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Abstract

Asymmetric contest games are examined under conditions which guarantee the existence of a unique pure Nash equilibrium. Conditions are derived for the local asymptotical stability of the equilibrium under continuous and discrete dynamics with gradient adjustments. In both cases, a crucial assumption is the nonexistence of a dominant player at the equilibrium level. In the case of continuous time scales, this is sufficient for stability, and in the discrete case, the speeds of adjustments have to be sufficiently small. As special cases, symmetric and semisymmetric games are analysed in details.

Keywords: Asymmetric contests, Dynamic models, Discrete dynamics, Continuous dynamics, Stability analysis

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1 Introduction

Contest games model situations when the players invest in order to increase the probability of winning a given prize. Contest games are closely related to oligopolies with hyperbolic price functions (Bischi et al., 2010), market share attraction games (Hanssens et al., 1990), rent seeking games (Tullock, 1980), to mention only a few. Many studies have examined contest and related games with exogenous prize, Pérez-Castrillo and Verdier (1992), Szidarovszky and Okuguchi (1997), Cornes and Hartley (2005) and Yamazaki (2008).

However in many cases (such as R&D contest, war, armament) the size of the prize depends on the efforts of the players. For example, higher efforts make more valuable prize available to the players. Chung (1996) was the first who examined rent-seeking games with an endogenous prize (rent in this case) which was assumed to be an increasing function of the aggregate effort of all players. For this case, Okuguchi (2005) and Corchón (2007) proved the existence of a unique symmetric pure Nash equilibrium. In the model developed by Shaffer (2006), an increased effort has a decreasing effect on the value of the prize. In these studies identical players were assumed in the valuation of the prize as well as abilities.

The heterogeneity of players can be divided into three types: in different valuations of the prize (e.g., Hillman and Riley, 1989), in different abilities to convert higher expenditures to higher productivity (e.g., Baik, 1994) and also in different financial constraints (e.g., Che and Gale, 1997).

Szidarovszky and Okuguchi (1997) proved the existence of a unique pure Nash equilibrium in special asymmetric rent-seeking games. This result was later generalized for contest games by Hirai (2012) and Hirai and Szidarovszky (2013).

Dynamic rent-seeking games were first examined in Okuguchi and Szidarovszky (1999) where the local asymptotic stability of the pure Nash equilibrium was proved by using linearization around the equilibrium. This result was reconsidered to discrete time scales and the global stability of the equilibrium was analyzed in both discrete and continuous time sales in Bischi et al. (2010). In these studies instantaneous information was assumed about the actions of the competitors as well as on available data and expectations of the players. Time delays were introduced in many variants of the oligopoly model including the cases of hyperbolic price function in Matsumoto and Szidarovszky (2018).

In this paper the model of Hirai and Szidarovszky (2013) is reconsidered by introducing its dynamic extensions in both discrete and continuous time scales. The stability analysis is more general than in earlier studies. The paper is developed as follows. In Section 2, the basic model is described. Sections 3 and 4, discuss the stability properties of the pure Nash equilibrium with continuous and discrete time scales. Conclusions and future research directions are given in Section 5.

2 The Basic Model

There are n players in a contest game, which are risk-neutral. If x_i denotes the expenditure of player i and $\varphi_i(x_i)$ is its production function for lotteries, then the probability that player i wins the prize is

$$p_i = \frac{\varphi_i(x_i)}{\sum_{j=1}^{n} \varphi_j(x_j)}.$$

Let L_i denote the budget of player i implying that its set of feasible strategies is the closed interval $[0, L_i]$. In Hirai and Szidarovszky (2013) the following assumptions were made:

Assumption 1. For all players i, function φ_i is twice differentiable,

$$\varphi_i(0) = 0,\ \varphi_i'(x_i) > 0 \text{ and } \varphi_i''(x_i) < 0 \text{ for } x_i \in [0, L_i].$$

The special form of $\varphi_i(x_i) = a_i x_i$ was studied earlier by several authors (e.g., Skaperdas, 1996, Clark and Riis, 1998). Introducing the new variables $y_i = \varphi_i(x_i)$ which represent the effective effort of player i, the expected payoff of player i can be given as follows:

$$\Pi_i = R_i(y_i + Q_i)\frac{y_i}{y_i + Q_i} - g_i(y_i) \tag{1}$$

where

$$Q_i = \sum_{j \neq i}^{n} y_j,\ g_i(y_i) = \varphi_i^{-1}(y_i)$$

and the prize as the function of the aggregate effort is

$$R_i(y_i + Q_i).$$

Notice that Assumption 1 implies that

$$g_i(0) = 0,\ g_i'(y_i) > 0 \text{ and } g_i''(y_i) > 0 \text{ for all } y_i \in [0, \varphi_i(L_i)].$$

Function $g_i(y_i)$ can be considered as the total cost of player i to generate level y_i of effort. Let

$$Q = \sum_{i=1}^{n} y_i$$

be the aggregate effort of all players.

Assumption 2. For all players i, the prize $R_i(Q)$ is twice differentiable,

$$R_i(Q) > 0 \text{ and } R_i''(Q) \leq 0 \text{ for all } Q \in [0, \sum_{i=1}^{n} \varphi_i(L_i)].$$

This assumption allows both positive and negative externalities of the aggregate effort as the linear function

$$R_i(Q) = a_i + b_i Q$$

shows with $b_i > 0$ and $b_i < 0$. Notice in addition that Assumptions 1 and 2 imply that $\Pi_i(y_i.Q_i)$ is strictly concave in y_i. Hirai and Szidarovszky (2013) proved that under Assumptions 1 and 2, there is a unique pure Nash equilibrium. The asymptotic properties of this equilibrium will be examined in the next sections.

3 Continuous Dynamics

Considering gradient adjustments, we notice first that by differentiation,

$$\frac{\partial \Pi_i}{\partial y_i} = R_i'(y_i + Q_i)\frac{y_i}{y_i + Q_i} + R_i(y_i + Q_i)\frac{Q_i}{(y_i + Q_i)^2} - g_i'(y_i), \tag{2}$$

which will be denoted by $f_i(y_i, Q_i)$ for notational simplicity. Then the gradient dynamics is described by the following system of ordinary differential equations:

$$\dot{y}_i = K_i f_i(y_i, Q_i) \quad (i = 1, 2, \ldots, n) \tag{3}$$

where $K_i > 0$ is the speed of adjustment of player i. The local asymptotic behavior of the equilibrium can be examined by linearization. Notice that

$$\frac{\partial f_i}{\partial y_i} = \frac{(R_i^{''} y_i + R_i^{'})(y_i + Q_i) - R_i' y_i}{(y_i + Q_i)^2} + \frac{R_i^{'} Q_i (y_i + Q_i)^2 - 2R_i Q_i (y_i + Q_i)}{(y_i + Q_i)^4} - g_i^{''}(y_i), \tag{4}$$

and with $j \neq i$,

$$\frac{\partial f_i}{\partial y_j} = \frac{R_i^{''} y_i (y_i + Q_i) - R_i^{'} y_i}{(y_i + Q_i)^2} + \frac{(R_i^{'} Q_i + R_i)(y_i + Q_i)^2 - 2R_i Q_i (y_i + Q_i)}{(y_i + Q_i)^4}. \tag{5}$$

Let S_i and T_i denote the right hand sides of (4) and (5), respectively. Notice first that

$$S_i = T_i + u_i \tag{6}$$

with

$$u_i = \frac{R_i' Q - R_i}{Q^2} - g_i^{''}. \tag{7}$$

We will now prove that $u_i < 0$. Consider function

$$h_i(Q) = R_i^{'}(Q)Q - R_i(Q).$$

Clearly, $h_i(0) = -R_i(0) \leqq 0$ and $h_i'(Q) = R_i^{''}(Q)Q \leqq 0$, therefore $h_i(Q) \leqq 0$ for all $Q \geqq 0$. Since $g_i^{''}(y_i) > 0$, the value of u_i is always negative.

Next we will find conditions which guarantee that $T_i \leqq 0$, implying that $S_i < 0$. Its numerator can be rewritten as

$$\begin{aligned} R_i^{''} y_i Q^2 - R_i^{'} y_i Q + R_i' Q_i Q + R_i Q - 2R_i Q_i \\ = R_i^{''} y_i Q^2 - R_i^{'} Q(y_i - Q_i) + R_i(Q - Q_i) - R_i Q_i \\ = R_i^{''} y_i Q^2 + (Q_i - y_i)(R_i^{'} Q - R_i). \end{aligned}$$

Since $R_i^{''} \leqq 0$ and $R_i^{'} Q - R_i \leqq 0$, the value of T_i is less than or equal to zero if $Q_i \geqq y_i$ for all i, that is, there is no dominant player.

The Jacobian of system (4) at the equilibrium has the special form

$$\boldsymbol{J}_C = \begin{pmatrix} K_1 S_1 & K_1 T_1 & \dots & K_1 T_1 \\ K_2 T_2 & K_2 S_2 & \dots & K_2 T_2 \\ \vdots & \vdots & & \vdots \\ K_n T_n & K_n T_n & \dots & K_n S_n \end{pmatrix}. \tag{8}$$

where S_i and T_i are now at their equilibrium levels.

By introducing

$$\boldsymbol{D} = \operatorname{diag}(K_1 u_1, \dots, K_n u_n), \ \ \boldsymbol{a} = \begin{pmatrix} K_1 T_1 \\ \vdots \\ K_n T_n \end{pmatrix} \text{ and } \mathbf{1}^T = (1, \dots, 1).$$

Here u_i denotes its equilibrium level. We have

$$\boldsymbol{J}_C = \boldsymbol{D} + \boldsymbol{a}\mathbf{1}^T \tag{9}$$

with characteristic polynomial

$$\begin{aligned} \varphi_C(\lambda) &= \det(\boldsymbol{D} + +\boldsymbol{a}\mathbf{1}^T - \lambda \boldsymbol{I}) \\ &= \det(\boldsymbol{D} - \lambda \boldsymbol{I}) \det(\boldsymbol{I} + (\boldsymbol{D} - \lambda \boldsymbol{I})^{-1} \boldsymbol{a}\mathbf{1}^T) \\ &= \Pi_{i=1}^n (K_i u_i - \lambda) \left[1 + \sum_{i=1}^n \frac{K_i T_i}{K_i u_i - \lambda} \right] \end{aligned}$$

where we used the well known fact that if $\boldsymbol{u}$ and $\boldsymbol{v}$ are n-element column vectors and $\boldsymbol{I}$ is the $n \times n$ identity matrix, then $\det(\boldsymbol{I} + \boldsymbol{u}\boldsymbol{v}^T) = \mathbf{1} + \boldsymbol{v}^T \boldsymbol{u}$, (Okuguchi and Szidarovszky (1999)). The potential eigenvalues are the $K_i u_i < 0$ quantities and the roots of equation

$$\sum_{i=1}^n \frac{K_i T_i}{K_i u_i - \lambda} + 1 = 0. \tag{10}$$

Let $k_C(\lambda)$ denote the left hand side, then clearly

$$\lim_{\lambda \to \pm\infty} k_C(\lambda) = 1, \quad \lim_{\lambda \to K_i u_i - 0} k_C(\lambda) = -\infty, \quad \lim_{\lambda \to K_i u_i + 0} k_C(\lambda) = +\infty$$

and

$$k'_C(\lambda) = \sum_{i=1}^{n} \frac{K_i T_i}{(K_i u_i - \lambda)^2} < 0$$

unless all $T_i = 0$. In this case the eigenvalues are $K_i S_i < 0$ for all i. The poles of $k_C(\lambda)$ are the $K_i u_i < 0$ values. Figure 1 shows the graph of $k_C(\lambda)$ by assuming that $n = 4$ and the players are numbered so that $K_1 u_1 < K_2 u_2 < \ldots < K_n u_n$. If some of the $K_i u_i$ values are equal, then the same argument can be used with minor modifications.

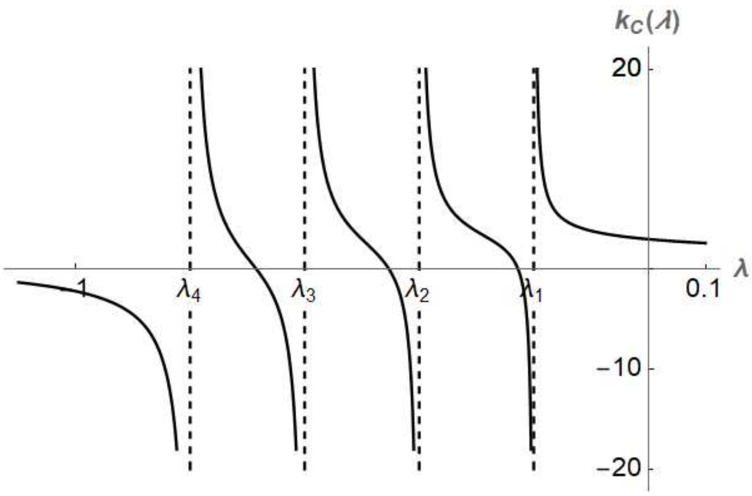

Figure 1. $n = 4$ and $K_{1u_1} < K_{2u_2} < K_{3u_3} < K_{4u_4}$

Notice that (10) is equivalent to a polynomial equation of degree n. There is one root before $K_1 u_1$, and one root inside each interval $(K_i u_i, K_{i+1} u_{i+1})$ for $i = 1, 2, \ldots, n-1$. So we found n real negative roots. So all eigenvalues are negative implying the following result.

Proposition 1 *The equilibrium is always locally asymptotically stable with continuous time dynamics (3), if no dominant player is present.*

4 Discrete Dynamics

In the case of discrete time scales system (3) is modified as

$$y_i(t+1) = y_i(t) + K_i f_i(y_i(t), Q_i(t)) \tag{11}$$

with Jacobian matrix

$$\boldsymbol{J}_D = \boldsymbol{I} + \boldsymbol{J}_C = \boldsymbol{I} + \boldsymbol{D} + \boldsymbol{a}\boldsymbol{1}^T. \tag{12}$$

The characteristic polynomial has the form

$$\begin{aligned}\varphi_D(\lambda) &= \det(\boldsymbol{I} + \boldsymbol{D} + \boldsymbol{a}\mathbf{1}^T - \lambda \boldsymbol{I}) \\ &= \det(\boldsymbol{I} + \boldsymbol{D} - \lambda \boldsymbol{I}) \det(\boldsymbol{I} + (\boldsymbol{I} + \boldsymbol{D} - \lambda \boldsymbol{I})^{-1} \boldsymbol{a}\mathbf{1}^T) \\ &= \Pi_{i=1}^n (1 + K_i u_i - \lambda) \left[1 + \sum_{i=1}^n \frac{K_i T_i}{1 + K_i u_i - \lambda} \right].\end{aligned}$$

The possible eigenvalues are $1 + K_i u_i$ for $i = 1, 2, \ldots, n$ and the solutions of equation

$$k_D(\lambda) = \sum_{i=1}^n \frac{K_i T_i}{1 + K_i u_i - \lambda} + 1 = 0. \tag{13}$$

Similarly to function $k_C(\lambda)$ it is easy to see that

$$\lim_{\lambda \to \pm\infty} k_D(\lambda) = 1, \quad \lim_{\lambda \to 1 + K_i u_i - 0} k_D(\lambda) = -\infty, \quad \lim_{\lambda \to 1 + K_i u_i + 0} k_D(\lambda) = +\infty,$$

$$k_D'(\lambda) = \sum_{i=1}^n \frac{K_i T_i}{(1 + K_i u_i - \lambda)^2} \leqq 0,$$

and the poles are the $1 + K_i u_i$ values. The graph of function $k_D(\lambda)$ is shown in Figure 2. There is one root before $1 + K_1 u_1$ and a root between each interval $(1 + K_i u_i, 1 + K_{i+1} u_{i+1})$. All roots are therefore less than 1, and they are larger than -1 if for all i, $1 + K_i u_i > -1$ and $k_D(-1) > 0$.

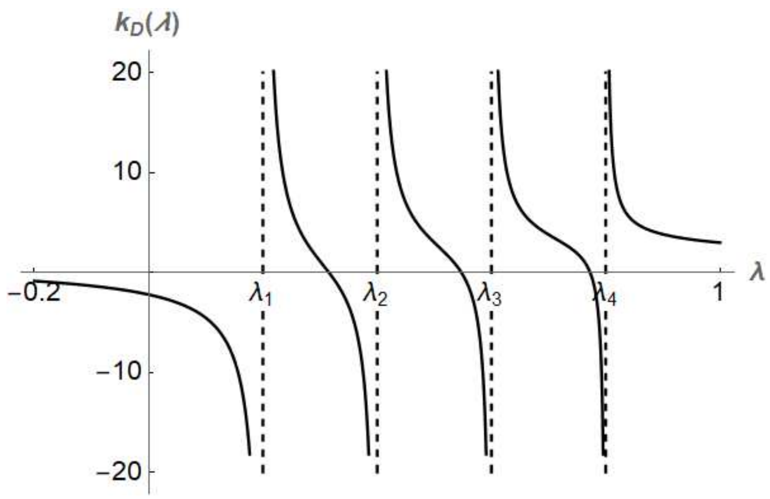

Figure 2. Graph of function $k_D(\lambda)$

These conditions can be written as

$$K_i < \frac{-2}{u_i} \quad \text{for all} \quad i, \tag{14}$$

and

$$\sum_{i=1}^{n} \frac{K_i T_i}{2 + K_i u_i} > -1. \tag{15}$$

when the denominator is positive by (14).

The left hand side of (15) is negative unless all $T_i = 0$, in which case the eigenvalues are the $1 + K_i u_i$ values, which are between -1 and $+1$ if (14) holds. Relation (15) can be interpreted as the speeds of adjustments have to be sufficiently small.

Proposition 2 *Assume that no dominant player is present. The equilibrium with the discrete time dynamics is locally asymptotically stable if (14) and (15) hold.*

In the symmetric case $T_i \equiv T$, $u_i \equiv u$, $K_i \equiv K$, $y_i(0) \equiv y(0)$. Clearly there is no dominant player, so the continuous dynamics is locally asymptotically stable. Relations (14) and (15) are simplified as

$$K < \frac{-2}{u} \tag{16}$$

and

$$\frac{nKT}{2 + Ku} > -1,$$

that is,

$$K < \frac{-2}{nT + u}, \tag{17}$$

which is the stability condition for the discrete time system.

Consider next the semi-symmetric case where for players i $(1 \leqq i \leqq m)$,

$$T_i \equiv T, u_i \equiv u, K_i \equiv K$$

and for players j $(m + 1 \leqq j \leqq n)$,

$$T_i = \bar{T}, u_j \equiv \bar{u} \quad \text{and} \quad K_j \equiv \bar{K}.$$

Then relation (14) has the forms

$$K < -\frac{2}{u} \quad \text{and} \quad \bar{K} < \frac{-2}{\bar{u}} \tag{18}$$

and (15) can be rewritten as

$$\frac{mKT}{2 + Ku} + \frac{(n - m)\bar{K}\bar{T}}{2 + \bar{K}\bar{u}} > -1$$

or

$$K(2mT + 2u) + \bar{K}(2(n - m)\bar{T} + 2\bar{u}) + K\bar{K}(mT\bar{u} + (n - m)\bar{T}u + u\bar{u}) > -4 \tag{19}$$

The coefficients of K and $\bar{K}$ are negative with a positive coefficient of $K\bar{K}$. This relation can be rewritten as

$$\bar{K}\left[K\left(mT\bar{u}+(n-m)\bar{T}u+u\bar{u}\right)+\left(2(n-m)\bar{T}+2\bar{u}\right)\right]>-4-K\left(2mT+2u\right)$$

or

$$\bar{K}\left(AK-B\right)>-4+CK \tag{20}$$

with

$$A=mT\bar{u}+(n-m)\bar{T}u+u\bar{u},$$
$$B=-\left[2(n-m)\bar{T}+2\bar{u}\right]$$

and

$$C=-\left(2mT+2u\right)$$

all being positive.

Relation (20) can be rewritten as

$$\bar{K}\begin{cases} > \dfrac{-4+CK}{AK-B} & \text{if } AK-B<0 \\ < \dfrac{-4+CK}{AK-B} & \text{if } AK-B>0 \end{cases} \tag{21}$$

It is easy to show that

$$\frac{4}{C}<\frac{B}{A} \text{ and } \frac{4}{B}<\frac{C}{A}.$$

Moreover

$$-\frac{2}{u}>\frac{B}{A} \text{ and-} -\frac{2}{\bar{u}}>\frac{C}{A},$$

furthermore point

$$\left(-\frac{2}{u},-\frac{2}{\bar{u}}\right)$$

is on the upper part of the hyperbola. So the stability region is the lower shaded area in Figure 3.

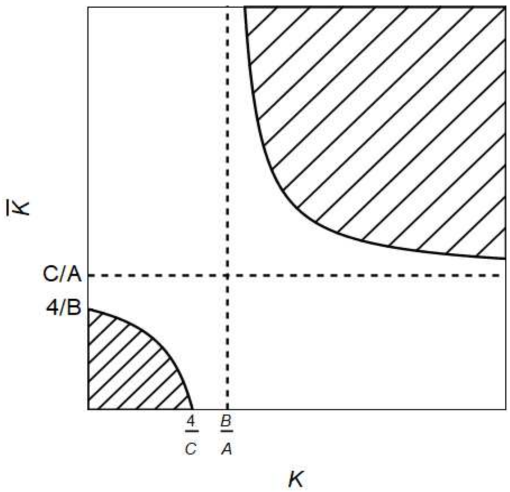

FIgure 3. Stability region in the $(K, \bar{K})$ plane

5 Conclusions

This paper examined the local asymptotical stability of the unique Nash equilibrium of asymmetric contest games with continuous and discrete time scales under gradient adjustments. In both cases the critical condition was that the game had no dominant player at the equilibrium level, which guaranteed the local asymptotical stability at the equilibrium in continuous dynamics. In the discrete case, in addition, the speeds of adjustments had to be sufficiently small to guarantee local asymptotical stability. Symmetric and semisymmetric cases were analyzed in detail.

Further research can continue in two different directions. First, the global asymptotical stability of the equilibrium can be examined, and second, time delays can be introduced into the models and an interesting question is to see how the delays can influence the asymptotical behavior of the equilibrium.

Conflict of Interest: The authors declare that they have no conflict of interest.

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