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Existence and Uniqueness of Pure Nash Equilibrium in Asymmetric Contests with Endogenous Prize

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Abstract

This paper considers a contest with an endogenous prize, which is increasing in aggregate efforts of the players. Each player may have a different valuation of the prize and a different ability to convert expenditures to productive efforts. Under standard assumptions in the literature, we prove that there exists a unique pure Nash equilibrium in an asymmetric contest with endogenous prize.

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1 Introduction

A contest is a strategic game in which players expend costly effort in order to increase their probability of winning a given prize. Since the pioneering work of Tullock (1980) and Dixit (1987), there is now a large and growing literature on the theory and application of contests.[1] One of the most important questions is the existence and uniqueness of pure Nash equilibrium. It has been extensively studied under the assumption of an exogenous prize; see e.g. Pérez-Castrillo and Verdier (1992), Szidarovszky and Okuguchi (1997), Cornes and Hartley (2005), and Yamazaki (2008).

However, many contests, such as R&D contest and labor tournament, involve a form of effort that changes the size of the total prize as well as its distribution.[2] Chung (1996) has first analyzed a rent-seeking contest with an endogenous prize (rent), which is increasing in aggregate efforts of the players. Okuguchi (2005) and Corchón (2007) showed that there exists a unique symmetric pure Nash equilibrium in Chung's endogenous contest with a general contest success function. In these studies, players are assumed to be identical in terms of abilities and valuations of the prize.

In many situations, each player may have a different valuation of the prize (e.g., Hillman and Riley, 1989). In addition each player may have a different ability to convert expenditures to productive efforts (e.g., Baik, 1994). Hence, in this paper, we prove that there exists a unique asymmetric pure Nash equilibrium in an endogenous contest with heterogeneity of players' abilities and valuations of the prize. The method used by Cornes and Hartley (2003, 2005) will be used to show the existence and uniqueness of the pure Nash equilibrium.

The rest of the paper is organized as follows. Section 2 explains the basic model and the assumptions. In section 3, we prove that there exists a unique pure Nash

[1] See the excellent surveys by Nitzan (1994) and Konrad (2007).

[2] For instance, the research activity will influence not only the probability of winning, but also the value of the winner's prize in R&D contests (Baye and Hoppe, 2003). This is because players' R&D efforts have a positive externality on the value of the prize .

equilibrium.

2 The Model

Let n be the number of players in a contest. Players are assumed to be risk-neutral. Player $i(= 1, \cdots, n)$ independently chooses a level of effort in order to seek the prize. Our analysis of contests is formulated as a simultaneous-move game and the solution concept we use throughout paper is that of a pure-strategy Nash equilibrium.

If x_i is player i's expenditure in contests, then the probability for winning the prize is given as

$$p_i = \frac{f_i(x_i)}{\sum_{j=1}^{n} f_j(x_j)} \tag{1}$$

where $f_i(\cdot)$ is an increasing function for all i.[3] Szidarovszky and Okuguchi (1997) called $f_i(\cdot)$ player i's production function for lotteries. In line with most of the existence literature, we adopt the following assumption.

Assumption 1. *For all i the function f_i satisfies the following conditions: f_i is twice differentiable, $f_i(0) = 0$, and $f_i'(x_i) > 0$, $f_i''(x_i) < 0$ for all $x_i > 0$.*

Notice that players' production functions do not necessarily have to be identical. A particularly well-studied form for f_i is $f_i(x_i) = a_i x_i^r$, where $r > 0$ and $a_i > 0$. This asymmetric form was given an axiomatic foundation by Clark and Riis (1998), following an earlier axiomatization by Skaperdas (1996) of the symmetric form.

It will prove convenient to change variables by setting $y_i = f_i(x_i)$ for each i. Then the function $f_i(\cdot)$ may be thought of as transforming individual expenditure x_i into effective efforts y_i. We will henceforth refer to x_i as the *expenditure*, and y_i as the *effort*, of player i. Since f_i is monotonic, it has a well-defined inverse function, $g_i(y_i) = f_i^{-1}(y_i)$. Then, Assumption 1 (A.1 in what follows) implies that

$$g_i(0) = 0, \text{ and } g_i'(y_i) > 0, \; g_i''(y_i) > 0 \text{ for all } y_i \in [0, f_i(\infty)). \tag{2}$$

[3] Another interpretation of p_i is that each player i receives a fraction $\frac{f_i(x_i)}{\sum_{j=1}^{n} f_j(x_j)}$ of the contested prize.

The function $g_i(y_i)$ describes the total cost to player i of generating the level y_i of effort.

Next, we introduce the following assumptions on the prize as a function of the aggregate effort by all players. Set $Y = \sum_{j=1}^{n} y_j$ for aggregate effort.

Assumption 2. *For all i the value of the prize is endogenously determined by the aggregate effort: $R_i(Y)$. $R_i(Y)$ is twice differentiable and satisfies $R_i(Y) > 0$ for $Y \geq 0$ and $R_i'(Y) > 0$, $R_i''(Y) \leq 0$ for all $Y > 0$.*

Our characterization of endogenous prize in A.2 follows Chung (1996), Okuguchi (2005) and Corchón (2007), but we assume that players' valuations of the prize may be different. For example, a functional form of R_i is $R_i(Y) = \bar{R}_i + b_i Y$, where $\bar{R}_i > 0$, $b_i > 0$. $\bar{R}_i$ is player i's intrinsic value of the prize and b_i is i's coefficient of enhancement of the prize by aggregate efforts. A.2, together with A.1 , ensures that a player's expected payoff is strictly concave function of her own effort. In addition, A.2 implies that the elasticity of the prize with respect to change the aggregate effort is less than 1 for positive Y. We will write $\epsilon_i = YR_i'/R_i$ for the elasticity of the prize of player i. Notice that ϵ_i needs not necessarily be constant.

Consequently, the expected payoff of player i is described by

$$\pi_i(y_i, Y_{-i}) \quad = \quad R_i(Y)p_i - x_i = R_i(y_i + Y_{-i})\frac{y_i}{y_i + Y_{-i}} - g_i(y_i), \tag{3}$$

where $Y_{-i} = \sum_{j \neq i}^{n} y_j$. Player i is assumed to maximize (3) with respect to y_i subject to $y_i \geq 0$. The expression (3) applies provided at least one player makes a positive effort. If $y_i = 0$ for all i we assume that no player wins the prize so that $\pi_i(0, 0) = 0$. Finally, for the sake of simplicity, we will assume that all players have initial wealth large enough such that the budget constraint dose not bind at all.

3 Existence Analysis

We can now calculate the best response of player i. Assume first that $Y_{-i} > 0$, so that the other players spend a positive amount of resources on contest activities.

Then, the first-order condition for the maximization of (3) with respect to $y_i \geq 0$ yields

$$[R_i'(y_i + Y_{-i})\frac{y_i}{y_i + Y_{-i}} + R_i(y_i + Y_{-i})\frac{Y_{-i}}{(y_i + Y_{-i})^2} - g_i'(y_i)] \leq 0 \quad \text{and} \quad [\cdots]y_i = 0. \tag{4}$$

As the second-order condition we get

$$R_i''(y_i + Y_{-i})\frac{y_i}{y_i + Y_{-i}} - 2R_i(y_i + Y_{-i})\frac{Y_{-i}}{(y_i + Y_{-i})^3}(1 - \epsilon_i) - g_i''(y_i) < 0. \tag{5}$$

Under A.1 and A.2, the second-order condition (5) is satisfied. Notice next that if $Y_{-i} = 0$, agent i's payoff has a maximum at a finite and positive value of input, which can be obtained from the first-order condition with $Y_{-i} = 0$ due to A.1 and A.2. Hence, it follows from (4) that given $Y_{-i} \geq 0$, agent i's best response function $y_i = \phi_i(Y_{-i})$ is well defined and continuous in Y_{-i}. It is well known that a vector $(y_1^*, \cdots, y_n^*)$ is an equilibrium if and only if for all i, y_i^* is the best response with fixed values of Y_{-i}^*.

We can rewrite the best responses of the players in terms of aggregate effort by rewriting the first-order conditions (4) in the form of

$$[R_i'(Y)\frac{y_i}{Y} + \frac{R_i(Y)}{Y}(1 - \frac{y_i}{Y}) - g_i'(y_i)] \leq 0 \quad \text{and} \quad [\cdots]y_i = 0 \tag{6}$$

Since $Y = 0$ can never be an equilibrium in our game, application of the implicit function theorem to (6) enable us to express y_i as a function of Y, namely $y_i = \Phi_i(Y)$. Following Wolfstetter (1999, p. 91), we call this function the *inclusive reaction function* of player i.[4]

Rather than use the inclusive reaction function directly, however, we will examine properties of player i's *share function* $s_i(Y) = \frac{\Phi_i(Y)}{Y}$, which is proposed by Cornes and Hartley (2003, 2005). It can be readily checked that Nash equilibrium values of Y occur where the *aggregate share function* equals unity. That is, $\sum_{i=1}^n s_i(Y^*) = 1$. Given Y^*, the corresponding equilibrium $(y_i^*, \cdots, y_n^*)$ is found by multiplying Y^* by each player's share evaluated at Y^*: $y_i^* = Y^* s_i(Y^*)$. This result enables us to prove

[4]Szidarovszky and Okuguchi (1997) have adapted this function to prove that there exists a unique equilibrium in a rent-seeking contest with exogenous rents.

the existence of a unique equilibrium by showing that the aggregate share is equal to one at a single value of Y. We define player i's share value as $\sigma_i = \frac{y_i}{Y}$ and rewrite (6) as

$$\left[R_i'(Y)\sigma_i + \frac{R_i(Y)}{Y}(1-\sigma_i) - g_i'(\sigma_i Y)\right] \leq 0 \quad \text{and} \quad [\cdots]\sigma_i = 0. \tag{7}$$

This condition leads directly to the next lemma.

Lemma 1. *Under A.1 and A.2 there exists a share function:* $s_i(Y)$*.* $s_i(Y)$ *satisfies* $s_i(Y) = 0$ *if and only if* $f_i'(0) < \infty$ *and* $Y \geq R_i(Y) f_i'(0)$*. Otherwise,* $s_i(Y) = \sigma_i$*, where* σ_i *is the unique solution of*

$$R_i'(Y)\sigma_i + \frac{R_i(Y)}{Y}(1-\sigma_i) = g_i'(\sigma_i Y). \tag{8}$$

Proof. Let us denote the left-hand side of (8) by $h_i(\sigma_i)$ and the right-hand side by $z_i(\sigma_i)$. An intersection of these two functions, if any, which is a solution of (8), determines share values. The function $h_i(\sigma_i)$ has the following properties in light of A.2.

$$\begin{aligned} h_i(0) &= \frac{R_i(Y)}{Y} > 0, \quad h_i(1) = R_i'(Y) > 0, \quad h_i(0) - h_i(1) = \frac{R_i(Y)}{Y}(1-\epsilon_i) > 0, \\ h_i'(\sigma_i) &= -\frac{R_i(Y)}{Y}(1-\epsilon_i) < 0. \end{aligned}$$

Then, the function $h_i(\sigma_i)$ is strictly decreasing in σ_i, and is bounded from above and below. In contrast, the function $z_i(\sigma_i)$ has the following properties due to A.1 or (2).

$$\begin{aligned} z_i(0) &= g_i'(0) > 0, \quad z_i(1) = g_i'(Y) > 0, \quad z_i(0) - z_i(1) = g_i'(0) - g_i'(Y) < 0, \\ z_i'(\sigma) &= g_i''(\sigma_i Y) Y > 0. \end{aligned}$$

The function $z_i(\sigma_i)$ is strictly increasing in σ_i. Notice in addition that $z_i(\sigma_i)$ exceeds the left at $\sigma_i = 1$ for some $Y > 0$ by A.1 and A.2. Thus, we may conclude that there is a unique share value in interval $(0, 1)$; it is zero if and only if $R_i(Y) \leq g_i'(0) Y$. The proof is completed by observing that $g_i'(0) = [f_i'(0)]^{-1}$. □

We may use this lemma to infer the crucial qualitative properties of the share function derived under A.1 and A.2. The full details are set out in the following lemma.

Lemma 2. *Under A.1 and A.2, the share function $s_i(Y)$ has the following properties:*

1. $s_i(Y)$ *is continuous,*

2. $\lim_{Y \to 0} s_i(Y) = 1$,

3. $s_i(Y)$ *is strictly decreasing where positive,*

4. *if* $f_i'(0) < \infty$, $s_i(Y) > 0$ *for* $0 < Y < R_i(Y) f_i'(0)$ *and* $s_i(Y) = 0$ *if* $Y \geq R_i(Y) f_i'(0)$, *and*

5. *if* $f_i'(0) = \infty$, $s_i(Y) > 0$ *for all* $Y > 0$ *and* $s_i(Y) \to 0$ *as* $Y \to \infty$.

Proof. First, note that the shares are continuous (indeed differentiable where positive) by the implicit function theorem, establishing Part 1. Second, since $g_i'(0)$ is finite, letting $Y \to 0$ on both side of (8) shows that the share must approach one as Y approaches zero, giving Part 2. To justify Part 3, we investigate the slope of s_i. The total differential of (8) has the following form:

$$\Big(R_i' - \frac{R_i}{Y} - g_i^{''} Y\Big) d\sigma_i = \Big(\frac{1}{Y}(\sigma_i Y (g_i^{''} - R_i^{''}) + (1-\sigma_i)(\frac{R_i}{Y} - R_i'))\Big) dY.$$

Using the elasticity of prize of player i, ϵ_i, we can then express the slope of s_i as follows:

$$s_i'(Y) = \frac{(g_i^{''} - R_i^{''})\sigma_i Y + \frac{R_i(1-\sigma_i)}{Y}(1-\epsilon_i)}{-R_i(1-\epsilon_i) - g_i^{''} Y^2} < 0.$$

The inequality follows since the denominator is negative by A.1 and A.2. The numerator is positive in light of A.1 and A.2. We may deduce that the positive shares are strictly decreasing in Y, establishing Part 3. The fourth part is an immediate consequence of Lemma 1. Finally, suppose that the marginal product $f_i'(0)$ is unbounded, which implies $g_i'(0) = 0$. Then (8) can hold as $Y \to \infty$ only if the share function approaches zero. In fact, (8) can be rewritten as

$$1 - (1-\epsilon_i)\sigma_i = g_i'(\sigma_i Y) \frac{Y}{R_i(Y)}.$$

Notice that an increase in Y implies an increase in the right-hand side of the above equation due to A.1 and A.2. On the other hand, the left-hand side of it is bounded above (i.e., 1) in light of A.2. Hence, as $Y \rightarrow \infty$, for (8) to be satisfied we must have $\sigma_i \longrightarrow 0$.

This completes the proof of the lemma. □

Recall that a Nash equilibrium Y^* corresponds to the solution to $\sum_{i=1}^{n} s_i(Y^*) = 1$. It follows from Lemma 2 that the aggregate share function is continuous, exceeds 1 for small enough Y, is less than 1 for large enough Y and is strictly decreasing when positive. Therefore, the equilibrium value is unique. Then, a unique Y^* implies a unique strategy profile $(y_1^*, \cdots, y_n^*)$, and we have the following result.

Theorem 1. *Under A.1 and A.2, there exists a unique pure Nash equilibrium in an asymmetric contest with endogenous prize.*

Finally, notice that for all player i and any fixed value of Y_{-i}, the solution $y_i = 0$ always gives zero payoff value for this player. Therefore, at the best response, her expected payoff must not be negative. Hence, under A.1 and A.2, each player enjoys nonnegative expected payoff at the equilibrium.

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