Coexistence of Multiple Business Cycles in Goodwin's 1951 Model*

Akio Matsumoto
Department of Economics
Chuo University

Mami Suzuki
Department of Management
Aichi-Gakusen University

January 15, 2006

Abstract

It is well-known that Goodwin's 1951 business cycle model gives rise to cyclical fluctuations when its stationary point is unstable. This study shows coexistence of multiple cycles when it is stable: the stable region of a stationary point is bounded by a unstable limit cycle that is, in turn, surrounded by a stable limit cycle. In this case, the model is globally stable in a sense that it does not diverge for any disturbances but its dynamics is different for different disturbances. That is, dynamics is stable and returns to the stationary point for small disturbances but unstable and exhibits persistent fluctuations for large disturbances. This result implies two issues: Goodwin's mode has the robustness of cyclic fluctuations regardless of local dynamical properties and corridor stability.

Key words: corridor stability, coexistence of multiple limit cycles, nonlinear delayed differential equation, hopf bifurcation.

*The earlier version of the paper, entitled "Reconsiderations of Goodwin's Model," was presented at the Goodwin workshop in EAEPE conference held in Bremen, Germany, on 10-12 November 2005, and Chuo Meeting for Economics of Time and Space (Chuo-Mets) held in Tokyo, Japan, on 30-31 August 2005. Both authors appreciate financial supports from Ministry of Education, Culture, Sports, Science and Technology (Grand-in-Aid for Scientifit Reseach B, 15330037) and Chuo University (Joint Reseach Project 0382). The paper benefited from comments by Serena Sordi, Peter Flaschel, Reiner Franke and Willi Semmler. The usual disclaimer applies.

1 Introduction

In this study, we re-examine the dynamics of the business cycle model constructed by Goodwin (1951) and exhibit coexistence of multiple limit cycles when a stationary point is locally stable. Goodwin develops a nonlinear accelerator business cycle model and shows that it can bring about a stable limit cycle when the stationary state is locally unstable. This study focuses on the unstable case on which very few attempts have been made. Thus it aims to complement Goodwin's dynamical analysis on nonlinear business cycle.

Goodwin proposes five different versions of his business cycle model. The first version assumes the piecewise linear function with three levels of investment. It is a text-book model that can give a simple exhibition of the value non-linearities have for generating endogenous cycles without relying on structually unstable parameters, exogenous shocks, etc. The piecewise linear investment function, which can be thought as the crudest or simplest of the non-linear accelerator, is replaced with a smooth nonlinear investment function in the second version. Cyclical fluctuations of output are shown to exist and be independent of initial conditions. However the second version includes a unfavorable phenomenon, namely, investments jump discontinuously, which is not observed in the real economic world. "In order to come close to reality" (p.11, Goodwin(1951)), the production lag is introduced in the third version. Although no dynamical considerations are given to the third version, an existence of a stable limit cycle is examined in the forth version, which is a linear approximation of the third version with respect to the production lag. The fifth version is recently reconsidered by Lorenze (1987) as a forced oscillator system in which the emergence of chaotic motion is demonstrated.

Thus far, a lot of efforts have been devoted to investigate the dynamic structure of the fourth version. Recently Sasakura (1996) gives an elegant proof of the stability and uniqueness of Goodwin's cycle. Thus it has been confirmed that Goodwin's model possesses a unique stable limit cycle. Since all these results are obtained under the condition for which a stationary point is locally unstable, we can ask a natural question: *is cyclic behavior robust under locally stable circumstances?*

The main result of this study is to provide a positive answer to this question. For this purpose, we augment the fourth version by introducing an nonlinear investment function of arctangent type and demonstrate coexistence of a stable stationary point, a unstable limit cycle, and a stable limit cycle. The Poincaré-Bendixson theorem is a usual procedure for detecting a cycle in a two-dimensional continuous dynamical model whose trajectories are unstable in a vicinity of the stationary point. However, the theorem is rendered inapplicable since we deal with the local stability case. We combine the theorem with the Hopf bifurcation theorem and then characterize the global dynamics to show coexistence. Coexistence of multiple cycles is also shown in Kaldor's business cycle model by Grasman and Wentzel (1994) and in a Metzlerian inventory cycle model by Matsumoto (1996) whose approach we resume here.

We also turn our attentions to dynamics of the second version with a unstable stationary point to which only limited efforts has been devoted. Since it is a nonlinear differential equation, analytical considerations seem fruitless. Yet, we perform numerical simulations to find what effects the production lag produces on the characteristics (i.e., length and amplitude) of cyclical fluctuations of output.

In what follows, Section 2 overviews three versions from the second to the fourth of Goodwin's business cycle model and reveals the characteristics of Goodwin's cycle. Section 3 points to the new results that the approximated version exhibits corridor stability in which it could be stable for small shocks but unstable and generates multiple limit cycles for large shocks. Section 4 makes concluding remarks.

2 Goodwin Business Cycle Model

This section is divided into three parts. Each of three versions of Goodwin's model are reviewed in each subsection. In particular, we recapitulate the basic element of the second version. Assuming an explicit form of the nonlinear investment function, we numerically simulate the model to see what dynamics it can generate in Section 2.1. Then we introduce the production lag into the second version to get the third version and perform, again, numerical simulations to find how the lag affects the characteristics(i.e., the length of a period and the amplitude) of endogenous cycles in Section 2.2. Finally, we derive the most popular version by expanding the third version with respect to the lag and reveal the stability condition in Section 2.3.

2.1 Basic Model

The second model, which we call the basic model, is summarized as follows.

$$\begin{cases} \varepsilon\dot{y}(t) = \dot{k}(t) - (1-\alpha)y(t), \\ \dot{k}(t) = \varphi(\dot{y}(t)). \end{cases} \tag{1}$$

k is capital stock, y income, α the marginal propensity to consume to be positive and less than unity, and ε the output adjustment coefficient. The first equation defines an adjustment process of national income in a such a way that national income rises or falls if and only if investment is greater or less than savings. The second is an adjustment process of capital stock based on the acceleration principle, according to which investment depends on the rate of change in national income. On the other hand, we depart from Goodwin's non-essential assumption of positive autonomous expenditure and will work with zero-autonomous expenditure for the reason of simplicity. An direct consequence is that an equilibrium solution or a stationary point of the basic model is $y(t) = \dot{y}(t) = 0$ for all t. Inserting the second equation into the first and arranging terms gives dynamics of the national income,

$$\varepsilon\dot{y}(t) - \varphi(\dot{y}(t)) + (1-\alpha)y(t) = 0. \tag{2a}$$

This is a first-order nonlinear differential equation. Once we specify an explicit form of the investment function and set an initial income, dynamics of y is accordingly determined.

About the induced investment, $\varphi(\dot{y}(t))$, we adapt a *non-linear* acceleration principle, by which we mean that the investment can be considered to be proportional to the change in national income in a neighborhood of the equilibrium income but get inflexible (i.e., less elastic) for the extreme large or small values of income. Goodwin assumes a piecewise linear investment function in his simulations. Following his spirit, but for the sake of analytical convenience, we assume a smooth nonlinear investment function;

Assumption 1. The investment function takes a form of an arctangent type,

$$\varphi(\dot{y}(t)) = \delta\left\{\tanh(\dot{y}(t)-a) - \tanh(-a)\right\}, \quad \delta > 0 \text{ and } a > 0.$$

This function has a "ceiling" and a "floor" and is asymmetric when a parameter, a, is a non-zero. In what follows, we set $a = 1$ to shift the function upward by $\frac{\pi}{4}$ and rightward so as to make the function pass through the origin and $\delta = \frac{12}{\pi}$ so that the ceiling is three time higher than the floor as in Goodwin. We assume, in this and the next subsection, the stationary point to be locally unstable by making

Assumption 2. $\dfrac{d\varphi(0)}{d\dot{y}}$ is greater than ε.

To find out what dynamics the basic model can generate, we perform numerical simulations. An initial point is set at $y(0) = 10$, and in throughout numerical examples below, all the parameters are fixed at $\varepsilon = 0.5$ and $\alpha = 0.6$ as in Goodwin. The numerical results are given in Figure 1 in which an endogenous cycle is illustrated in its right and the corresponding time path in its left. Along the mirror imaged N-shaped $\dot{y} = 0$ locus in the right, the initial point denoted by I_0 is displaced slightly upward to point A, at which investment switches discontinuously from positive to negative. In consequence the orbit jumps from point A to point B. With negative $\dot{y}(t)$ at point B, the national income gradually declines from point B to point C. Once point C is reached, investment, again, switches discontinuously from negative to positive. In other word, the orbit jumps again to point D from point C, from which the national income glides toward point A, and then the process repeats itself. Thus we have a closed orbit constituting a self-sustaining cycle. The points A and C are critical points at which one of the variables makes a discontinuous jump. The same dynamics is depicted from a different view point in the left of Figure 1. There the corresponding time path form $t = 0$ to $t = 20$ is illustrated. It can be seen that the time path has a kink at the turning point, which is an alternative expression of the discontinuous jump. If we interpret one time period to be one calendar year as in Goodwin, five cycles are observed in 20 years in the left. The length of one business cycle in this simulation is approximately 4 years. The numerical simulations imply that the basic model can generate an endogenous cycle of output due to the nonlinearity involved in the investment accelerator, and investment suffers discontinuous jumps. We sum up the results.

Proposition 1 *Due to the nonlinear investment accelerator, a slow-rapid limit cycle is brought about in the basic model.*

Insert Figure 1 Here.

2.2 Delayed Model

In order to come closer to reality and to get rid of discontinuous jumps, we introduce the production lag, θ, between decisions to invest and the corresponding outlays into the basic model. As a result, the third version is

$$\epsilon\dot{y}(t+\theta) + (1-\alpha)y(t+\theta) = \varphi(\dot{y}(t)). \tag{3}$$

This is a first-order delayed nonlinear differential equation, which we call the delayed model. Since a cycle has been shown to exist, our next concern is whether such a lag affects characteristics of the slow-rapid cycle observed in the basic model. To put it more concretely, we will address the questions: *what effects the lag causes on the kink, length and amplitude of the cyclical fluctuations?*

Goodwin refers nothing to dynamics generated by the delayed model. Furthermore, to the best of our knowledge, any analytical solutions of the delayed model are not successfully constructed yet. However, it is possible to investigate dynamics of the model with the aid of numerical simulations. For this reason, we conduct simulations with various values of the production lag and then compare the one result with the other. By doing so, we can reveal what effects the lag can produce.

In Figure 2, we exhibit four time paths from $t = 0$ to $t = 20$ associated with different values of θ in the left and the corresponding limit cycles in the right.[1] For all these simulations, we choose the same initial condition (i.e., $y(0) = 10$) and the same parameter values as in the last simulations but the different values of the production lags. As can be observed in the left of each figure, the length of a one-period gets longer as the production lag gets larger.[2] To put it more concretely, we have the following numerical results. In Figure 2A, $\theta = 0.125$ and there are four cycles within the 20 time-period, so that one cycle is lengthened to 5 years. In Figure 2B, the lag is doubled to be $\theta = 0.25$ and there are approximately three and a quarter cycles implying the 6 years cycle. In Figure 2C, $\theta = 0.5$ and there approximately two and a three-quarter cycles, so that the length of the cycle is approximately 7 years. In Figure 2D, $\theta = 1$ and there are a bit-longer-than-two cycles, so that the length of the cycle is 9 years or so. It is found from these simulation results with the one of the basic model that the length of the business cycles changes, roughly speaking, from 4 years to 10 years according as the length of production lag gets larger from zero to one. In his simulations, Goodwin points out that the length of the whole cycle is 10.7 years, the upswing is prolonged to 6.5 years and the downswing shorted to 4.2 years. Although it is not clear from these time paths in Figure 2 below, the asymmetry of the upswing and downswing is also observed in our simulations. When $\theta = 1$, careful observation reveals that the cycle has a 9-years period, its upswing is lengthened to 4.8 years and its downswing to 4.2 years. The similar results

[1] Since one unit of time is assumed to equal to one year, "20 years" is long enough to see the lag-effects on business cycles.

[2] As explained in Goodwin (1951, p.15), there are economic justification to set values of structural parameters at $\epsilon = 0.5$, $\alpha = 0.6$ and $\theta = 1$. It may be unfair to Goodwin to change values of θ, taking other parameter values equal. We, however, believe to derive the qualitatively similar results even if we make proper adjustments.

are obtainable in other simulations.

Insert Figure 2 Here

In the right of each figure, a limit cycle is illustrated in the phase space in which we plot the national income on the vertical axis and its time derivative on the horizontal axis. Figure 3 juxtaposes these limit cycles. Comparing these limit cycles, we find that the limit cycle changes its shape from a parallelogram-wise cycle to a vertically longer ellipse as the production lag increases from zero to unity (that is, the width of the cycle gets smaller and the length gets longer). Since the peak and bottom of the output-cycle are reached at the point where the time derivative is zero, the amplitude of the cycle is the distance between two points at which the limit cycle crosses the vertical axis. We also find that the amplitude of the cycles grows larger and the rate of output change becomes less as the production lag gets longer. These numerical results are summarized in

Proposition 2 *In the delayed model, the production lag has the effects to make the length of the business cycle longer and the amplitude larger.*

Insert Figure 3 Here.

2.3 Approximated Model

To investigate the effect of the production lag on the emergence of cyclical development of output, Goodwin expands the delayed nonlinear differential equation with respect to θ and then gets the following second-order nonlinear differential equation as his forth version,

$$\epsilon\theta\ddot{y}(t) + [\epsilon + (1-\alpha)\theta]\dot{y}(t) - \varphi(\dot{y}(t)) + (1-\alpha)y(t) = 0. \tag{4}$$

We call it the approximated model. One way to learn about the movement of output is to integrate numerically on a computer for any given set of initial conditions on y and $\dot{y}$. Our interest in this subsection is not on the specifics of how y wonders about in the short run but is in understanding whether the displaced output eventually return to its stationary point or not. Namely, we raise the question of the stability, which can be resolved without solving the nonlinear equation.

To this purpose, we introduce a new variable, $x(t) = \dot{y}(t)$, and then transform the second-order nonlinear differential equation into a two-dimensional nonlinear system that can be put, after arranging terms, into the form,

$$\begin{cases} \dot{x}(t) = \dfrac{1}{\epsilon\theta}\left\{\varphi(x(t)) - [\epsilon + (1-\alpha)\theta]x(t) - (1-\alpha)y(t)\right\}, \\ \\ \dot{y}(t) = x(t). \end{cases} \tag{5}$$

On the (x, y) plane, the $\dot{y}(t) = 0$ locus is identical with the vertical axis while the $\dot{x}(t) = 0$ locus becomes the mirror imaged N-shaped curve and passes through the origin. Thus the 2D-dynamical system possesses a unique stationary point, $(x, y) = (0, 0)$, at which both loci cross each other.

We now proceed to stability analysis of the approximated system to give insight into the qualitative behavior in a neighborhood of the stationary point. By developing a local version of the system around the stationary point, we obtain the Jacobi matrix of the planar system,

$$J = \begin{bmatrix} \dfrac{1}{\epsilon\theta}\left\{\varphi^{\prime}(0) - [\epsilon + (1-\alpha)\theta]\right\} & -\dfrac{1-\alpha}{\epsilon\theta} \\ \\ 1 & 0 \end{bmatrix}.$$

The characteristic equation is

$$\lambda^2 - \frac{k}{\epsilon\theta}\lambda + \frac{1-\alpha}{\epsilon\theta} = 0,$$

where we set $k = \varphi^{\prime}(0) - [\epsilon + (1-\alpha)\theta]$. The characteristic roots are computed to be

$$\lambda_{1,2} = \frac{1}{2}\left\{\frac{k}{\epsilon\theta} \pm \sqrt{\left(\frac{k}{\epsilon\theta}\right)^2 - \frac{4(1-\alpha)}{\epsilon\theta}}\right\}.$$

It follows that the product of the characteristic roots is positive due to the assumptions imposed on parameters, $0 < \alpha < 1$ and $(\epsilon, \theta) > 0$,

$$\lambda_1\lambda_2 = \frac{1-\alpha}{\epsilon\theta} > 0,$$

and thus excludes that the stationary point is a saddle point. It also follows that the sum of the characteristic root can be either positive or negative according to whether the slope of the investment function at the stationary point is larger or smaller than the sum of other parameter values,

$$\lambda_1 + \lambda_2 = \frac{1}{\epsilon\theta}\left\{\varphi^{\prime}(0) - [\epsilon + (1-\alpha)\theta]\right\} \gtrless 0.$$

The stationary point is locally stable or unstable according to whether the sum of terms in braces is negative or positive.

Putting this result in a different way, we can say that the parameter region is divided into two subregions, the stable one and the unstable one, by the critical line satisfying

$$v = [\epsilon + (1 - \alpha)\theta], \tag{6}$$

where $v = \varphi^{\theta}(0)$. It depends on the value of the discriminant of the characteristic equation whether the local dynamics is oscillatory or monotonic. The locus along which the discriminant is zero is determined by

$$v = [\epsilon + (1 - \alpha)\theta] \pm 2\sqrt{(1 - \alpha)\varepsilon\theta}, \tag{7}$$

that distinguishes the parameter region for real roots from the one for complex roots. These two loci divide the parameter region as shown in Figure 4 in which MS and MU mean monotonic stable and unstable while OS and OU mean oscillatory stable and unstable. There, we plot θ on the horizontal axis and v on the vertical axis. For combinations of parameters, θ and v, in either the light-gray region or the dark-gray region, the characteristic roots are complex and thus the system produces fluctuations. Moreover, the fluctuations are explosive for the combinations in the light-gray region and damped for the one in the dark-gray region. On the other hand, for the combinations of θ and v in the white regions, the characteristic roots are real and thus the system produces monotonic dynamics that is convergent or divergent according to whether the combination is in the lower-white region or the upper-white region. Figure 4 also implies that the approximated model may produce qualitatively the same dynamics as v increases irrespective of a choice of the production lag. Namely, given θ, it generates stable monotonic dynamics, stable oscillatory dynamics, unstable oscillatory dynamics and then stable monotonic dynamics as v increases from zero.

Insert Figure 4 Here

Thus far, there have been a lot of studies that focus on cyclic dynamics of the approximated model when the stationary point is locally unstable, that is, the parameters are selected from the region above the critical line (6) of Figure 4. On the other hand, little attention has been given to the case in which the stationary point is stable. In consequence, little is known about dynamics for the parameter configurations selected from the region below the critical line (6). We move one step forward and address the following question: *what can be said about the global dynamics of the approximated model when the stationary point is locally stable?*

3 Coexistence of Multiple Cycles

To answer the question we raise just above, we first assume that the fixed point is locally stable and then demonstrate coexistence of a stable stationary point, a unstable limit cycle and a stable limit cycle. In what follows, we select a slope of the investment function evaluated at the stationary point as a bifurcation parameter and concern a possibility of the supercritical bifurcation in Section 3.1. With use of an unstable limit cycle to be shown to exist, we construct an invariant set in the state space and apply the Poincaré-Bendixson theorem to find a stable limit cycle that encloses the unstable limit cycle in Section 3.2.

3.1 Hopf Cycle

We investigate, with the help of Hopf bifurcation theorem, whether there exists an unstable limit cycle in the approximated model. According to the theorem, the Hopf bifurcation occurs if the complex conjugate roots cross the imaginary axis. Apparently, the characteristic roots are complex conjugate with zero real part if $k = 0$. As there are no other roots in two-dimensional system, the consideration of the existence of closed orbits is complete if the eigenvalues cross the imaginary axis with non-zero speed at the bifurcation point. Though there may exist several possibilities to parametrize the approximated model, it seems interesting to choose the slope of the investment function evaluated at the stationary point as the bifurcation parameter.

Since we assume the local stability (i.e., $\varphi^{\theta}(0) < \epsilon + (1 - \alpha)\theta$ where α, ϵ and θ are given) in this section, it can be directly seen that there exists a value $v_0 = \varphi^{\theta}(0)$ for which

$$v_0 - [\epsilon + (1 - \alpha)\theta] = 0,$$

implying that the complex conjugate cross the imaginary axis. As for $v > v_0$, respectively $v < v_0$, the real part becomes positive, respectively negative. Hence, v_0 is indeed a bifurcation value of the approximated model. According to the Hopf bifurcation theorem, these establishes the existence of closed orbits in a neighborhood of the stationary point $(0, 0)$ at $v = v_0$.[3]

The Hopf theorem, however, is somewhat ambiguous about the nature of the closed orbit. There are two possibilities, one is that orbits spiral outward from the stationary point toward a stable limit cycle, namely, the *subcritical*

[3]See Lorenz (1993) for the Hopf bifurcation theorem, the stability index and the following coordinate transformations..

bifurcation, and the other is that all orbits starting inside the cycle spiral in toward the stationary point, namely, the *supercritical bifurcation*. To make the distinction between the sub- and super-critical Hopf bifurcation, we check the stability of the cycle by calculating the stability index.

The approximated model can be written as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = J \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x) \\ 0 \end{pmatrix}$$

where J is the Jacobian defined above, and $f(x)$ is a nonlinear term that can be derived as

$$f(x) = \frac{1}{\varepsilon\theta} \{\varphi(x) - \varphi'(0)x\}.$$

In order to transform the approximated model into the normal form, we make the coordinate transformation,

$$\begin{pmatrix} x \\ y \end{pmatrix} = D \begin{pmatrix} u \\ v \end{pmatrix} \text{ with } D = \begin{pmatrix} 0 & 1 \\ -\sqrt{\dfrac{\varepsilon\theta}{1-\alpha}} & 0 \end{pmatrix}.$$

Since the matrix D transforms the coordinate system (x, y) into a new coordinate system (u, v), the approximated model is put into

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & -\sqrt{\dfrac{1-\alpha}{\varepsilon\theta}} \\ \sqrt{\dfrac{1-\alpha}{\varepsilon\theta}} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} 0 \\ f(v) \end{pmatrix}$$

It is well-known that the stability of the emerging cycle depends on up to third-order derivatives of the nonlinear function, $f(v)$. In the approximated model, the stability index is calculated as

$$I = \frac{1}{16\varepsilon\theta}\varphi'''(0) > 0.$$

The direction of the inequality indicates that the emerging cycle encloses the stable stationary point and is repelling (i.e., unstable). It follows that a subcritical Hopf bifurcation occurs for $v < v_0$ in the approximated model. We sum up the result obtained.

Proposition 3 *Given the stability of the stationary point, there exists an unstable limit cycle that encloses a stable equilibrium in the approximated model.*

3.2 Stable and Unstable Limit Cycles

Taking v to be less than v_0 for which the stationary point is stable, we concern a number of a limit cycle in this subsection. As shown above, a unstable limit cycle exists for $v < v_0$ due to the Hopf theorem. We proceed to show an existence of a stable limit cycle that encloses the unstable Hopf cycle when the stationary point is locally stable.

We construct an invariant set in such a way that once an orbit enters the set, it cannot later escape at some future time and then apply the Poincaré-Bendixon theorem to examine whether a stable cycle can arise in the set. A typical application of the theorem is to the case in which there is a single unstable equilibrium in some invariant set. Remove a neighborhood of the stationary point in which all orbits move away from it. Then the theorem ensures that the remaining set must contain a limit cycle.[4] Since the stationary point is now assumed to be stable in the approximated model, we need another considerations for searching for such an invariant set.

In Figure 5, we may find a point A so high on the vertical axis that an orbit starting there comes back to meet the vertical axis at a lower point E after crossing not only the horizontal axis at points B and D but also the vertical axis at point C. Consider a region bounded by $ABCDEA$ and an open region bounded by the unstable limit cycle that is shown to exist in Section 3.1. Then the invariant set to be considered can be constructed by deleting the latter region from the former region. The result is shown in Figure 5 in which the shaded region represents the invariant set. The boundary of the inner white region corresponds to the unstable limit cycle that srrounds the stable stationary point. The shaded region thus has no stationary point, and any orbit starting inside the region stays within its inside. Two orbits, one starting at point a and the other at point b, are examples that are shown to remain inside of the invariant set and converge to the same limit cycle, illustrated as a bold curve. On the other hand, it can be seen that a dotted orbit starting at point c located inside of the white region spirals toward the stationary point. Formally, the Poincaré-Bendixson theorem guarantees an existence of one stable cycle in the shaded region. Summing up the results with Proposition 3 yields the following results.

Proposition 4 *Given the stability of the stationary point, a stable limit cycle coexists with an unstable limit cycle that encloses a stable equilibrium point in the approximated model.*

Insert Figure 5 Here

[4] See Chaper 2.2 of Lorenze (1993) for an example.

In Figure 6, we present a bifurcation diagram in which the amplitude of the cycle is on the vertical axis and v as bifurcation parameter on the horizontal axis. v_u is the critical value for which the approximated model loses its stability, and its stationary point is replaced by a limit cycle. For $v > v_u$, the model is destabilized for small perturbations so that any orbit moves away from the stationary point. Nonlinearity of the model prevents it from diverging globally but leads to a unique stable limit cycle. This is essentially the same cycle as the one that Goodwin demonstrates by applying the Lienard method. On the other hand, for $v < v_u$, it is locally stabilized but generates an unstable cycle as well as a stable cycle for v in the interval $[v_s, v_u]$. It can be seen that as v decreases from v_u, the amplitude of the inner unstable cycles increases and the one of the outer stable cycle decreases. v_s is the other critical value for which two cycles coincide each other. For a lower $v < v_s$, limit cycles no longer exit.

Coexistence of multiple cycles and a stable stationary point reminds us "corridor stability," the notion of which was introduced by Leijonhufvud [1973]. It implies that a dynamical system is stable for small perturbations but unstable for large perturbations. The inside of white circle region in Figure 5 is the corridor in which the stationary point is stable. Thus, if perturbations around the stationary point are small enough not to take orbits out of the corridor, the restoring effect of the model works to return to the stationary point. To the contrary, if perturbations are large enough to take orbits outside the corridor, the lasting effect of the model leads to persistent cyclic fluctuations. The foregoing numerical analysis indicates that non-linearity of investment function can be a source of corridor stability in Goodwin's approximated model.

Insert Figure 6 Here

4 Concluding Remarks

We investigate the characteristics and the robustness of Goodwin's business cycle in this study. It is confirmed numerically that the nonlinearity of the investment function generates cyclical fluctuations of output as shown in Figure 1 and summarized in Proposition 1. It is also confirmed numerically that the longer production lag makes the amplitude of the cycle larger and the length of its period longer as shown in Figure 2 and summarized in Proposition 2. Further, it is seen in Figure 6 that the amplitude of the cycle is sensitive to the nonlinearity of investment function and gets larger as the nonlinearity gets stronger. It is demonstrated analytically that cyclic fluctuations of output are brought about even when the stationary state is locally stable as summarized in Propositions 2 and 3. Coexistence of multiple cycles and the stable stationary point implies the corridor stability of Goodwin's business cycle. This study complements the existing studies on Goodwin's business cycles, the studies on which focus mainly on circumstances in which the stationary state is locally unstable.

References

[1] Goodwin, R., "The Nonlinear Accelerator and the Persistence of Business Cycles," *Econometrica*, vol.19, 1-17, 1951.

[2] Grasman, J., and J. Wentzel, "Co-existence of a limit cycle and an equilibrium in Kaldor's Business Cycle Model and its Consequences," *Journal of Economic Behavior and Organization*, vol.24, 369-377,1994.

[3] Leijonhufvud, A., "Effective Demand Failures," *Swedish Journal of Economics*, 27-48, 1973.

[4] Lorenz, H-W., *Nonlinear Dynamical Economics and Chaotic Motion*, Springer-Verlag, 1993.

[5] Lorenz, H-W., "Goodwin's Nonlinear Accelerator and Chaotic Motion," *Journal of Economics*, vol. 47, 413-418, 1987.

[6] Matsumoto, A., "A Non-linear Macro Model of Endogenous Inventory Oscillations," *Research in Economics*, vol.51, 101-129, 1997.

[7] Puu, T., *Attractors, Bifurcations, and Chaos, Second Edition*, Springer-Verlag, 2003.

[8] Sasakura, K., "The Business Cycle Model with a Unique Stable Limit Cycle," *Journal of Economic Dynamics and Control*, vol.20, 1763-1773, 1996.

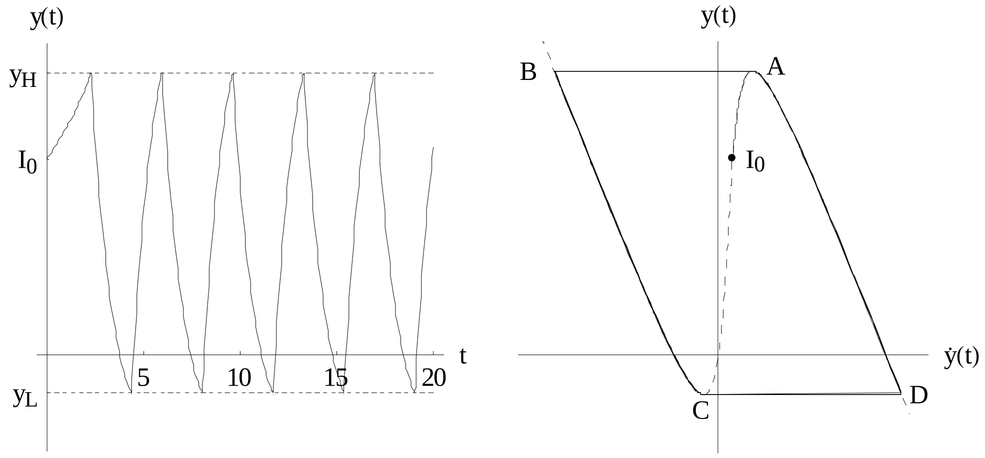

Figure 1. Endogeneous Cycle with Discontinuous Jumps.

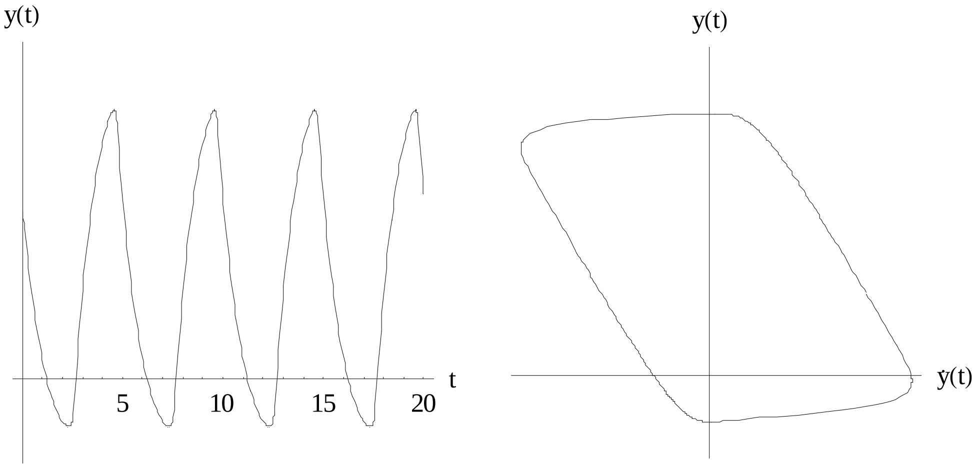

Figure 2A. Cyclical Fluctuations with θ = 0.125

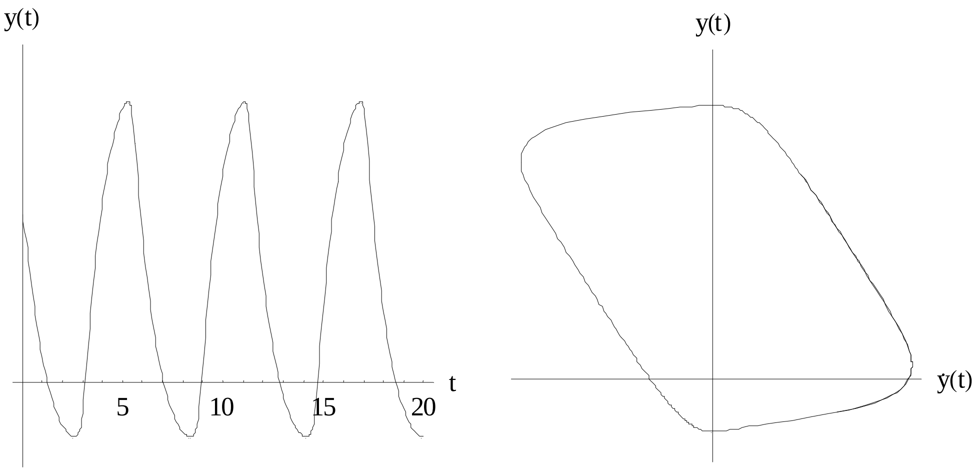

Figure 2B. Cyclical Fluctuations with θ = 0.25.

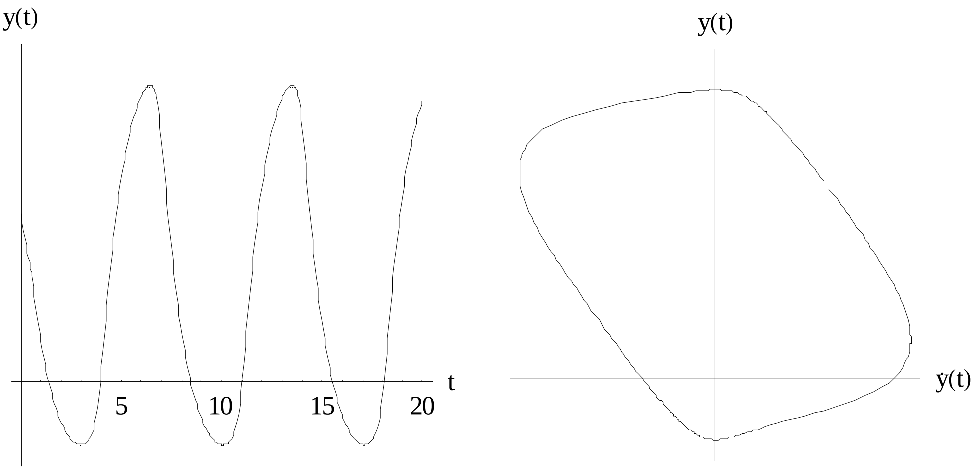

Figure 2C. Cyclical Fluctuations with $\theta = 0.5$.

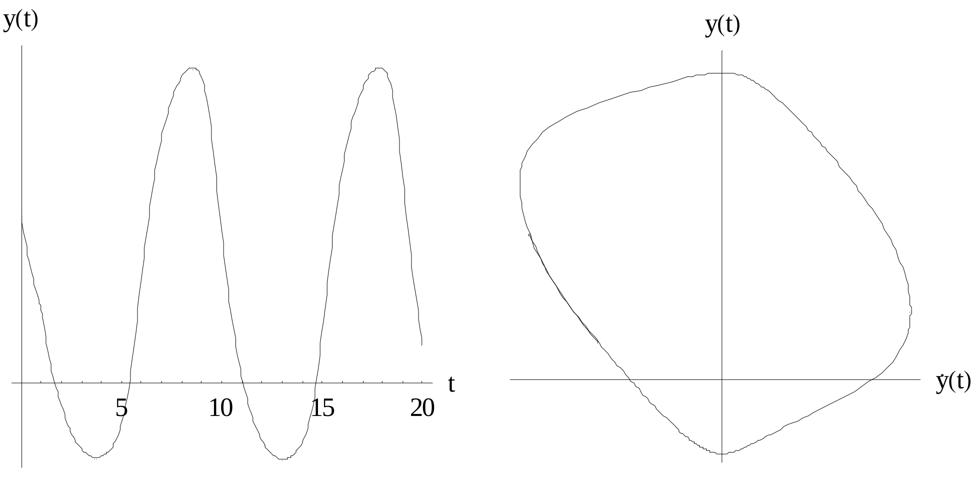

Figure 2D. Cyclical Fluctuations with $\theta = 1$.

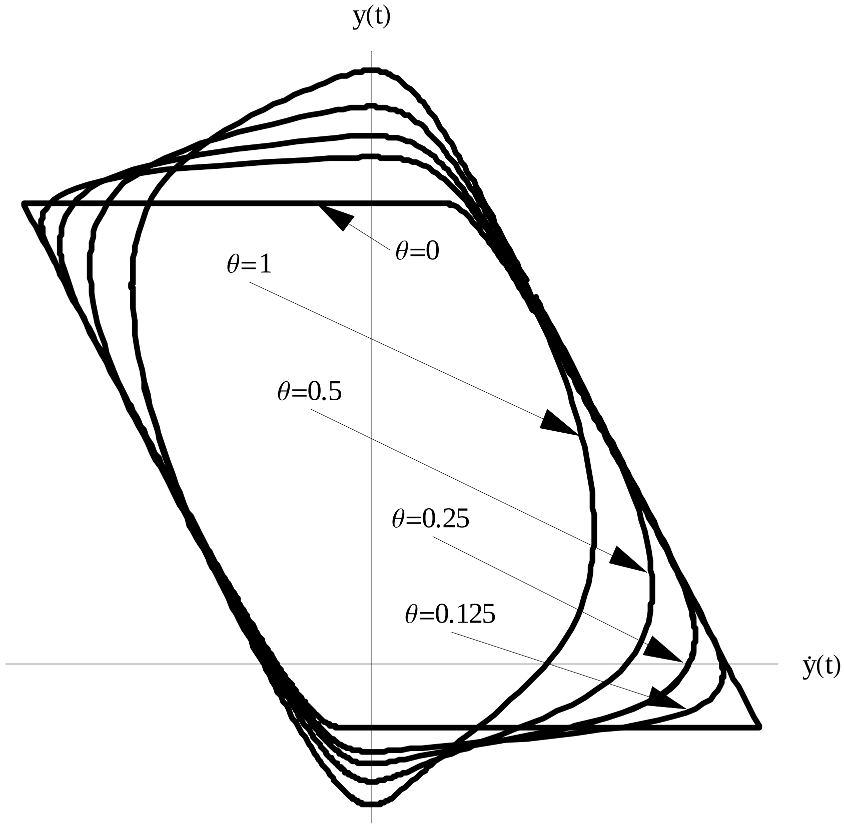

Figure 3. Juxtaposition of limit lycles with different lag.

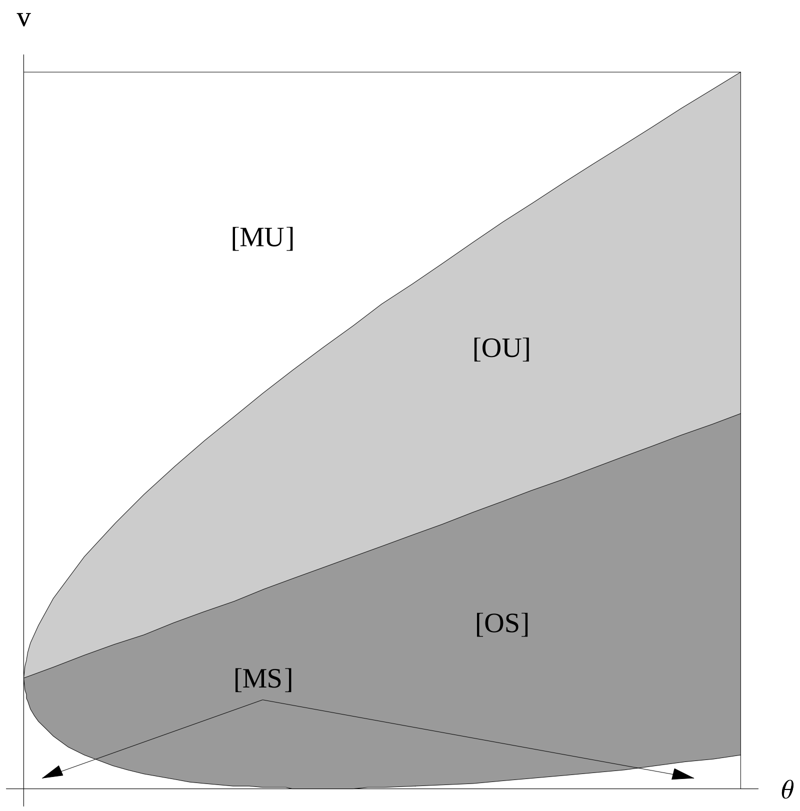

Figure 4. Division of the Parameter Region.

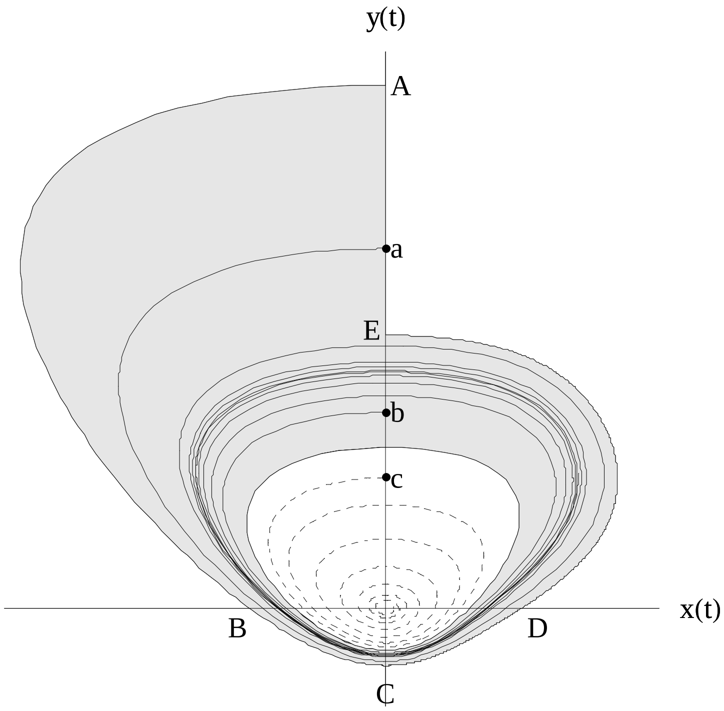

Figure 5. Coexistence of Limit Cycles.

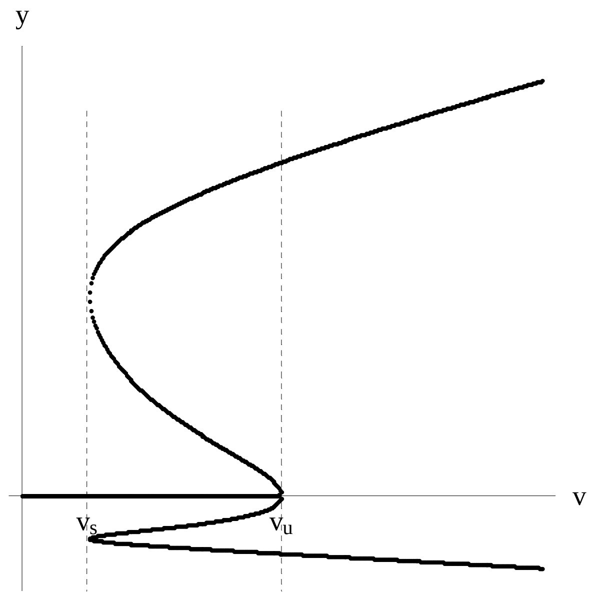

Figure 6. Bifurcation Diagrame.