A Special Labor-Managed Oligopoly

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Abstract

A special labor-managed oligopoly without product differentiation is considered. The existence of the equilibrium is first proved and a simple example is presented to show the possibility of multiple equilibrium. The local asymptotic stability of the equilibria is next examined, the stability conditions are derived in both discrete and continuous time scales.

1 Introduction

The theory of oligopoly is one of the most frequently discussed subjects of mathematical economics. Since the pioneering work of Cournot (1838), many researchers examined the classical Cournot model and its extensions including models without and with product differentiation, multi-product oligopolies, rent-seeking games, labor-managed models, oligopolies with intertemporal demand interaction and models with product adjustment costs. A comprehensive summary of the most significant works up to the mid 70s is given in Okuguchi (1976). Multiproduct models with futher extensions and applications are disccussed in Okuguchi and Szidarovszky (1999), and nonlinear dynamic oligopolies are examined in detail in Bischi *et al.* (2009).

In this paper a special labor-managed model is examined. The seminal work of Ward (1958) is considered to be the first to introduce labor-managed oligopolies. Hill and Waterson (1983) have formulated profit-maximizing and labor-managed models without product differentiation and with symmetric firms and compared the long-term behavior of these models. Neary (1984) generalized this work to the nonsymmetric case. The existence and uniqueness of the equilibrium was proved by Okuguchi (1991) under rather restrictive conditions, furthermore in Okuguchi (1993) comparative statics are presented for profit-maximizing and labor-managed oligopolies. The existence of equilibrium was also shown in Okuguchi and Szidarovszky (1999) under slightly more general conditions, and the asymptotical properties of the equilibrium were also investigated with discrete and continuous time scales. These results were further

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generalized in Li and Szidarovszky (1999). Okuguchi and Szidarovszky (2008) have presented a general existence and uniqueness theorem for the equilibrium of labor-managed oligopolies.

In contrary to earlier works, the payoff functions are not concave and the strategy sets are not compact in the model to be presented in this paper, so general existence theorems cannot be used. This paper is organized as follows. In Section 2 the mathematical model will be presented and the existences of the equilibrium will be proved. The uniqueness of the equilibrium cannot be guaranteed in general as it will be demonstrated by a simple example. The local asymptotical stability of the equilibrium will be examined with discrete and continuous time scales in Section 3. Section 4 concludes the paper.

2 Mathematical Model and Existence Theorem

Consider a single product oligopoly without product differentiation. Let N denote the number of firms, and let L_k be the capacity limit of firm k. If x_k denotes the output of firm k and $Q = \sum_{k=1}^{N} x_k$ is the output of the industry, then the price function is assumed to be $f(Q) = A - BQ$ with $A > 0$ and $B > 0$. In order to guarantee nonnegative price, we assume that $A/B \geq \sum_{k=1}^{N} L_k$. The amount of labor necessary for producing output x_k by firm k is denoted by $h_k(x_k) = p_k x_k^2$. Let $d_k > 0$ be the fixed cost of firm k, and w the competitive wage rate. The surplus of firm k per unit labor is given as

$$\varphi_k(x_1, ..., x_N) = \frac{x_k(A - Bx_k - BQ_k) - wp_k x_k^2 - d_k}{p_k x_k^2}, \tag{1}$$

where $Q_k = \sum_{\ell \neq k} x_\ell$ is the output of the rest of the industry. With these notations, an N-person game can be defined in which the set of strategies of player k is the closed interval $[0, L_k]$ and its payoff function is φ_k. Notice that $\varphi_k \to -\infty$ as $x_k \to 0$, so φ_k is not defined for $x_k = 0$.

The best response of firm k can be obtained by maximizing φ_k with fixed value of Q_k. Since

$$\varphi_k(x_1, ..., x_N) = \frac{A - BQ_k}{p_k x_k} - \frac{d_k}{p_k x_k^2} - \left(\frac{B}{p_k} + w\right),$$

we have

$$\begin{aligned} \frac{\partial \varphi_k}{\partial x_k} &= -\frac{A - BQ_k}{p_k x_k^2} + \frac{2d_k}{p_k x_k^3} \\ &= \frac{1}{p_k x_k^3}\left(2d_k - (A - BQ_k)x_k\right). \end{aligned} \tag{2}$$

Therefore φ_k increases as $x_k < 2d_k/(A - BQ_k)$, decreases if $x_k > 2d_k/(A - BQ_k)$, so the best response of firm k is the following:

$$R_k(Q_k) = \begin{cases} x_k^* \text{ if } x_k^* \leq L_k \\ \\ L_k \text{ otherwise,} \end{cases} \tag{3}$$

where

$$x_k^* = \frac{2d_k}{A - BQ_k}.$$

The first case occurs when

$$Q_k \le \frac{AL_k - 2d_k}{BL_k}.$$

The graph of R_k is shown in Figure 1.

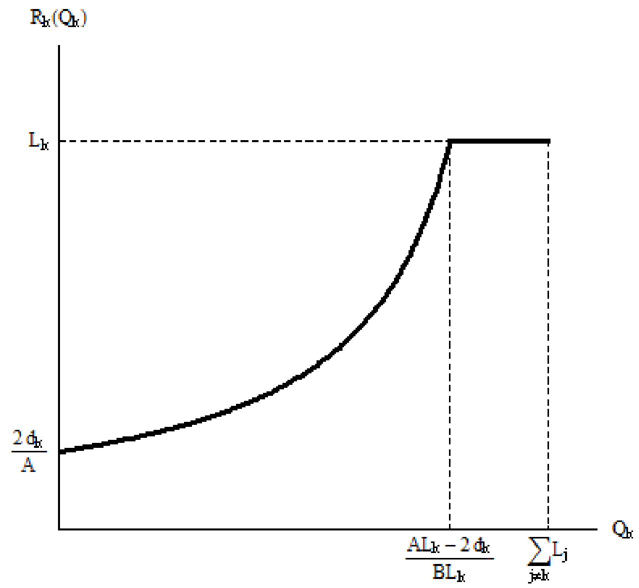


Figure 1. Piecewise continuous best response curve

Notice that the best response mapping $\left(R_1\left(\sum_{\ell\neq 1} x_\ell\right), ..., R_N\left(\sum_{\ell\neq N} x_\ell\right)\right)$ maps the compact, convex set $\prod_{k=1}^{N}[0, L_k]$ into itself, and since it is continuous, the Brouwer fixed point theorem guarantees the existence of at least one fixed point.

Similarly to classical Cournot models, we can present a constructive existence proof which can be also used to compute the equilibria.

If Q denotes the total output of the industry, then from (3), we have

$$x_k^* = \frac{2d_k}{A - BQ + Bx_k^*}$$

in the first case, so x_k^* solves the quadratic equation

$$Bx_k^{*2} + (A - BQ)x_k^* - 2d_k = 0 \tag{4}$$

and therefore

$$x_k^* = \frac{-(A - BQ) + \sqrt{(A - BQ)^2 + 8Bd_k}}{2B}.$$

Hence the best response of firm k can be rewritten as

$$\bar{R}_k(Q) = \begin{cases} x_k^* \text{ if } x_k^* \leq L_k, \\ \\ L_k \text{ otherwise.} \end{cases} \tag{5}$$

From (4) we conclude that $\bar{R}'_k(Q) = 0$ or

$$\begin{aligned} \bar{R}'_k(Q) &= \frac{1}{2B}\left(B - \frac{2(A-BQ)B}{2\sqrt{(A-BQ)^2 + 8Bd_k}}\right) \\ &= \frac{1}{2}\left(1 - \frac{A-BQ}{\sqrt{(A-BQ)^2 + 8Bd_k}}\right) \end{aligned}$$

implying that in this case

$$0 < \bar{R}'_k(Q) \leq \frac{1}{2}.$$

In the second case of (5), $\bar{R}''_k(Q) = 0$ and in the first case the sign of $\bar{R}''_k(Q)$ is the same as the sign of the following expression:

$$-\left\{(-B)\sqrt{(A-BQ)^2 + 8Bd_k} - (A-BQ)\frac{2(A-BQ)(-B)}{2\sqrt{(A-BQ)^2 + 8Bd_k}}\right\},$$

which is the same as the sign of

$$(A-BQ)^2 + 8Bd_k - (A-BQ)^2 = 8Bd_k > 0.$$

Therefore in the first case of (5), $\bar{R}_k(Q)$ strictly increases and is strictly convex, and its graph has a similar shape as $R_k(Q_k)$ shown in Figure 1. The equilibrium industry output is the solution of equation

$$\sum_{k=1}^{N} \bar{R}_k(Q) - Q = 0. \tag{6}$$

The left hand side is continuous, at $Q = 0$ it is nonnegative and at $Q = \sum_{k=1}^{N} L_k$ it is nonpositive implying the existence of at least one equilibrium.

The uniqueness of the equilibium cannot be guaranteed as it is shown in the following example.

Example 1. Let $N \geq 3$ be arbitrary, $L_k = 4$, $d_k = 8.5$ for all k, $A = 12$ and $B = \frac{2}{N-1}$. Clearly all conditions are satisfied: $A > 0$, $B > 0$ and

$$\sum_{k=1}^{N} L_k = 4N \leq \frac{A}{B} = 6(N-1).$$

We can show that both

$$\bar{x}_k^S = 3 - \sqrt{0.5} \text{ and } \bar{x}_k^M = 3 + \sqrt{0.5}$$

are symmetric interior equilibria. Notice first that for all k,

$$0 < \bar{x}_k^S < L_k \text{ and } 0 < \bar{x}_k^M < L_k,$$

and they satisfy the best response equations $x_k = R_k(Q_k)$, since

$$\bar{Q}_k^S = (N-1)(3-\sqrt{0.5}) \text{ and } \bar{Q}_k^M = (N-1)(3+\sqrt{0.5}),$$

furthermore in these cases

$$R_k(\bar{Q}_k) = \frac{2d_k}{A - B\bar{Q}_k} = \frac{17}{12 - 2(3 \mp \sqrt{0.5})} = 3 \mp \sqrt{0.5} = \bar{x}_k.$$

It can be also shown that $\bar{x}_k^L = L_k$ is a symmetric boundary equilibrium, since in this case the second case of (3) occurs:

$$x_k^* = \frac{17}{12 - \frac{2}{N-1}4(N-1)} = \frac{17}{4} > 4 = L_k.$$

3 Local Stability Analysis

From equation (3) we see that $R_k'(Q_k) = 0$ or

$$R_k'(Q_k) = \frac{2d_k B}{(A - BQ_k)^2}. \tag{7}$$

Consider the second case. At the best response

$$R_k'(Q_k) = \frac{x_k^{*2} B}{2d_k} = \frac{2d_k - (A - BQ)x_k^*}{2d_k},$$

where we used equation (4). This expression implies that

$$0 < R_k'(Q_k) \leq 1.$$

Consider first continuous time scales and assume that the firms use adaptive adjustment toward best responses. Then their outputs satisfy the differential equation system

$$\dot{x}_k = K_k \left(R_k \left(\sum_{\ell \neq k}^{N} x_\ell \right) - x_k \right) \text{ for } k = 1, 2, ..., N, \tag{8}$$

where $K_k > 0$ is the speed of adjustment of firm k. The equilibria of the labor-managed oligopoly game coincide with the steady states of this system. The local asymptotic stability of this system can be examined by linearization. The Jacobian has the special form

$$\mathbf{J} = \begin{pmatrix} -K_1 & K_1\bar{r}_1 & \cdot & K_1\bar{r}_1 \\ K_2\bar{r}_2 & -K_2 & \cdot & K_2\bar{r}_2 \\ \cdot & \cdot & \cdot & \cdot \\ K_N\bar{r}_N & K_N\bar{r}_N & \cdot & -K_N \end{pmatrix} = \mathbf{D} + \mathbf{a}\mathbf{1}^T,$$

where $\bar{r}_k$ is the value of R'_k at the equilibrium,

$$\mathbf{D} = diag\left(-K_1(1+\bar{r}_1), ..., -K_N(1+\bar{r}_N)\right),$$

$$\mathbf{a} = \left(K_1\bar{r}_1, ..., K_N\bar{r}_N\right)^T$$

and

$$\mathbf{1}^T = (1, ..., 1).$$

The characteristic polynomial of this matrix can be written in the following form:

$$\begin{aligned} \varphi(\lambda) &= \det(\mathbf{D} + \mathbf{a}\mathbf{1}^T - \lambda\mathbf{I}) \\ &= \det(\mathbf{D} - \lambda\mathbf{I}) \det(\mathbf{I} + (\mathbf{D} - \lambda\mathbf{I})^{-1}\mathbf{a}\mathbf{1}^T) \\ &= \prod_{k=1}^{N} \left(-K_k(1+\bar{r}_k) - \lambda\right) \left(1 - \sum_{k=1}^{N} \frac{K_k\bar{r}_k}{K_k(1+\bar{r}_k)+\lambda}\right). \end{aligned} \tag{9}$$

For the sake of simplicity, we assume that the $K_k(1+\bar{r}_k)$ values are different, the other case can be discussed in the same way. The eigenvalues are $\lambda = -K_k(1+\bar{r}_k) < 0$ and the roots of equation

$$g(\lambda) = \sum_{k=1}^{N} \frac{K_k\bar{r}_k}{K_k(1+\bar{r}_k)+\lambda} - 1 = 0. \tag{10}$$

It is easy to see that

$$\lim_{\lambda\to\pm\infty} g(\lambda) = -1, \quad \lim_{\lambda\to -K_k(1+\bar{r}_k)\pm 0} g(\lambda) = \pm\infty$$

and

$$g'(\lambda) = \sum_{k=1}^{N} \frac{-K_k\bar{r}_k}{\left(K_k(1+\bar{r}_k)+\lambda\right)^2} < 0,$$

so g strictly decreases locally. All poles are negative, and since equation (10) is equivalent to a polynomial equation of degree N, there are N real or complex roots. There is one root between each adjacent pair of poles and one additional root after the largest pole. So we found all roots, they are real, and they are negative if $g(0) < 0$ or

$$\sum_{k=1}^{N} \frac{\bar{r}_k}{1+\bar{r}_k} < 1. \tag{11}$$

Consider next discrete time scales. Then system (9) is replaced by the following system of difference equations:

$$x_k(t+1) = x_k(t) + K_k\left(R_k\left(\sum_{\ell\neq k}^{N} x_\ell(t)\right) - x_k(t)\right) \text{ for } k = 1, 2, ..., N, \tag{12}$$

where $0 < K_k \leq 1$. The Jacobian of this system is $\mathbf{I}+\mathbf{J}$, the eigenvalues of which are inside the unit circle if the eigenvalues of $\mathbf{J}$ are between -2 and 0 which is the case if all poles are larger than -2 and $g(0) < 0$. These conditions can be rewritten as

$$K_k(1 + \bar{r}_k) < 2 \tag{13}$$

and

$$\sum_{k=1}^{N} \frac{\bar{r}_k}{1 + \bar{r}_k} < 1. \tag{14}$$

Notice first that the stability condition (11) (or (14)) does not depend on the speeds of the adjustments of the firms. They are satisfied if the $\bar{r}_k$ derivative values are sufficiently small. Since $0 < \bar{r}_k \leq 1$ and by resonable assumption $0 < K_k \leq 1$, condition (13) is almost always satisfied. The only exception is the case of best response dynamics with $Q = A/B$ where all firms receive zero price. This is a very unrealistic equilibrium. Hence the stability conditions for discrete time and continuous time systems are identical. In the case of duopolies and $Q \neq A/B$, condition (11) (or (14)) holds, since $\bar{r}_k/(1 + \bar{r}_k) < 1/2$, showing the local asymptotic stability of the equilibrium with both constinuous and discrete time scales.

If $N \geq 3$, the model may have multiple equilibria and the local stability of these equilibria may not be guaranteed as shown in the following example.

Example 2. Take $N = 3$. The model specified in Example 1 has three equilibria:

$$\bar{x}_k^S = 3 - \sqrt{0.5},\ \bar{x}_k^M = 3 + \sqrt{0.5} \text{ and } \bar{x}_k^L = L_k$$

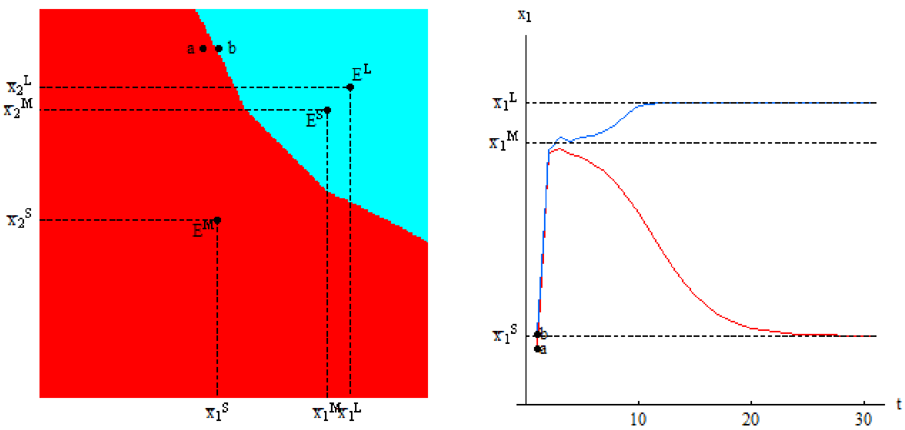
which we call the smallest, middle and largest equilibria. It is apparent that the largest equilibrium is locally stable as $R_k'(Q_k) = 0$. To examine the stability of the other two equilibria, we calculate the derivative values evaluated at the equilibria,

$$\bar{r}_k^S = \frac{17}{(6 + \sqrt{2})^2} \simeq 0.31 \text{ and } \bar{r}_k^M = \frac{17}{(-6 + \sqrt{2})^2} \simeq 0.81.$$

According to (11) (or (14)), the smallest equilibrium is locally stable whereas the middle equilibrium is locally unstable.

Example 2 confirms the local stability of the smallest and the largest equilibrium but does not say anything about the global behavior of the trajectories. The basin of attraction for the discrete dynamic system (12) with $N = 3$, $K_k = 0.8$ for all k and $x_3(0) = 5.1$ is illustrated in Figure 2. The shape of the basin depends on the value of $x_3(0)$. It is the set of points in the output space (x_1, x_2) such that initial points chosen in the red (darker) region converges to the smallest equilibrium and those in the blue (lighter) region evolve to the largest equilibrium. We choose two points, denoted as a in the red region and b in the blue region in Figure 2A, and perform numerical simulations to confirm the convergence, which are depicted in Figure 2B. In particular, although the initial point a is close to b, the trajectory starting at point a eventually converges to the smallest equilibrium and so does the trajectory starting at point

b to the largest equiliubrium.



(A) Basin of Attraction (B) Globally stable time trajectories

Figure 2. Global stability

4 Conclusions

The existence of a special labor-managed oligopoly is proved by using fixed point theorems and also by introducing a solution algorithm. The uniqueness of the equilibrium is not guaranteed in general. Conditions are derived for the local asymptotical stability of the equilibrium, and a simple numerical example illustrates the theoretical results.

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