpha\beta\overline{S}(t-\tau). \end{cases} \tag{11}$$

Put $\begin{pmatrix} \overline{S}(t) \\ \overline{L}(t) \end{pmatrix} = \mathsf{C}e^{\lambda t}$ where $\mathsf{C} \in \mathsf{C}^2$ and $\lambda \in \mathsf{C}$. Comparing the linear terms, we have

$$\mathsf{C}\lambda e^{\lambda t} = \begin{pmatrix} r(1-\frac{2S^e}{K}) & -\alpha\beta S^e \\ 0 & 0 \end{pmatrix}\mathsf{C}e^{\lambda t} + \begin{pmatrix} -\alpha\beta L^e & 0 \\ \phi\alpha\beta L^e & 0 \end{pmatrix}\mathsf{C}e^{\lambda(t-\tau)}.$$

In consequence, for (10), we get the characteristic equation of the linearized system around (S^e, L^e),

$$\lambda^2 + \{-r(1-\frac{2S^e}{K}) + \alpha\beta L^e e^{-\lambda\tau}\}\lambda + (\alpha\beta)^2\phi S^e L^e e^{-\lambda\tau} = 0,$$

and substitute (8) into it to have

$$\lambda^2 + \{-(r + \frac{2cr}{\phi\alpha\beta K}) + (r + \frac{rc}{\phi\alpha\beta K})e^{-\lambda\tau}\}\lambda + (\alpha\beta)^2\phi S^e L^e e^{-\lambda\tau} = 0. \quad (12)$$

When $\tau = 0$, (12) becomes

$$\lambda^2 - \frac{cr}{\alpha\beta\phi K}\lambda + (\alpha\beta)^2\phi S^e L^e = 0. \quad (13)$$

Since $c < 0$ and $(\alpha\beta)^2\phi S^e L^e > 0$, all the characteristic roots of (13) have negative real parts, by which, in this case, the equilibrium point (S^e, L^e) is locally asymptotically stable for (10).

For the sake of notational simplify, we introduce new variables, p and q, defined as

$$p = r + \frac{rc}{\alpha\beta\phi K} \text{ and } q = (\alpha\beta)^2\phi S^e L^e > 0.$$

Substituting the new variables into (12), we can rewrite the characteristic equation with delay as

$$\lambda^2 + (-(2p - r) + pe^{-\lambda\tau})\lambda + qe^{-\lambda\tau} = 0. \quad (14)$$

Substituting $\lambda = iy$ into (14) gives

$$py \sin y\tau + q \cos y\tau = y^2, \quad (15)$$

$$py \cos y\tau - q \sin y\tau = (2p - r)y. \quad (16)$$

Squaring and adding (15) and (16) yields

$$y^4 + (3p^2 - 4pr + r^2)y^2 - q^2 = 0. \quad (17)$$

Let $Y = y^2 \geq 0$ where the direction of inequality is due to $y \in R$. By the way, if $Y = 0$, we have $q = 0$ which is a contradiction. Thus we are concerned with $Y > 0$ and have roots y_0 of (17),

$$y_0 = \pm\left(\frac{-(3p^2 - 4pr + r^2) + \sqrt{(3p^2 - 4pr + r^2)^2 + 4q^2}}{2}\right)^{\frac{1}{2}}. \quad (18)$$

From (15) and (16),

$$\cos(y_0\tau) = \frac{y_0^2(q + 2p^2 - pr)}{p^2y_0^2 + q^2} \text{ and } \sin(y_0\tau) = \frac{y_0\{py_0^2 - q(2p - r)\}}{p^2y_0^2 + q^2},$$

which imply that there is a τ_0 such that

$$\tau_0 = \frac{1}{|y_0|}\arcsin\frac{y_0^2(q + 2p^2 - pr)}{q^2 + p^2y_0^2} = \frac{1}{|y_0|}\arccos\frac{|y_0|\{py_0^2 - q(2p - r)\}}{p^2y_0^2 + q^2}. \quad (19)$$

Then we have the following theorem.

Theorem 2 *The delayed dynamical system (10) is unstable if* $\tau > \tau_0$ *where*

$$\tau_0 = \frac{1}{|y_0|}\arcsin\frac{y_0^2(q+2p^2-pr)}{q^2+p^2y_0^2} = \frac{1}{|y_0|}\arccos\frac{|y_0|\{py_0^2-q(2p-r)\}}{q^2+p^2y_0^2},$$

$$p = r + \frac{rc}{\alpha\beta\phi K} > 0,\ q = (\alpha\beta)^2\phi S^e L^e > 0$$

and

$$y_0 = \pm\left(\frac{-(3p^2-4pr+r^2)+\sqrt{(3p^2-4pr+r^2)^2+4q^2}}{2}\right)^{\frac{1}{2}}.$$

Proof. See Appendix. ■

To the best of our knowledge, a nonlinear dynamical system with time delay does not have an analytical solution.[5] Nevertheless, it is possible to examine its dynamical behavior by simulating the system numerically. Since we have performed two simulations without delay in the last section, we conduct simulations with production delay and then compare the results with time delay with the one without it. By doing so, we can detect effects on long-run dynamics of population and natural stocks caused by the delay in production.

Figure 3 presents simulation results that occur when the delay production is introduced into the first example, *ceteris paribus.* Real lines show time series of population and natural resource obtained in the current simulation while dotted lines are reproductions of simulation results depicted in Figure 1. With delay in production, the populations and the resource stock generates more volatile fluctuations relative to the ones without delay. Population reaches a much higher peak and much lower trough, jumping to 17,500 and falling quickly down to near zero while the natural resource declines more rapidly and gets closer to zero stock. These numerical simulations indicate the destabilizing effect caused delay in production in the oscillatory case.

[5] It is possible to construct an analytical solution when a nonlinear dynamical system is discrete. See Suzuki [1996, 2000].

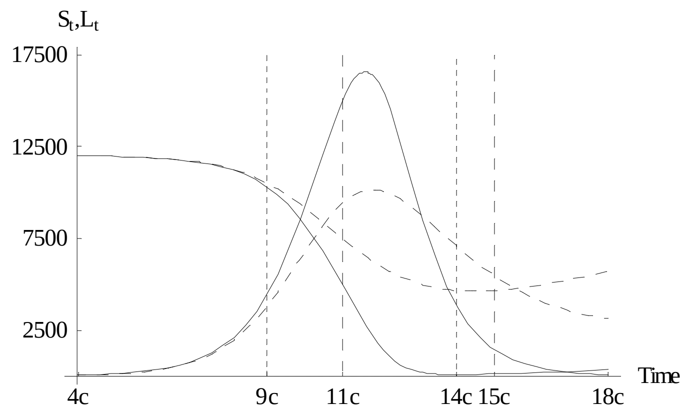


Figure 3. Production Delay in Fluctuating Economy

Figure 4 presents simulations that occur when the delay production is introduced into the second example, *ceteris paribus*. As in Figure 3, real lines show the simulation results with the delay in production and dotted lines the ones without it, which reproduce the results depicted in Figure 2. Comparing these results, we observe firstly the earlier and much more volatile fluctuations in population as well as natural resource; secondly the almost exhaustion of natural resource and the much more severe falls in population at the beginning of 18th century. This numerical simulations again indicate the destabilizing effect of the delay even on otherwise monotonic dynamics.

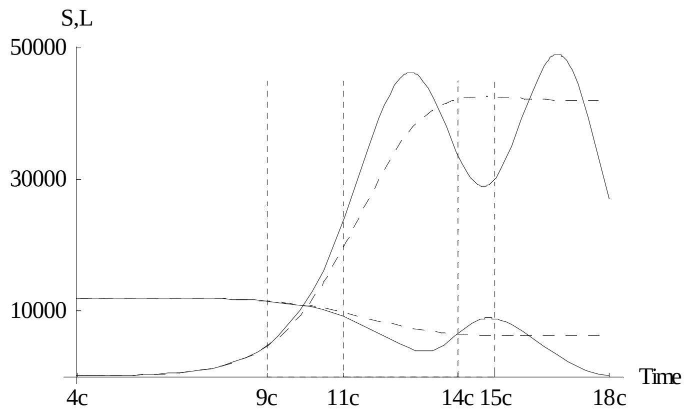
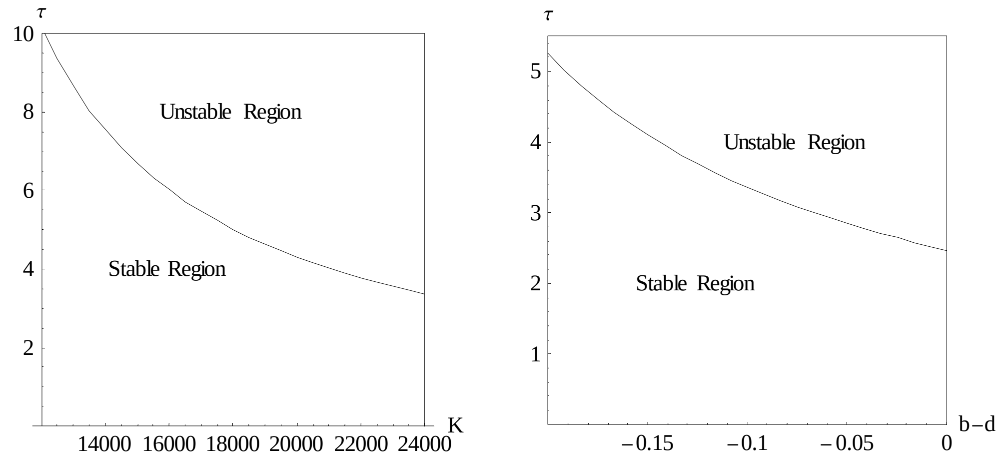


Figure 4. Production Delay in Stable Economy

In these numerical simulations, we adopt the parameterization used by BT for which the critical value of the delay is $\tau_0 = 10.1652$.[6] This is indeed a large delay, but we should not be too much concerned about this. As seen in (19), the critical value depends on any other parameters. We should thus be

[6]See Appendix for calculations.

content with the *existence* of the critical value, trusting that it might be considerately small under slightly different parameter specifications. Although it may be possible to derive analytically its dependency on parameters, we can expect the computations to become messy and thus confirm it numerically. Figure 5(A) illustrates the relationship between the delay and the maximum size of the natural resource, *ceteris paribus*. It displays the downward sloping borderline between stable region and unstable region. It is observed that the critical value of τ gets smaller as the size of K becomes larger. Figure 5(B) illustrates the relationship between the delay and the exogenous net birth rate. We can find the same property that τ_0 is negatively related to $b - d$. A combination of larger K and smaller $b - d$ in absolute value can lead to smaller τ_0.



(A) the size of resource stock (B) the net birth rate

Figure 5. Downward sloping τ_0 curves.

4 Concluding Remarks

This study has investigated the long-run dynamics of renewable natural resource and population introducing a delay in agricultural production activities. It first confirms that the basic mode without delay exhibits stable dynamics. Then it analytically derives a critical value of the delay in production for which the loss of stability occurs. Subsequent two numerical simulations show the destabilizing effect caused by the delay in production on evolutions of the small island economy. It makes fluctuations of the resource stock and the population much more widely over time when the growth rate of the natural resource is low (see Figure 3) and it also generates fluctuations even in a constantly-growing economy with the higher growth rate (see Figure 4).

Appendix

To prove Theorem 2, we apply the method used in Saito [2002. 116-122], which makes it possible to check the simplicity of a characteristic root on the imaginary axis without tedious calculations. Let

$$P(\lambda, \tau) = \lambda^2 + (-(2p - r) + pe^{-\lambda\tau})\lambda + qe^{-\lambda\tau},$$

and $\tau_n = \tau_0 + 2\pi n$ $(n = 0, 1, 2, \cdots)$. Then, $P(iy_0, \tau_n) = 0$ and we obtain the following from (14),

$$\frac{\partial P(iy_0, \tau_n)}{\partial \tau} = iy_0[-y_0^2 - (2p - r)iy_0],$$
$$\frac{\partial P(iy_0, \tau_n)}{\partial \lambda} = 2iy_0 - (2p - r) + [p - \tau_n(q + piy_0)]e^{-iy_0\tau_n}.$$

Clearly, $\frac{\partial P(iy_0,\tau_n)}{\partial \tau} \neq 0$. We now consider the following value:

$$K = \frac{p^2y_0^4 + 2q^2y_0^2 + (2p - r)^2q^2}{[((2p - r)y_0)^2 + y_0^4][q^2 + (py_0)^2]}.$$

We get $K > 0$. Furthermore, from (14),

$$\begin{aligned}
\mathrm{sign}K &= \mathrm{sign}\left[Re\left(\frac{2iy_0 - (2p - r)}{-iy_0(-y_0^2 - (2p - r)iy_0)} + \frac{p}{iy_0(q + piy_0)}\right)\right] \\
&= \mathrm{sign}\left[Re\left(\frac{2iy_0 - (2p - r)}{-iy_0(-y_0^2 - (2p - r)iy_0)} + \frac{pe^{-iy_0\tau_n}}{iy_0(q + piy_0)e^{-iy_0\tau_n}} - \frac{\tau_n}{iy_0}\right)\right] \\
&= \mathrm{sign}\left[Re\left(\frac{2iy_0 - (2p - r) + pe^{-iy_0\tau_n}}{-iy_0(-y_0^2 - (2p - r)iy_0)} - \frac{\tau_n}{iy_0}\right)\right] \\
&= \mathrm{sign}\left[Re\left(-\frac{\frac{\partial P(iy_0,\tau_n)}{\partial \lambda}}{\frac{\partial P(iy_0,\tau_n)}{\partial \tau}}\right)\right].
\end{aligned}$$

Hence, we can obtain $\frac{\partial P(iy_0,\tau_n)}{\partial \lambda} \neq 0$ and, by the well-known implicit function theorem, we have

$$\begin{aligned}
\mathrm{sign}\left[Re\left(\left.\frac{d\lambda}{d\tau}\right|_{\lambda = iy_0, \tau = \tau_n}\right)\right] &= \mathrm{sign}\left[Re\left(-\frac{\frac{\partial P(iy_0,\tau_n)}{\partial \tau}}{\frac{\partial P(iy_0,\tau_n)}{\partial \lambda}}\right)\right] \\
&= \mathrm{sign}\left[Re\left\{\left(-\frac{\frac{\partial P(iy_0,\tau_n)}{\partial \tau}}{\frac{\partial P(iy_0,\tau_n)}{\partial \lambda}}\right)^{-1}\right\}\right] \\
&= \mathrm{sign}K > 0.
\end{aligned}$$

This implies that (S^e, L^e) becomes unstable if $\tau > \tau_0$ holds (see, for example Kuang [1993]).

Under the current parameter specification in which $K = 12000$, $c = -0.1$, $r = 0.04$, $\phi = 4$, $\alpha = 0.00001$ and $\beta = 0.4$, we have $p = 0.0191667, q = 0.00191667$ and $y_0 = 0.0459087$. Then solving either $\arcsin[\tau_0 y_0] = 0.449917$ or $\arccos[0.466673] = \tau_0 y_0$ for τ_0 yields

$$\tau_0 = 10.1652.$$

If we take $\tau > 10.1652$, then the dynamical system (10) becomes unstable as shown in Figure 3 and Figure 4.

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