

# Japanese contributions to Dynamic Economic Theories in the 1940s – the 1970s : A Historical Survey

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## Abstract

Until the end of the World War II, Japan was pure importer of economic theory and Japan had nothing to export except only a few exceptional contributions such as those by Kei Shibata on the Marxian Economics and Walrasian/Casselian general equilibrium analysis and Masazo Sono's contribution to mathematical consumer theory. However, the situation dramatically changed soon after the World War II especially in the research area of mathematical economics. In this paper, we survey the international contributions of the distinguished Japanese theoretical/mathematical economists of the first generation after the World War II, especially focusing on the development of the dynamic economic theories in the 1940s – the 1970s. We take up the theoretical contributions by Takuma Yasui, Michio Morishima, Hirofumi Uzawa, Takashi Negishi, Nobuo Okishio and Hukukane Nikaido.

## 1. Introduction

Although this paper is mainly concerned with the international contributions by Japanese theoretical economists in the period after the World War II (after 1945), in this section, we shall briefly refer to some important contributions before 1945. Until the end of the World War II, Japan was pure importer of economic theory and Japan had nothing to export except only a few exceptional contributions.<sup>1</sup> Kei Shibata and Masazo Sono are two such exceptional Japanese scholars who could add something new to modern economic theory at that period.

Shibata was very creative theoretical economist of Kyoto University. He studied mathematical Walrasian/Casselian general equilibrium theory as well as Marxian economic theory. In 1930, He found that Cassel(1918)'s system of equations of general economic equilibrium may not have meaningful solution because there is the possibility that a subsystem of Cassel's equations becomes over-determinate. Unfortunately, his paper(Shibata 1930) was written in Japanese, so that his contribution was unknown among the western economists. But, his assertion was in fact the same as that of von Stackelberg(1933) which appeared three years *after* the publication of Shibata(1930).<sup>2</sup> Shibata(1933), which was written in English, tried to formulate Marx's scheme of reproduction by means of simplified two sector Walrasian general equilibrium system of equations, and this paper was referred to by Oscar Lange(1935). He also published a critical review of Keynes' General Theory(Keynes 1936) in English only a year after the publication of Keynes' book(cf. Shibata 1937). Masazo Sono was a mathematician rather than economist, but his mathematical analysis of the 'separable goods', which was done during the World War II, is an important contribution to consumer theory(cf. Sono 1943).<sup>3</sup> The contributions by such economists as Shibata and Sono were the pioneering works which foreshowed the striking international contributions by several Japanese theoretical economists after the World War II, especially in the period between the 1940s and the 1970s.

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<sup>1</sup> The newest imported economics in this period in Japan was Marxian economics, but the Japanese government oppressed it because the government considered it as politically dangerous, and the Japanese mainstream economics in this period was quite domestic, nationalistic, and illogical economics written in Japanese, which had no international influence.

<sup>2</sup> Zeuthen(1933) presented correct way to solve this problem by means of the system of inequalities rather than equations, and Wald(1936) managed to solve the problem mathematically. See also Yasui, Kumagai and Fukuoka(1977) for the related topics.

<sup>3</sup> The goods X and Y are said to be separable from the good Z if the marginal rate of substitution between X and Y is independent of the quantity of Z. (cf. Ikee 2006 p. 99. According to Ikee 2006, this is the definition by Takuma Yasui..) Sono's paper in Japanese(1943) was later translated into English(Sono 1961).

In the following sections, we shall consider international contributions by Japanese economists to the development of economic theory after the World War II, especially focusing on the development of dynamic economic theories in the 1940s – the 1970s. We take up the theoretical contributions by Takuma Yasui, Michio Morishima, Hirofumi Uzawa, Takashi Negishi, Nobuo Okishio and Hukukane Nikaido.<sup>4</sup>

## 2. Takuma Yasui as a pioneer of dynamic analysis of economic stability and nonlinear analysis of the business cycle

Takuma Yasui, who was a professor of Economics at Tohoku University, Osaka University and other universities in Japan, is considered to be the father of mathematical economics in Japan. He began to study Walrasian general equilibrium theory during the World War II. But, he was frustrated because he could not find the correct *method* to deal with the dynamic analysis of Walras' *tâtonnement* (trial and error) process of the price adjustment. Soon after the World War II, he found the correct method in the mathematical theory of differential and difference equations. He published a series of papers which deal with the mathematical conditions for the dynamic stability in Japanese(cf. Yasui 1948a, 1948b, 1950). Yasui(1948b) described the Routh-Hurwitz conditions for the local stability of an n-dimensional system of differential equations<sup>5</sup> and provided their economic applications to Walrasian/Hicksian general equilibrium system and Keynesian system. Yasui(1948b) also clearly described

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<sup>4</sup> Other important economists who are not taken up in this paper are Ichiro Nakayama(study of Walrasian general equilibrium theory and Schumpeter's economics), Eiichi Sugimoto(synthesis of Marshallian and Marxian economics), and Hideo Aoyama(study of the theory of imperfect competition and the sociological study of Max Weber), because most of their contributions were written in Japanese and it seems that they had little international impact. As for the expositions of their contributions, see Ikeo(ed.)(1999) and Ikeo(2006). Another influential economist who is not taken up in this paper is Shigeto Tsuru, a Marxian-oriented 'institutional' economist who is internationally well known(cf. Tsuru 1993).

<sup>5</sup> Let us consider the following n-th order polynomial characteristic equation.

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_{n-1}\lambda + a_n = 0$$

The Routh-Hurwitz conditions mean that all of the roots of this characteristic equation have negative real parts *if and only if* the following set of inequalities are satisfied(the following expressions are the same as those adopted by Yasui 1948b).

$$a_1 > 0, \quad \begin{vmatrix} a_1 & a_3 \\ 1 & a_2 \end{vmatrix} > 0, \quad \begin{vmatrix} a_1 & a_3 & a_5 \\ 1 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} > 0, \quad \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ 1 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & 1 & a_2 & a_4 \end{vmatrix} > 0, \quad \dots$$

the Cohn-Schur conditions for the local stability of an n-dimensional system of difference equations without economic application.<sup>6</sup> In the postscript to his collected papers( Yasui 1971), he noted that these papers were written *before* he read Samuelson's famous book *Foundations of Economic Analysis*(Samuelson 1947), and Cohn-Schur conditions are not referred to by Samuelson(1947) although the Routh-Hurwitz conditions are contained in Samuelson(1947). In Yasui(1950), he referred to Liapunov's theorem on the global stability of a system of differential equations due to Liapunov(Liapounoff)(1907).<sup>7</sup> At that time, this theorem was not well known even among western mathematical economists. This means that Yasui is one of

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<sup>6</sup> The Cohn-Schur conditions mean that all the roots of the characteristic equation

$$\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_{n-1}\lambda + a_n = 0$$

have the absolute values less than 1 *if and only* if the following set of inequalities are satisfied(again the following expressions are the same as those by Yasui 1948b).

$$\begin{vmatrix} 1 & a_n \\ a_n & 1 \end{vmatrix} > 0, \quad \begin{vmatrix} 1 & a_1 & a_n & 0 \\ 0 & 1 & a_{n-1} & a_n \\ a_n & a_{n-1} & 1 & 0 \\ 0 & a_n & a_1 & 1 \end{vmatrix} > 0, \quad \begin{vmatrix} 1 & a_1 & a_2 & a_n & 0 & 0 \\ 0 & 1 & a_1 & a_{n-1} & a_n & 0 \\ 0 & 0 & 1 & a_{n-2} & a_{n-1} & a_n \\ a_n & a_{n-1} & a_{n-2} & 1 & 0 & 0 \\ 0 & a_n & a_{n-1} & a_1 & 1 & 0 \\ 0 & 0 & a_n & a_2 & a_1 & 1 \end{vmatrix} > 0, \quad \dots$$

<sup>7</sup> We can summarize a version of Liapunov's theorem as follows.

Let  $\dot{x} = f(x)$ ,  $x = [x_1, x_2, \dots, x_n] \in R^n$  be an n-dimensional system of differential equations that have the unique equilibrium point  $x^* = [x_1^*, x_2^*, \dots, x_n^*]$  such that  $f(x^*) = 0$ . Suppose that there exists a scalar function  $V = V(x - x^*)$  with continuous first derivatives and with the following properties (1) – (5).

- (1)  $V \geq 0$ ,
- (2)  $V = 0$  if and only if  $x_i - x_i^* = 0$  for all  $i \in \{1, 2, \dots, n\}$ ,
- (3)  $V \rightarrow +\infty$  as  $\|x - x^*\| \rightarrow +\infty$ ,
- (4)  $\dot{V} = \sum_{i=1}^n \frac{\partial V}{\partial (x_i - x_i^*)} \dot{x}_i \leq 0$ ,
- (5)  $\dot{V} = 0$  if and only if  $x_i - x_i^* = 0$  for all  $i \in \{1, 2, \dots, n\}$ .

Then, the equilibrium point  $x^*$  of the above system is globally asymptotically stable.

The above function  $V = V(x - x^*)$  is called the 'Liapunov function'. By the way, this version of Liapunov's theorem is not quoted from Yasui(1950) but it is quoted from Gandolfo(1996) p. 410. In the postscript to Yasui(1970), he confessed that he found Liapunov's paper in French which was published in 1907 in the library of Tohoku University and he struggled with this huge paper with 260 pages long to write Yasui(1950).

the economists who realized the analytical power of Liapunov's theorem for economic analysis for the first time, as Weintraub(1987, 1991) correctly pointed out. Although Yasui(1950) did not provide economic application of this theorem, later this theorem was effectively used by Arrow and Hurwicz(1958) and Arrow, Block and Hurwicz(1959) to prove the global stability of Walrasian competitive equilibrium under the assumption of gross substitutability.

Another important Yasui's contribution to dynamic economic theory is a reformulation of Kaldor(1940)'s business cycle theory by means of nonlinear differential equation. Kaldor(1940) presented an ingenious graphical presentation of his business cycle theory by means of S-shaped nonlinear investment function(cf. Fig. 1). In Fig. 1,  $Y$  is real national income, the line  $ff$  represents the saving function  $S = S(Y)$ , and the S-shaped curves  $\varphi_{ei}$  ( $i = 1, 2, 3, \dots$ ) represent the investment function  $I = I(Y, K)$ , where the real investment  $I$  is supposed to be an S-shaped increasing function of real national income  $Y$  and decreasing function of real capital stock  $K$ . Kaldor(1940) assumes that the speed of the quantity adjustment in the goods market is very high, while the real capital stock moves slowly. In Fig. 1, the downward(upward) shift of investment function is caused by the increase(decrease) of  $K$ , and the endogenous business cycle along the closed curve in Fig. 1 occurs in his model.

Yasui(1953) reformulated Kaldor(1940)'s theory by means of the following system of dynamic equations omitting the asymmetry between the movement of income and that of capital stock.

$$\dot{i} = f(y) - \mu k, \quad f'(y) > 0, \quad \mu > 0 \quad (1)$$

$$\varepsilon \dot{y} = i - sy, \quad \varepsilon > 0, \quad 0 < s < 1 \quad (2)$$

$$\dot{k} = i - \delta k, \quad 0 < \delta < 1 \quad (3)$$

$$y = Y - Y_0, \quad i = I - I_0, \quad k = K - K_0 \quad (4)$$

where  $Y$  = real national income,  $I$  = real investment demand,  $K$  = real capital stock. The values  $Y_0$ ,  $I_0$ , and  $K_0$  are equilibrium values of these variables. The parameters  $\mu$ ,  $\varepsilon$ ,  $s$ , and  $\delta$  are the sensitivity of investment with respect to the changes of the capital stock, the reciprocal of the speed of the quantity adjustment in the goods market, the average propensity to save, and the rate of capital depreciation respectively. Eq. (1) is the Kaldorian investment function where  $f(y)$  is S-shaped. Eq. (2) is the disequilibrium adjustment equation in the goods market. Eq. (3) is the capital accumulation equation.

Yasui(1953) derived the following single nonlinear second-order differential equation from the above system of equations.

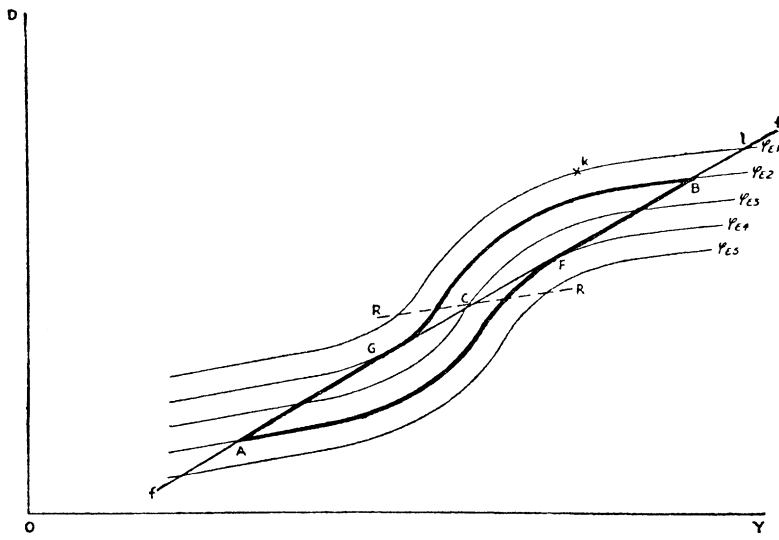


Fig.1. Kaldor's business cycle model  
 Source : Kaldor (1960) P.189

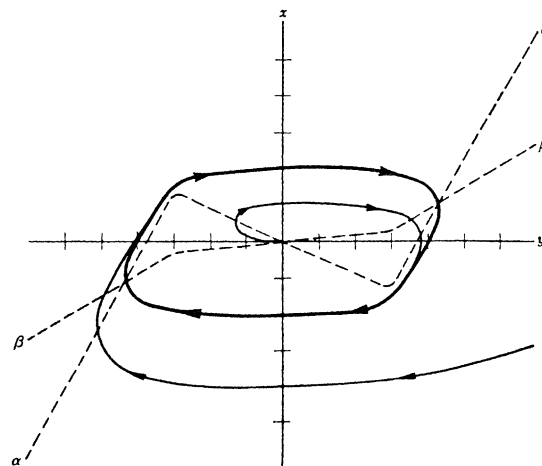


Fig.2. Yasui's graphical representation  
 Source : Yasui (1971) P.236

$$\varepsilon \ddot{y} + [\varepsilon(\mu + \delta) + s - f'(y)]\dot{y} + s(\mu + \delta)y - \mathcal{F}(y) = 0 \quad (5)$$

Under some additional assumptions, he showed that this equation is mathematically equivalent to the following type of the generalized van der Pole equation (or Liénard equation) with  $zF(z) > 0$  for  $|z| > 0$ .

$$\ddot{z} + F(z)\dot{z} + G(z) = 0 \quad (6)$$

Yasui(1953) noted that this type of equation was already treated by Goodwin(1951) and it produces the dynamic behavior of cyclical fluctuations by means of the limit cycles as represented by Fig. 2 under some additional assumptions.<sup>8</sup>

It is worth noting that Kaldor himself referred to Yasui(1953)'s paper in the preface of his collected papers(Kaldor 1960). Kaldor wrote as follows.

“ The paper on ‘A Model of the Trade Cycle’, written shortly before the war, put forward a new idea – that the postulate of non-linear investment and/or savings functions produces a ‘limit cycle’ which is neither damped, nor anti-damped, and which does not presuppose for its operation that the various parameters assume values which fall within narrow range, as was the case with the earlier models of Frisch, Tinbergen or Kalecki. This idea attracted relatively little attention until some ten years later when Goodwin and Hicks published, quite independently, models of the trade cycle based on non-linear investment functions and Yasui showed that the model could be translated into mathematical terms with the aid of van der Pol’s theory of relaxation oscillations.” (Kaldor 1960 p. 9)<sup>9</sup>

By the way, Yasui(1953)'s model as well as Kaldor(1940)'s original model produces the cycles around the constant equilibrium values of income and capital. In other words, these models cannot produce cyclical growth (cycles around the growth trend). Yasui(1954) tried to reformulate such a Kaldorian model to produce cyclical growth. We shall interpret Yasui(1954)'s model after we interpret Morishima(1955)'s model of cyclical growth, because both models are closely related to each other.

### 3. Micho Morishima as a pioneer of nonlinear analysis of business cycle and cyclical

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<sup>8</sup> In Fig 2, we have  $x = \dot{y}$ .

<sup>9</sup> At the footnotes on this sentence, Kaldor referred to Yasui(1953) and Morishima(1958) as well as Hicks(1950) and Goodwin(1951). Although he did not refer to the paper by Kalecki in the footnote, apparently he talked about Kalecki(1935) in the text which we quoted.

## growth

Michio Morishima is a distinguished Japanese mathematical economist, although he was a professor of London School of Economics during the period 1970 – 1988. He entered into Kyoto University(Kyoto Imperial University at that time) in 1942 as a student, and studied Hicks' *Value and Capital* (Hicks 1939) under the guidance of his teacher Hideo Aoyama. After the World War II, he taught economic theory in Kyoto University, Osaka University, University of Essex and London School of Economics. In the 1950s and the 1960s, he was a colleague of Takuma Yasui at Osaka University. In 1964, he became the first Japanese economist who played the role of the president of the Econometric Society. At that time, he was 41 years old.<sup>10</sup> Since the 1950s, he published numerous quite original and creative papers and books on dynamic economic theory both in English and in Japanese (cf. Morishima 1950, 1953, 1955, 1958, 1964, 1969, 1996). In particular, he studied nonlinear macrodynamic models of economic fluctuations and multisectoral dynamic models of economic growth à la von Neumann. He also studied mathematical approaches to Marxian, Ricardian, and Walrasian economic theories (cf. Morishima and Seton 1957, Morishima and Catephores 1978, and Morishima 1973, 1977, 1989).<sup>11</sup>

In this section, we take up Morishima's contribution to nonlinear business cycle theories in the 1950s which are closely related to Yasui's works which we considered in the previous section. First, we shall take up the model of Morishima(1953, 1958), which was called 'Ichimura-Morishima model' by Velupillai(2008).<sup>12</sup>

Morishima(1953, 1958)'s model consists of the following system of equations(cf. Velupillai 2008).

$$C_t = \alpha Y_{t-\tau} + \beta(t), \quad 0 < \alpha < 1 \quad (7)$$

$$I_t = v(Q_t + L_t - K_t), \quad v > 0 \quad (8)$$

$$\dot{Q}_t = \phi(\dot{Y}_t), \quad \phi' > 0 \quad (9)$$

$$Y_t = C_t + I_t + l(t), \quad l(t) = \dot{L}_t \quad (10)$$

where  $Y$  = national income,  $C$  = consumption expenditure,  $I$  = induced investment

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<sup>10</sup> The second and the third Japanese economists who became the president of the Econometric Society are Hirofumi Uzawa and Takashi Negishi.

<sup>11</sup> With respect to mathematical study of Marxian economics, it seems that Morishima was much influenced by another very creative Japanese economist Nobuo Okishio, who is true pioneer of mathematical Marxian economics.

<sup>12</sup> This name stems from the fact that Shinichi Ichimura, who was a colleague of Morishima at Osaka University, published a paper with the similar idea(cf. Ichimura 1955).



expenditure,  $Q$  = desired capital stock,  $L$  = capital stock required for innovational investment,  $K$  = actual capital stock, and  $l(t)$  = autonomous or innovational investment expenditure. The subscript denotes time period.

Eq. (7) is the consumption function with expenditure time lag. Eq. (8) together with Eq. (9) is the investment function of the nonlinear accelerator type. Eq. (10) is the equilibrium condition of the goods market.

The above system is a nonlinear system of mixed difference and differential equations. But, Morishima approximates Eq. (7) by the following equation expanding Eq. (7) in a Taylor series and neglecting the terms of higher order than the first.<sup>13</sup>

$$C_t = \alpha Y_t - \varepsilon \dot{Y}_t + \beta(t), \quad \varepsilon = \alpha \tau > 0 \quad (11)$$

Furthermore, he adopted the simplifying assumption that  $\beta(t)$  and  $l(t)$  are constant. Then, he derived the following single nonlinear second order differential equations from the system of equations (8) – (11).

$$\varepsilon \theta \ddot{y} + [\varepsilon + (1 - \alpha)\theta - \phi(\dot{y})/\dot{y}] \dot{y} + (1 - \alpha)y = 0 ; \quad \theta = 1/v, \quad y = Y - Y^* \quad (12)$$

where  $Y^*$  is the equilibrium value of  $Y$ .<sup>14</sup> As Morishima(1958) noted, this is a Rayleigh type nonlinear differential equation rather than Liénard equation. He proved the existence of the limit cycle in this model under some restrictive assumptions. As Velupillai(2008) pointed out, however, it is worth to note that we can prove the existence of the limit cycle in Morishima(1953, 1958)'s model as well as Yasui(1953)'s model under less restrictive assumptions than those imposed by them by applying Hopf-bifurcation theorem.<sup>15</sup>

If it is assumed that the autonomous expenditures  $\beta(t)$  and  $l(t)$  are increasing functions of time, the equilibrium level of national income itself becomes the increasing function of time. In this case, the system of equations (8) – (11) can produce the cyclical growth around the moving equilibrium. Morishima(1955) Chap. 4 applied the same idea to Kaldor(1940)'s model of business cycle. He wrote as follows.

“ Basic consumption increases as the population and the standard of living increase, while the population increases as the time passes, and the standard of living also tends to increase as the time passes, so that we can suppose that the basic consumption increases as the time passes. Therefore,  $b(t)$  becomes the increasing function of  $t$ ,

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<sup>13</sup> This procedure was also adopted by Goodwin(1951). Needless to say, this procedure can be rationalized only when the time lag  $\tau$  is sufficiently small.

<sup>14</sup> In this expression, we omit the subscript which describes time period for simplicity of the notation.

<sup>15</sup> See Gandolfo(1996) p. 477 for the exposition of the Hopf-bifurcation theorem.

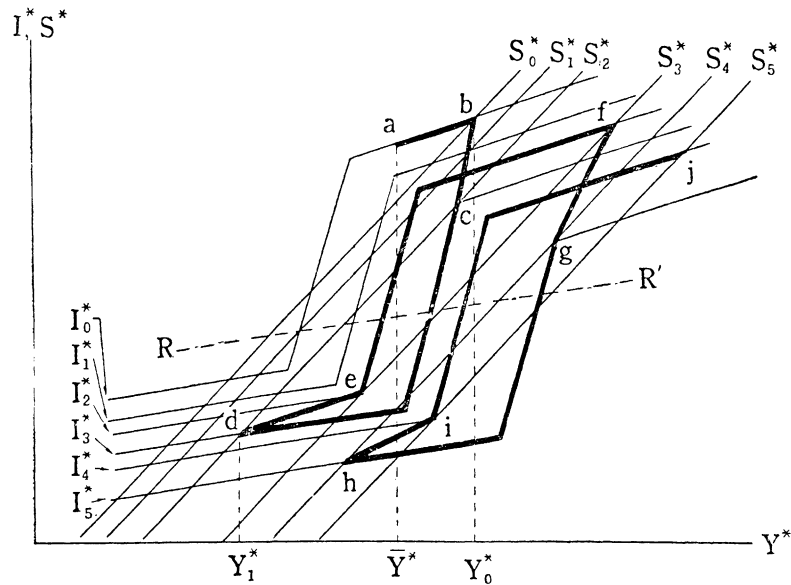


Fig.3. Morishima's model of cyclical growth  
 Source : Morishima (1955) P.106

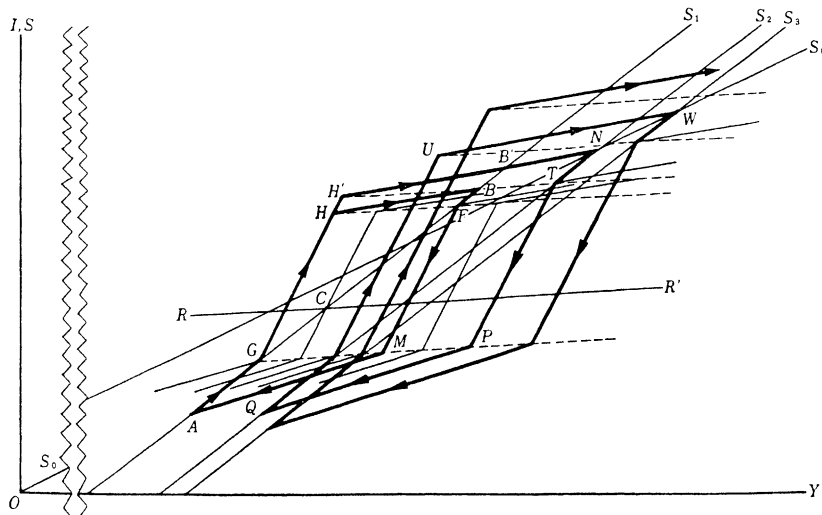


Fig.4. Yasui's model of cyclical growth  
 Source : Yasui (1971) P.255

where  $b(t)$  is the basic consumption at the time period  $t$ ." (Morishima 1955 p. 102, original text is in Japanese)

Then, Morishima(1955) presented Fig. 3 which describes the process of cyclical growth in Kaldorian setting. In this figure, the increase of basic consumption induces the rightward shift of the saving function  $S_i^*$ , so that the cyclical growth along the trajectory  $abcdef\dots$  is caused.

Yasui(1954) provided more complex version of Kaldorian cyclical growth, which is represented by Fig. 4. It must be noted that Yasui(1954)'s model is an adaptation of Morishima(1955)'s model and not the other way round although Yasui's paper was published a year before the publication of Morishima's book. In fact, Yasui(1954) referred to Morishima's forthcoming book at that time. In Morishima(1955)'s model, it is simply assumed that the saving function continues to shift rightward because of the increase of the autonomous consumption expenditure, while Yasui(1954) adopted more sophisticated Duesenberry(1949)'s 'ratchet effect' approach, which means that the consumption does not return to the previous low level even if the national income decreases because of the inertia of consumers' custom. Another difference between these two models is that in Yasui(1954)'s model the cyclical growth can persist forever because the trajectory moves toward the north-east direction, while in Morishima(1955)'s model the process of the cyclical growth must stop before long because the trajectory moves toward the south-east direction.

#### **4. Hirofumi Uzawa's investment function and Tobin's q theory**

Hirofumi Uzawa is one of the most distinguished Japanese mathematical economists together with Michio Morishima and Hukukane Nikaido. He studied mathematics in University of Tokyo, and moved to Stanford University to complete joint research of mathematical programming with Kenneth Arrow and Leonid Hurwicz in 1956. Outputs of this joint research were published in Arrow, Hurwicz and Uzawa(1958). He became a professor of Economics at University of Chicago in 1964. At that time, he was 36 years old. In 1969 he went back to Japan and became a professor of Economics at University of Tokyo. In 1977 he was selected as the president of the Econometric Society. After retiring teaching Economics at University of Tokyo, he taught Economics at Niigata University and Chuo University. Most of his important theoretical works are collected in three books (Uzawa1988a, 1988b, 1990). In the 1960s he published a series of important papers on Neoclassical growth theory (Uzawa1961a, 1961b, 1963, 1965,

1969) and Walrasian general equilibrium theory (Uzawa1962). In particular we shall refer to three important papers (Uzawa 1962, 1965, 1969) in this section.

It is well known that the existence of Walrasian competitive equilibrium can be proved by means of Brouwer's fixed point theorem.<sup>16</sup> This means that Brouwer's fixed point theorem implies Walras's existence theorem. On the other hand, Uzawa(1962) proved that the existence of Walrasian competitive equilibrium implies Brouwer's fixed point theorem. This implies that Brouwer's fixed point theorem and Walras's existence theorem are logically equivalent. This is the famous Uzawa's theorem.

In Uzawa(1961a), he studied the mathematical condition for the 'neutral' technical progress. In this paper, however, the rate of technical progress was treated as an exogenous variable. On the other hand, in Uzawa(1965), the rate of technical progress becomes an endogenous variable which depends on the amount of the investment on human capital(education of the workers). In this sense, Uzawa(1965) is a pioneering work which became a predecessor of the so called 'endogenous growth theory' by Lucas(1988), Romer(1990) etc. which flourished in the 1980s and the 1990s. In fact, Lucas(1988) quoted Uzawa(1965) as the direct source of his idea.<sup>17</sup>

The above Uzawa's works which we referred to are very famous among economic theorists. Now, we shall take up relatively less known Uzawa's contribution, that is, Uzawa(1969). In this paper, he presented a dynamic optimization model of Keynesian type investment function by adopting the hypothesis of increasing adjustment cost of investment.<sup>18</sup> This contribution is quite important because later Yoshikawa(1980) and Hayashi(1982), both of them were Uzawa's students at University of Tokyo, showed that Uzawa(1969)'s investment function provides a microeconomic foundation of Tobin's q theory due to Tobin(1969, 1980). In fact, they proved that Uzawa(1969)'s investment theory implies Tobin's q theory.

Uzawa(1969) formulated the following typical firm's dynamic optimization problem of investment.

$$\text{Maximize } \int_0^{\infty} \{r - \varphi(g(t))\} p_K K(t) e^{-\rho t} dt \quad (13)$$

$$\text{subject to } \dot{K}(t) = g(t)K(t), \quad K(0) = K_0 = \text{constant} . \quad (14)$$

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<sup>16</sup> See Nikaido(1968) for the exposition of Brouwer's fixed point theorem.

<sup>17</sup> Other two important predecessors of endogenous growth theory which entails endogenous technical progress are Kaldor(1957) and Arrow(1962).

<sup>18</sup> Uzawa(1968)' approach is called the 'adjustment cost approach' along with the similar approaches by Lucas(1967) and Treadway(1969).

where  $K$  = real capital stock,  $g$  = rate of investment (rate of capital accumulation),  $r$  = rate of profit,  $p_K$  = price of the capital goods,  $\rho$  = rate of discount, and  $t$  denotes time period. It is assumed that  $r$ ,  $p_K$ , and  $\rho$  are positive parameters. The function  $\varphi(g)$  is the adjustment cost function of investment with the following properties.<sup>19</sup>

$$\varphi(0) = 0, \quad \varphi'(g) > 0, \quad \varphi''(g) > 0, \quad \varphi'(0) = 1 \quad (15)$$

The control variable is the rate of investment ( $g$ ).

Uzawa(1969) derived the following optimal solution of this problem.<sup>20</sup>

For all  $t \geq 0$ , keep  $g(t) = g^*$  at the level such that

$$\frac{r - \varphi(g^*)}{\rho - g^*} = \varphi'(g^*) \quad (16)$$

Uzawa(1969) illustrated this solution by using a graph such as Fig. 5.

It follows from Fig. 5 that

$$g^* = g^*(r, \rho) ; \quad \partial g^* / \partial r > 0, \quad \partial g^* / \partial \rho < 0, \quad (17)$$

which is nothing but Uzawa(1969)'s investment function.

By the way, Yoshikawa(1980) showed that this investment function is equivalent to Tobin's  $q$  theory. We can summarize his argument as follows. Suppose that  $g(t) = g^* < \rho$  for all  $t \geq 0$ . In this case, the market value of the firm becomes

$$\begin{aligned} V(t) &= \int_t^\infty \{r - \varphi(g^*)\} p_K K(s) e^{-\rho(s-t)} ds \\ &= \int_t^\infty \{r - \varphi(g^*)\} p_K K(t) e^{-(\rho - g^*)(s-t)} ds = \frac{r - \varphi(g^*)}{\rho - g^*} p_K K(t). \end{aligned} \quad (18)$$

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<sup>19</sup> This means that the marginal adjustment cost as well as the total adjustment cost of investment increases as the rate of investment increases. Uzawa(1969) called this effect 'Penrose effect' after Penrose(1959).

<sup>20</sup> We can solve this problem by means of Pontryagin's maximum principle as follows (cf. Gandolfo 1996 for Pontryagin's maximum principle). First, we can define the current value Hamiltonian as

$$H(t) = \{r - \varphi(g(t))\} p_K K(t) + \mu(t) g(t) K(t),$$

where  $\mu(t)$  is the co-state variable. Then, a set of necessary conditions for optimal solution becomes as follows.

$$\dot{K}(t) = \partial H(t) / \partial \mu(t).$$

$$\dot{\mu}(t) = \rho \mu(t) - \partial H(t) / \partial K(t).$$

Select  $g(t)$  which maximizes  $H(t)$  for all  $t \geq 0$ .

$$\lim_{t \rightarrow \infty} \mu(t) e^{-\rho t} = 0.$$

Solving this system of equations, we have the result which is described in the text.

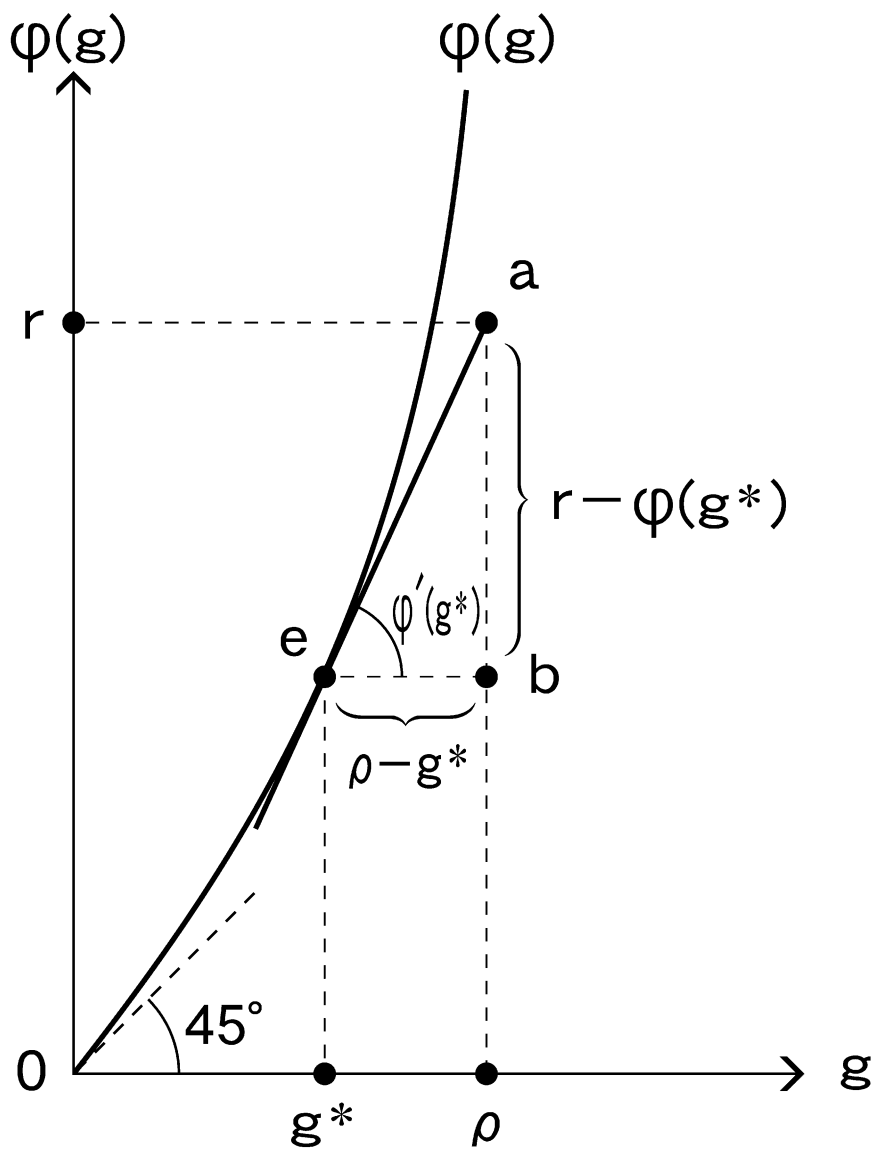


Fig.5. Determination of optimal investment in Uzawa's model

Therefore, Tobin's  $q$ , which is defined as the ratio of the market value of the firm to the value of the real capital stock, becomes as follows.

$$q = \frac{V(t)}{p_K K(t)} = \frac{r - \varphi(g^*)}{\rho - g^*} \quad (19)$$

From equations (16) and (19) we have

$$q = \varphi'(g^*) \quad (20)$$

Solving this equation with respect to  $g^*$ , we obtain the following famous 'q investment function'.

$$g^* = f(q) ; f'(q) > 0, f(1) = 0 \quad (21)$$

This means that Uzawa(1969)'s approach can serve as a microeconomic foundation of Tobin's  $q$  theory.

## 5. Takashi Negishi as a pioneer of dynamic stability analysis of the Walrasian competitive equilibrium

Takashi Negishi, who became a professor of Economics at University of Tokyo after studying Economics at University of Tokyo, published a series of pioneering mathematical papers on general equilibrium theory in the 1950s and the 1960s (cf. Negishi 1958, 1960, 1961, 1962). In the 1970s, he utilized the method of general equilibrium analysis to investigate the microeconomic foundation of Keynesian Macroeconomics (cf. Negishi 1979). In the 1980s, he shifted his attention to the study of the history of economic theory (cf. Negishi 1989). After retiring teaching Economics at University of Tokyo, he taught Economics at Aoyama Gakuin University. He became the president of the econometric society in 1993. In this section, we shall focus on Negishi's remarkable contributions on the dynamic stability analysis of the Walrasian competitive equilibrium in the 1950s and the 1960s.

Walras's tâtonnement process is the price adjustment process in which the prices of the goods with positive (negative) excess demand increase (decrease). But, there is no *a priori* reason to guarantee that this process is dynamically stable unless we impose some restrictions on the properties of the excess demand functions. Hicks(1939) proposed a set of particular mathematical 'stability conditions', but Samuelson(1944) pointed out that Hicks's stability conditions are not identical to the 'true' Routh-Hurwitz conditions for local stability of a system of differential equations. Metzler(1945) found that Hicks's stability conditions become identical to the 'true' local stability conditions if

it is assumed that all goods are ‘gross substitutes’ each other.<sup>21</sup>

In 1958, remarkable progress was made in the related research area by two papers which were published in *Econometrica*, that is, Hahn(1958) and Negishi(1958). They arrived at the same conclusion independently of each other. Namely, the ‘true’ local stability conditions are satisfied automatically in the system with ‘gross substitutes’. Next, we shall summarize Negishi(1958)’s approach. He effectively used the Perron-Frobenius theorem concerning the indecomposable nonnegative matrix to prove the local stability of the Walrasian price adjustment in the system with gross substitutes.<sup>22</sup>

Negishi(1958) considers the following Walrasian price adjustment process by means of a system of differential equations.

$$\dot{p}_j = E_j(p_1, p_2, \dots, p_{m-1}, p_m), \quad j = 1, 2, \dots, m-1 \quad (22)$$

$$p_m = 1 \quad (23)$$

$$\sum_{j=1}^m p_j E_j(p_1, p_2, \dots, p_{m-1}, p_m) = 0 \quad (24)$$

where  $p_j$  is the price of the good  $j$ , and  $E_j$  is the excess demand (the difference between demand and supply) of the good  $j$ . It is assumed that the excess demand function is differentiable and the following conditions for the gross substitutability are satisfied.

$$\frac{\partial E_j}{\partial p_j} < 0, \quad \frac{\partial E_j}{\partial p_k} > 0 \quad \text{if } j \neq k \quad (j, k = 1, 2, \dots, m-1, m) \quad (25)$$

Eq. (23) means that the  $m$ ’th good is selected as the numéraire and its price is always fixed at the level 1. In addition, it is assumed that there exists the unique strictly positive equilibrium price vector  $\bar{p} = [\bar{p}_1, \bar{p}_2, \dots, \bar{p}_{m-1}]'$  which satisfies<sup>23</sup>

$$E_j(\bar{p}) = 0 \quad \text{for all } j \in \{1, 2, \dots, m-1, m\}. \quad (26)$$

Eq. (24) is the so called ‘Walras law’, which means that the total excess demand is

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<sup>21</sup> The good  $i$  is said to be the gross substitute of the good  $j$  if the excess demand of the good  $i$  ( $E_i$ ) is an *increasing* function of the price of the good  $j$  ( $p_j$ ).

<sup>22</sup> See also Negishi(1965) for the restatement of the results.

<sup>23</sup> In this expression, the dash is the notation of transposition of the vector. In other words, we define the price vector  $p$  as the column vector.



always zero even at the disequilibrium situation.<sup>24</sup>

Expanding the right hand side of Eq. (22) in a Taylor series at the equilibrium point and neglecting the terms of higher order than the first, Negishi(1958) derived the following linear system of differential equations.

$$\dot{p}_j = \sum_{k=1}^{m-1} \frac{\partial E_j(\bar{p})}{\partial p_k} (p_k - \bar{p}_k), \quad j = 1, 2, \dots, m-1 \quad (27)$$

By using matrix notation, we can rewrite this system as follows.

$$\dot{p} = A(p - \bar{p}), \quad (28)$$

$$p = [p_1, p_2, \dots, p_{m-1}]' \quad (29)$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m-1} \\ a_{21} & a_{22} & \cdots & a_{2m-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m-11} & a_{m-12} & \cdots & a_{m-1m-1} \end{bmatrix} \quad (30)$$

$$a_{jk} = \frac{\partial E_j(\bar{p})}{\partial p_k} \quad (j, k = 1, 2, \dots, m-1) \quad (31)$$

The characteristic equation of this linear system of differential equation becomes

$$\Delta(\lambda) = |\lambda I - A| = 0. \quad (32)$$

It is well known that the equilibrium point of the original (nonlinear) dynamic system (22) is locally stable if all the roots of Eq. (32) have negative real parts. As Yasui(1948b) pointed out, such stability condition is equivalent to the quite complicated Routh-Hurwitz conditions. The direct application of the Routh-Hurwitz conditions to such a high-dimensional system is, however, highly impractical. Negishi(1958) solved the problem ingeniously by applying the Perron-Frobenis theorem on the indecomposable nonnegative matrix rather than the Routh-Hurwitz theorem to this system of equations. In his derivation, the Walras law (Eq. 24) as well as the gross substitutability postulate (inequalities 25) plays the crucial role.

By differentiating Eq. (24) with respect to  $p_k$ , he obtains the following expression.

$$\sum_{j=1}^m p_j \frac{\partial E_j(p)}{\partial p_k} + E_k(p) = 0 \quad (k = 1, 2, \dots, m-1) \quad (33)$$

At the equilibrium point  $E_k(\bar{p}) = 0$  is satisfied, so that he obtains

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<sup>24</sup> Usually this equality is derived from the budget constraints of the economic agents (cf. Negishi 1965).

$$-\sum_{j=1}^{m-1} \bar{p}_j \frac{\partial E_j(\bar{p})}{\partial p_k} = \frac{\partial E_m(\bar{p})}{\partial p_k} \quad (k = 1, 2, \dots, m-1) \quad (34)$$

or in matrix notation,

$$-xA = z, \quad (35)$$

$$x = \bar{p}' = [\bar{p}_1, \bar{p}_2, \dots, \bar{p}_{m-1}] > 0, \quad (36)$$

$$z = \left[ \frac{\partial E_m(\bar{p})}{\partial p_1}, \frac{\partial E_m(\bar{p})}{\partial p_2}, \dots, \frac{\partial E_m(\bar{p})}{\partial p_{m-1}} \right] > 0. \quad (37)$$

Since all of the non-diagonal elements of the matrix  $A$  are positive, he can define the following positive matrix  $M$  by selecting the sufficiently large positive scalar  $t$ .

$$M = tI + A > 0 \quad (38)$$

Substituting this equation into Eq. (35), he obtains the following equation.

$$x[tI - M] = z, \quad M > 0, \quad x > 0, \quad z > 0, \quad t > 0 \quad (39)$$

Now, let us consider the following characteristic equation.

$$\Delta(\rho) = |\rho I - M| = 0 \quad (40)$$

Since the positive matrix  $M$  is a special case of the indecomposable nonnegative matrix, we can use the following results of the Perron-Frobenius theorem concerning the indecomposable nonnegative matrix.<sup>25</sup>

- (1) The characteristic root of Eq. (40) which has the largest absolute value ( $\rho_M$ ), which is called the 'Frobenius root', is positive real root.
- (2) Suppose that the positive vectors  $x$ ,  $z$ , and positive real number  $t$  satisfy the relationship (39). Then, we have the inequality  $t > \rho_M$ .

Therefore, we have the following inequality for any characteristic root  $\rho$  of Eq. (40).

$$t > \rho_M \geq |\rho| \quad (41)$$

By the way, it follows from equations (32) and (38) that  $\rho = \lambda + t$ , so that inequality (41) implies that

$$t > |\lambda + t| \geq \operatorname{Re}(\lambda + t) = (\operatorname{Re} \lambda) + t \quad (42)$$

where  $\operatorname{Re} \lambda$  means the real part of the complex number  $\lambda$ .<sup>26</sup> Inequality (42) implies that we have  $\operatorname{Re} \lambda < 0$  for any characteristic root  $\lambda$  of Eq.(32). This completes

<sup>25</sup> See Nikaido(1968) chap. 2 for the exposition of the Perron-Frobenius theorem.

<sup>26</sup> Note that the absolute value of a complex number is not smaller than the value of its real part.

Negishi(1958)'s proof of local stability of the equilibrium point of the Walrasian dynamic adjustment process with gross substitutes (22) – (24).

Arrow and Hurwicz(1958) proved the *global* rather than local stability of the competitive equilibrium under Walrasian adjustment process with gross substitutability in a paper which was also published in *Econometrica* in 1958 by using the Liapunov's theorem, so that it seems that the proofs of *local* stability of the same system by Hahn(1958) and Negishi(1958) immediately became obsolete in 1958.<sup>27</sup> Nevertheless, Negishi(1958)'s proof has its own value as an example of the ingenious application of the Perron-Frobenius theorem concerning the indecomposable nonnegative matrix to economic analysis in the area other than Leontief-type input-output analysis. A few years later, Negishi published an exhaustive survey article on the stability of the Walrasian competitive equilibrium including the evaluations of the contributions by himself, Hahn, Arrow and Hurwicz and others, and his survey article(Negishi 1962) had much influence on the researchers of the related topics(cf. Weintraub 1991 chap. 7).

By the way, in the disequilibrium situation in the Walrasian tâtonnement process, transaction is not allowed for. On the other hand in the so-called non-tâtonnement process the transaction is allowed for even in the disequilibrium situation. Hahn and Negishi(1962) proved the dynamic stability of equilibrium in such a non-tâtonnement process.

## **6. Nobuo Okishio on the mathematical formulation of the Harrodian Instability**

Nobuo Okishio, who was a professor of Economic theory at Kobe University, is a world-famous pioneer of the mathematical Marxian economics. Mathematical study of the classical and Marxian economic systems began in the early period of the 20th century by a few economists such as Dmitriev and Bortkiewicz(cf. Dmitriev 1904 for the Ricardian system and Bortkiewicz 1907 for the Marxian system). But, up to the 1950s, much progress in this research area did not occur. In 1960, Sraffa published a small but important book on the mathematical study of the Ricardian system (Sraffa 1960) and the revival of the mathematical study of the classical economics occurred. But, in the 1950s, Okishio already began to study the Marxian theory of value and price by using the mathematical theorems on nonnegative matrices which became famous among the economists through the study of the Leontief-type input-output system. Earlier mathematical works by Okishio on Marxian value and price theory were published in

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<sup>27</sup> See also Arrow, Block and Hurwicz(1959).

Japanese(cf. Okishio 1957, 1965), but later his works which include the essence of his analytical results were published in English(cf. Okishio 1963 and Okishio 1993). In fact, Morishima's works on the mathematical Marxian Economics, for example, Morishima(1973), are much indebted on Okishio's works and not the other way round.

Another notable contribution by Okishio is the mathematical formulation of the Harrodian instability principle in the models of economic growth and economic fluctuations(cf. Okishio 1964, 1976, 1977). Okishio reconsidered Harrod(1948)'s dynamic theory of economic growth from a Marxian point of view. He thought that Harrod's idea on the instability of the capital accumulation process can be utilized to construct a mathematical Marxian theory of the reproduction process of the capitalist economy. Next, we shall reproduce the simplest version of Okishio's analysis of Harrodian instability by means of the difference equation which is contained in Okishio(1964).

Okishio(1964) summarizes the steady-growth solution of the so-called 'Harrod-Domar model' as follows.

$$sY = I \tag{43}$$

$$Y = \sigma K \tag{44}$$

$$I = \Delta K \tag{45}$$

where  $Y$  = real national income,  $I$  = real investment expenditure,  $K$  = real capital stock,  $s$  = average propensity to save ( $0 < s < 1$ ), and  $\sigma$  = output-capital ratio(average productivity of capital) in case of the full utilization of capital stock  $> 0$ . It is assumed that  $s$  and  $\sigma$  are constant parameters.

The system of equations (43) – (45) can determine the movement of the three endogenous variables  $Y$ ,  $I$ , and  $K$ . In fact, he obtains the following solution of the growth rates of these variables from the above system of equations.

$$s\sigma = \frac{\Delta K}{K} = \frac{\Delta I}{I} = \frac{\Delta Y}{Y} \tag{46}$$

This equation determines the equilibrium growth rates of capital, investment and income at the steady state with full capacity utilization. Okishio(1964) correctly pointed out, however, that there is no investment function in this model, and he wrote as follows.

“ Though this model determines the value of new investment  $I$ , this value is the required value of new investment to guarantee the steady growth and is not the actual value which capitalists invest. Therefore it remains as a problem whether the actual growth path converges to the steady growth.” (Okishio 1993 p. 191)

He pointed out that in the actual situations, there is no guarantee that the full utilization of the capital stock is realized and the degree of capital utilization must become an endogenous variable. Therefore, according to Okishio(1964), the actual growth process must be reformulated as follows.<sup>28</sup>

$$sY_t = I_t \quad (47)$$

$$Y_t = \delta_t \sigma K_t \quad (48)$$

$$I_t = \Delta K_t \quad (49)$$

where  $\delta$  is the degree of the utilization of the capital stock. This is a system of three equations with four endogenous variables ( $Y, I, K, \delta$ ). Okishio(1964) pointed out that one more equation is required to close the system and the missing additional equation is nothing but the firms' investment function. He proposed to close this system by adding the following type of the investment function.

$$\frac{I_{t+1}}{K_{t+1}} = \frac{I_t}{K_t} + F(\delta_t), \quad F(1) = 0, \quad F' > 0 \quad (50)$$

This equation means that the entrepreneurs determine their investment expenditures according to the degree of the capital utilization. In fact, it is possible to interpret this type of investment function as a mathematical formalization of Harrod(1948)'s idea.<sup>29</sup>

From equations (47) and (48) Okishio(1964) obtains the following expression.

$$s\sigma\delta_t = \frac{I_t}{K_t} \quad (51)$$

Substituting this equation into Eq. (50), he derived the following difference equation with the single endogenous variable  $\delta$ .

$$\delta_{t+1} = \delta_t + \frac{1}{s\sigma} F(\delta_t) \quad (52)$$

Differentiating this equation with respect to  $\delta_t$ , he derived the following

<sup>28</sup> The subscript  $t$  denotes time period in this formulation.

<sup>29</sup> It follows from Eq. (48) that  $C_t \equiv K_t / Y_t = 1/(\delta_t \sigma)$ ,  $C^* \equiv (K_t / Y_t)|_{\delta=1} = 1/\sigma$ , where  $C_t$  is the 'actual' capital-output ratio and  $C^*$  is the 'required' capital-output ratio. Therefore, we can rewrite Eq. (50) as follows.

$$\frac{I_{t+1}}{K_{t+1}} = \frac{I_t}{K_t} + H(C_t), \quad H(C^*) = 0, \quad H' < 0$$

This means that the entrepreneurs accelerate (decelerate) investment expenditure in case of the shortage (surplus) of the capital stock, which is nothing but the investment attitude of the entrepreneurs which Harrod(1948) postulated in his verbal explanation of the model.

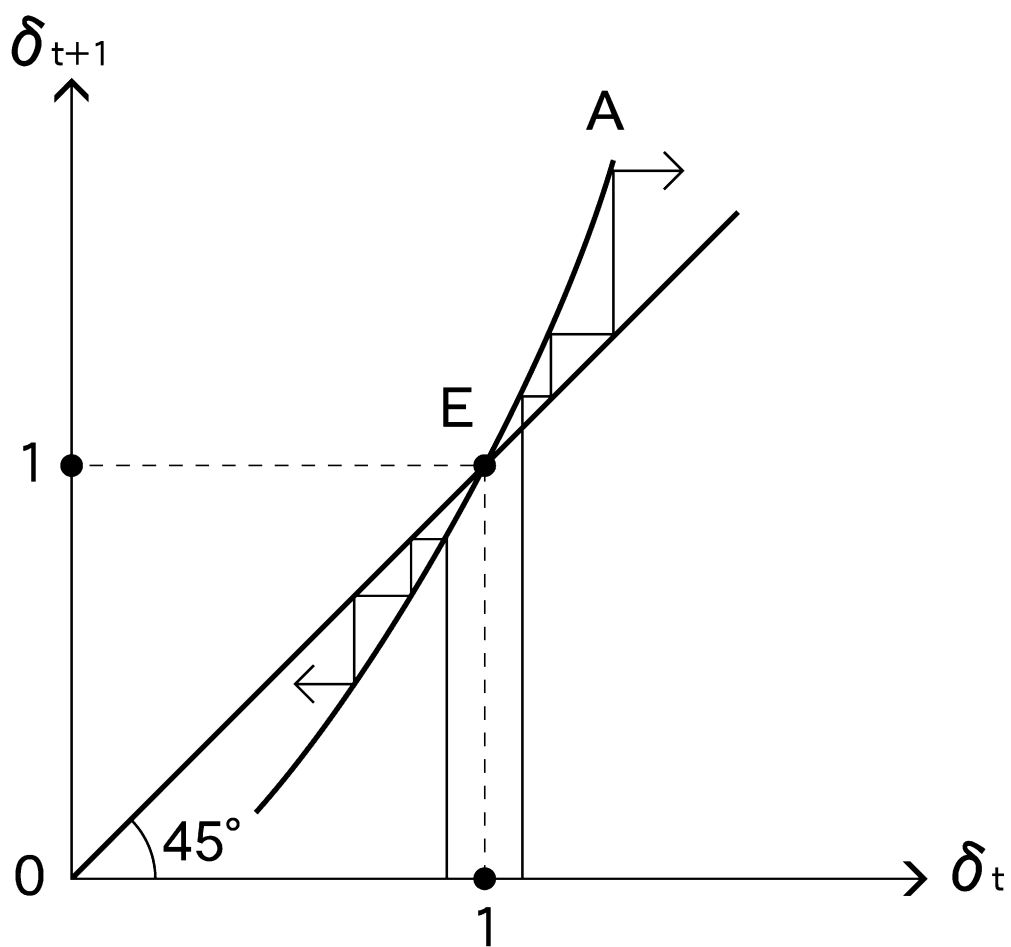


Fig.6. Harrodian instability in Okishio's model

inequality.

$$\frac{d\delta_{t+1}}{d\delta_t} = 1 + \frac{1}{s\sigma} F'(\delta_t) > 1 \quad (53)$$

This inequality means that the equilibrium point of this system which satisfies  $\delta = 1$  is dynamically unstable. Okishio(1964) illustrated this instability proposition by using such a graph as Fig. 6.

One of the causes of instability in this model is the dual characteristics of the investment. In other words, the increase of the investment means the increase of the effective demand as well as the increase of the productive capacity. The increase (decrease) of the rate of capacity utilization of capital stock which is caused by the increase (decrease) of the effective demand induces the increase (decrease) of the investment expenditure, which causes the increase (decrease) of the rate of capacity utilization through the increase (decrease) of the effective demand. This is the destabilizing positive feedback mechanism in this model. Another cause of instability is the postulated myopic bounded-rational investment behavior of the entrepreneurs, which exclusively depends on the current rate of capacity utilization of the capital stock. This process of the 'cumulative disequilibrium' will continue until the floor and the ceiling which are created by financial, labor or other markets prevent the investment expenditure from the upward or downward infinite divergence. Okishio(1976) developed his theory of business cycle by introducing such ceiling and floor into the model, which somewhat resembles Hicks(1950)'s theory of business cycle.

## **7. Hukukane Nikaido on the Harrodian instability and the irrelevance of smooth factor substitution**

Hukukane Nikaido, who was a professor of mathematical economics at Osaka University, Hitotsubashi University and other universities in Japan, is one of the most brilliant Japanese mathematical economists. After studying mathematics at University of Tokyo, he began to write a series of brilliant papers on mathematical economics in the 1950s. His earlier important contribution is Nikaido(1956), which provided a rigorous but elegant proof of the existence of the Walrasian competitive equilibrium by means of Kakutani's fixed point theorem independent of the papers by Arrow and Debreu(1954) and Gale(1955). He also wrote a joint paper with Uzawa on the stability of the competitive equilibrium (Nikaido and Uzawa 1960) and a joint paper with Gale on sufficient conditions for the uniqueness of the equilibrium solution (Gale and Nikaido 1965). He also succeeded to present a complete mathematically rigorous

characterization of Radner(1961)'s turnpike theorem (cf. Nikaido 1964). Nikaido(1968) is the exhaustive advanced textbook of mathematical economics, in which his original contributions up to this period are contained.

After the 1970s, he shifted his attention from the mathematical study of the Walrasian/Neoclassical microeconomic theories to the mathematical study of the imperfect competition and Keynesian/Harrodian/Marxian macrodynamic theories (cf. Nikaido 1975a, 1975b, 1979, 1980, 1983). In the introduction of his collected papers on economic dynamics which was published in 1996, he wrote as follows.

“These early results<sup>30</sup> dealt with statics and dynamics within the Hicksian week of a generic Walrasian Arrow-Debreu economy, one that is solely composed of production, exchange and consumption but devoid of capital formation, credit, and money. In my view, such models describe ideal resource allocation in an institutional framework of complete private ownership, rather than the working of the market economy in the real world. They are best thought of as normative, rather than positive, construction.” (Nikaido 1996 p. xiii)

“The principal view, which drives me most, is that the working of the real world capitalist economy can better be elucidated along the lines suggested by Keynes(1936) in his criticism of the Classical school rather than along the lines currently pursued by the rational expectations and new Classical as descendants of the old Classical. It is especially his grand vision that the economy is unstable and can be ill-behaved that seems most important.” (Nikaido 1996 p. xiv)

In this section, we consider Nikaido(1975b) which proved that the assumption of smooth factor substitution is not sufficient to deny the Harrodian instability principle contrary to the common belief among the economists.<sup>31</sup> Nikaido(1975b) formulates the following system of equations.

$$\dot{L} / L = n \tag{54}$$

$$y = \min[I / s, F(K, L)] \tag{55}$$

$$\dot{K} = sy \tag{56}$$

$$I / s = \theta F(K, L) = F(\theta K, \theta L) \tag{57}$$

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<sup>30</sup> ‘These early results’ mean the analytical results which Nikaido derived in his early works.

<sup>31</sup> Nikaido(1980) is the more complicated sequel to Nikaido(1975).



$$\frac{d}{dt}(I/K) = \phi(\theta - 1), \quad \phi(0) = 0, \quad \phi' > 0 \quad (58)$$

where  $L$  = labor supply,  $n$  = growth rate of labor supply  $> 0$ ,  $y$  = real national income,  $I$  = real investment demand,  $K$  = real capital stock,  $s$  = average propensity to save ( $0 < s < 1$ ),  $\theta$  = measure of the degree of the capital shortage, and  $F(K, L)$  is the Neoclassical linear homogeneous production function with smooth factor substitution. It is assumed that  $n$  and  $s$  are constant parameters.

Eq. (54) means that the labor supply grows at the constant rate. Eq. (55) means that the real national income ( $y$ ) is determined by the 'short side' between the Keynesian effective demand ( $I/s$ ) and the potential full employment level ( $F(K, L)$ ). Eq. (56) means that the capital accumulation is determined by the following rule.

$$\dot{K} = \begin{cases} sF(K, L) & \text{if } I/s > F(K, L) \\ I & \text{if } I/s \leq F(K, L) \end{cases} \quad (59)$$

Eq. (57) defines the measure of the degree of the capital shortage ( $\theta$ ). The inequality  $\theta > 1$  means capital shortage, while  $\theta < 1$  means excess capacity of capital. Eq. (58) is the Harrod-type investment function, which implies that the entrepreneurs accelerate (decelerate) investment in case of capital shortage (excess capacity of capital). The dynamic system of five equations (54) – (58) determines the movement of five endogenous variables ( $L, y, K, \theta, I$ ).

Nikaido(1975b) proved that the rate of investment ( $I/K$ ) continues to decrease indefinitely if the following set of inequalities are satisfied at the initial period (period 0).

$$I(0)/s < F(K(0), L(0)) \quad (60)$$

$$\dot{K}(0)/K(0) \leq \dot{L}(0)/L(0) = n \quad (61)$$

His reasoning is as follows. Since Eq. (57) implies that

$$I/s - F(K, L) = (\theta - 1)F(K, L), \quad (62)$$

we have  $\theta < 1$  at the initial period if the inequality (60) is satisfied. In this case, we

have  $\frac{d}{dt}(I/K) < 0$  at  $t = 0$ . This result assures that there exists the time period  $\tau > 0$

such that we have the set of inequalities

$$I(t)/s < F(K(t), L(t)), \quad \dot{K}(t)/K(t) < n \quad (63)$$

for all  $t \in (0, \tau)$ , which also implies that we have  $\frac{d}{dt}(I/K) < 0$  for all  $t \in (0, \tau)$ . On

the other hand, we have the expression

$$\frac{d}{dt} \left[ \frac{I}{sK} - \frac{F(K, L)}{K} \right] = \frac{1}{s} \frac{d}{dt} \left( \frac{I}{K} \right) - \left( \frac{L}{K} \right) \left( n - \frac{\dot{K}}{K} \right) \frac{\partial}{\partial(L/K)} F(1, L/K). \quad (64)$$

It follows from the inequalities (63) and Eq, (64) that we have

$$\frac{d}{dt} \left[ \frac{I}{sK} - \frac{F(K, L)}{K} \right] < 0 \quad (65)$$

for all  $t \in (0, \tau)$ . This inequality means that we can extend the time period  $\tau$  which satisfy a set of inequalities (63) for all  $t \in (0, \tau)$  indefinitely. Therefore, the rate of investment  $I/K$  continues to decrease indefinitely if a set of initial conditions (60) and (61) are satisfied. This completes Nikaido(1975b)'s proof of the Harrodian instability in a system with smooth factor substitution. This is the rationale of Nikaido's following assertion.

“The stability of the Solovian neoclassical growth equilibrium heavily relies upon its particular investment habits of capitalists in which investment automatically equals savings, rather than factor substitution.<sup>32</sup>

On the other hand, one of the most crucial points in Harrod's analysis is that the investment behaviors of capitalists are primarily responsible for the growth equilibrium balanced on the knife-edge. Thus instability is likely to occur even under flexible smooth factor substitution, depending on the investment habit.” (Nikaido 1996 p. 159)

## 8. Theoretical contributions of other Japanese economists in the 1950s – the 1960s

In this final section, we shall refer to some other important theoretical contributions of Japanese economists in the 1950s – the 1960s.

Kenjiro Ara, who was a professor of Economics at Hitotsubashi University, is the first Japanese economist who published a paper in *Economic Journal* (cf. Ara 1958). Ara(1958) is one of the earliest papers on the Neoclassical aggregative theory of economic growth which were formulated independent of Solow(1956) and Swan(1956). Ara(1959), which was published in *Econometrica*, is a theoretical contribution to the aggregation problem of the Leontief-type multisectoral input-output system, which is somewhat related to the paper by Morishima and Seton(1957).

Kenichi Inada and Yoichi Shinkai, both of them were professors of Economics at Osaka University, also contributed to the theory of economic growth in the 1960s (cf. Inada 1964, 1966 and Shinkai 1960). In particular, the name of Inada is well known because of the so called ‘Inada conditions’ concerning the shape of the Neoclassical

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<sup>32</sup> Needless to say, the word ‘Solovian’ that was used by Nikaido stems from Solow(1956)'s famous paper that formulated the prototype of the neoclassical growth model. See also Swan(1956) as an example of the similar formulation.

production function.<sup>33</sup>

In this paper, we concentrated on some notable contributions of the Japanese economists to dynamic economic theories in the 1940s – the 1970s. Although the contributions which were taken up in this paper are by no means exhaustive, it is unquestionable that all of them are important contributions to economic theory.

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<sup>33</sup> Neoclassical production function is formulated as  $y = f(k)$ ,  $f'(k) > 0$ ,  $f''(k) < 0$ , where  $y = Y/L$  is the output-labor ratio (average labor productivity) and  $k = K/L$  is the capital-labor ratio (capital intensity). The ‘Inada conditions’ are the additional conditions such that  $\lim_{k \rightarrow 0} f'(k) = +\infty$  and  $\lim_{k \rightarrow +\infty} f'(k) = 0$ . For example, the Cobb Douglas production function  $y = Ak^\alpha$ ,  $A > 0$ ,  $0 < \alpha < 1$  satisfies these conditions.

155 – 169.

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