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Delays Dynamics in Nonlinear Monopoly with Gradient Adjustment

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Delay Dynamics in Nonlinear Monopoly with Gradient Adjustment*

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Abstract

Nonlinear dynamic monopoly is considered, when the firm does not have an analytic form of its profit function, but it can observe its value at any time. This is the case when the price function is unknown. In applying gradient dynamics, the marginal profit is approximated by finite differences based on two past profit observations. Stability conditions are derived first with discrete time scales, which are also applied in special cases. Two models of continuous dynamics are then introduced. The first is a natural modification of the discrete model, and the other includes an inertia coefficient with the derivative. In each case a delay differential equation is obtained with two delays. Stability conditions are derived and the stability switching curves are constructed and illustrated.

Keywords Gradient adjustment, Boundedly rational monoply, Discrete-time dynamics, Continuous-time dynamics, Stability switching curves

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1 Introduction

It is well-known that a monopolist in an elementary textbook of microeconomics is assumed to be *rational* in the sense that it has the perfect information on the market and instantaneous responses to changing circumstances. Accordingly, such a monopolist can choose the levels of price and output that maximizes its overall profit and can adjust its decisions in no time if some exogeneous changes occur. It is also well-known that the decision makers in real world are *boundedly rational* and thus have to make a decision under limited information and delayed responses.[1] We can say this behavioral difference in other words. The rational monopolist can jump to the optimal point of output and price in one shot without any adjustments. In consequence, output as well as price will not change over time (i.e., no dynamic consideration is necessary) unless environmental phenomenon changes. The boundedly rational monopolist, on the other hand, can make a mistake. It might produce a different amount of output and set a different value of price other than the optimal ones. Noticing the mistake and revising the decision, it experiences time delays in collecting past data of price and output associated with uncertainty, information and implementation delays. Output (and price) will vary in every subsequent time period. The main purpose of this paper is to shed light on such an adjustment or dynamic process of output of the boundedly rational monopolist.

In the existing literature, the gradient method is often adopted to describe the adjustment process of the boundedly rational monopolist toward the profit maximizing output. Accordingly, the monopolist increases the output level if its marginal profit is positive, decreases if negative and maintains the same output level if zero. Two types of models are known to introduce the method, discrete-time models and continuous-time models. It is demonstrated that the former could generate choatic dynamics if the involved nonlinearities are strong enough. Among others, we mention Puu (1995) that follows Baumol and Quandt (1964) constructing a model of monopoly with a linear cost function and a cubic price function with inflection points. Naimzada and Ricchiuti (2008) replace Puu's price function with a cubic function having no inflection points. Askar (2013) assumes a general concave price function. Elsadany and Awad (2015) introduce a log-concave function. In a continuous-time framework, Matsumoto and Szidarovszky (2012, 2014) build a monopoly model, focusing on the effects caused by time delays and show the delay effect can be a source of complex dynamics as well as simple dynamics. In those studies, it is assumed that the form of the demand function could be known or estimated correctly by using the past history of output and price.[2] In this study, the form of the price function is considered unknown.

The rest of this paper is organized as follows. Section 2 considers the learning process in a discrete-time model. Section 3 considers the same subject in a

[1] Clower (1959) calls the former knowledgeable monopolist and the latter ignorant monopolist.

[2] Even if the price function is known, it might be possible that a monopolist is endowed with limited computational skills to solve the profit maximization problem.

continuous-time framework. Section 4 constructs a continuous-time model with inertia. Finally, the concluding remarks and future research directions are given in Section 5.

2 Model

Consider a monopoly that produces one good. Let x denote its output and $\pi(x)$ its profit. In the dynamic monopoly models, the firm adjusts its output in proportion to its marginal profit, which is usually called the *gradient adjustment*. In applying best response dynamics the best output selection is used at each time period, however by knowing the best choice, the firm will make this choice at all times, so there is no need to dynamic adjustments. It is assumed that the firm does not have the analytic form of its profit, it can only observe it at each time. Therefore the marginal profit cannot be computed, only its estimates can be assessed.

3 Discrete time dynamics

If discrete time scales are assumed, then the following model can be considered:

$$x(t) = x(t-\tau_1) + K\frac{\pi(x(t-\tau_1)) - \pi(x(t-\tau_2))}{x(t-\tau_1) - x(t-\tau_2)} \quad (\tau_1 \leq \tau_2) \tag{1}$$

where $t-\tau_1$ and $t-\tau_2$ are earlier time periods with known profit values, so τ_1 and τ_2 are nonnegative integers. In estimating $\pi'(x(t))$, it is logical to select the earlier time periods as close to t as possible, so model (1) becomes

$$x(t) = x(t-1) + K\frac{\pi(x(t-1)) - \pi(x(t-2))}{x(t-1) - x(t-2)} \tag{2}$$

where $K > 0$ is the adjustment coefficient. The asymptotical behavior of this nonlinear model can be obtained by linearization around the steady state x^*. At the steady state $x(t) = x(t-\tau_1) = x(t-\tau_2) = x^*$, and from (1), we have

$$x(t) = x(t-\tau_1) + K\pi'(z)$$

with z being between $x(t-\tau_1)$ and $x(t-\tau_2)$. At the steady state z has to be also x^*, therefore

$$x^* = x^* + K\pi'(x^*),$$

implying that x^* is a stationary point of the profit function. In order to guarantee that the first order condition at x^* provides maximum, we make the following assumption,

Assumption 1. $\pi''(x^*) < 0$.

Notice that

$$\frac{\partial x(t)}{\partial x(t-1)} = 1+K\frac{\pi'(x(t-1))\,[x(t-1)-x(t-2)]-[\pi(x(t-1))-\pi(x(t-2))]}{(x(t-1)-x(t-2))^2}$$

where the numerator can be written as

$$\pi'(x(t-1))\,(x(t-1)-x(t-2))+\left[\pi'(x(t-1))\,(x(t-2)-x(t-1))+\frac{\pi''(z)}{2}\,(x(t-2)-x(t-1))^2\right]$$

where z is between $x(t-1)$ and $x(t-2)$. Therefore

$$\frac{\partial x(t)}{\partial x(t-1)} = 1+K\frac{\pi''(z)}{2}.$$

At the equilibrium $z = x^*$, therefore this derivative becomes

$$\frac{\partial x(t)}{\partial x(t-1)} = 1+K\frac{\pi''(x^*)}{2}.$$

Similarly,

$$\frac{\partial x(t)}{\partial x(t-2)} = K\frac{\pi''(x^*)}{2}$$

at the steady state.

By introducing the notation

$$A = K\frac{\pi''(x^*)}{2},$$

the linearized equation becomes

$$x(t) = (A+1)\,x(t-1) + Ax(t-2) \tag{3}$$

with characteristic equation

$$\lambda^2 - (A+1)\,\lambda - A = 0. \tag{4}$$

The steady state is locally asymptotically stable if

$$\pm(A+1) - A + 1 > 0$$

$$-A < 1$$

which can be simplified as $-1 < A < 0$.

Proposition 1 *The steady state of dynamic equation (2) is locally asymptotically stable if $-1 < A < 0$ and locally unstable if $A < -1$*

Next we will show two examples with different forms of the price function where this general stability condition applies. Askar (2013) assumes a general concave price function,

$$p = a - bx^{\alpha},\ \ \alpha \in \mathbb{Z}^{+}. \tag{5}$$

With a marginal cost c, the profit function is

$$\pi(x) = (p - c)x = (a - c)x - bx^{1+\alpha}$$

and its second derivative at the equilibrium point x^* is

$$\pi''(x^*) = -\alpha b(1+\alpha)\left(\frac{a-c}{(1+\alpha)b}\right)^{\frac{\alpha-1}{\alpha}}$$

where x^* solves $\pi'(x^*) = 0$. Apparently the profit function satisfies the second-order condition for profit maximization $\pi''(x^*) < 0$, the stability condition is, according to Proposition 1,

$$-A = -\frac{K}{2}\pi''(x^*) = \frac{K}{2}\alpha b(1+\alpha)\left(\frac{a-c}{(1+\alpha)b}\right)^{\frac{\alpha-1}{\alpha}} < 1 \tag{6}$$

that is identical with the condition given in his Proposition. The price function assumed by Naimzada and Ricciuti (2008) is (5) with $\alpha = 3$ and the corresponding stability condition is obtained from (6) as

$$12Kb\left(\frac{a-c}{4b}\right)^{\frac{2}{3}} < 2$$

that is, needless to say, the same as the one given in their Proposition. Elsadany and Awad (2016) consider the case in which the price function is log-concave,

$$p = a - b\ln x.$$

The profit function is

$$\pi(x) = (a-c)x - bx\ln x$$

and its second-derivative at the equilibrium x^* is

$$\pi''(x^*) = -\frac{b}{x^*}$$

where x^* solves $\pi'(x^*) = 0$. They adopt the growth rate dynamics with the gradient method in which the adjustment coefficient K should be replaced by $Kx(t-1)$ in (2). Hence the stability condition is

$$-A = -\frac{Kx^*}{2}\pi''(x^*) = \frac{Kb}{2} < 1$$

which is the same as the stability condition given in their Theorem 1.

We now consider the stability conditions of the more general delay difference equation,

$$x(t) = \alpha x(t-m) + \beta x(t-k) \tag{7}$$

where k and m are coprime intergers, $k > m > 0,\ \alpha = A + 1$ and $\beta = A < 0$. If we assume $-1 < A < 0$, then

$$|\alpha| + |\beta| = |A + 1| + |A| = 1.$$

Under this special condition, according to Corollary 3.1 of Čermák and Jánský (2015), the steady state of (7) is asumptotically stable if $\alpha^k \beta^m < 0$ for any such k and m. Since $A + 1 > 0$ and $A < 0$, it is clear that

$$\alpha^k \beta^m = (A+1)^k A^m < 0 \text{ if } m \text{ is odd.}$$

We then have the following result with $m = 1$.

Proposition 2 *If $-1 < A < 0$, then the steady state of the following delay difference equation is locally asymptotically stable for any $k \geq 2$,*

$$x(t) = x(t-1) + K\frac{\pi(x(t-1)) - \pi(x(t-k))}{x(t-1) - x(t-k)}.$$

4 Continuous time dynamics I

Considering continuous time scales, the direction $\dot{x}(t)$ of the output change is determined, and in this case $x(t)$ might be already known. Equation (1) can be modified as

$$\dot{x}(t) = K\frac{\pi(x(t-\tau_1)) - \pi(x(t-\tau_2))}{x(t-\tau_1) - x(t-\tau_2)} \tag{8}$$

where the case of $\tau_1 = 0$ is possible. Similar to the discrete case, at the steady state

$$\frac{\partial \dot{x}(t)}{\partial x(t-\tau_1)} = K\frac{\pi''(x^*)}{2} = A$$

and

$$\frac{\partial \dot{x}(t)}{\partial x(t-\tau_2)} = K\frac{\pi''(x^*)}{2} = A.$$

So the linearized model has the form

$$\dot{x}(t) = Ax(t-\tau_1) + Ax(t-\tau_2). \tag{9}$$

with characteristic equation

$$\lambda - Ae^{-\lambda\tau_1} - Ae^{-\lambda\tau_2} = 0. \tag{10}$$

Case 0: $\tau_1 = 0,\ \tau_2 = 0.$

Consider first the no-delay case as a benchmark. The characteristic equation (10) with $\tau_1 = \tau_2 = 0$ is

$$\lambda = 2A.$$

The steady state is locally asymptotically stable since $A < 0$.

Case 1: $\tau_1 = 0,\ \tau_2 > 0$

Consider now the special case of $\tau_1 = 0$. Then equation (10) is reduced to a one-delay equation,

$$\lambda - A - Ae^{-\lambda\tau_2} = 0. \tag{11}$$

As is already seen, the steady state is locally asymptotically stable at $\tau_2 = 0$. As the value of τ_2 increases, stability might be lost, when $\lambda = i\omega$ ($\omega > 0$). Assuming positive value of ω does not restrict generality, since if λ is an eigenvalue, then its complex conjugate is also an eigenvalue. Substituting this value of λ into equation (11), we have

$$i\omega - A - A(\cos\omega\tau_2 - i\sin\omega\tau_2) = 0.$$

The separation of the real and imaginary parts gives

$$A + A\cos\omega\tau_2 = 0,$$

$$\omega + A\sin\omega\tau_2 = 0.$$

The first equation implies that $\cos\omega\tau_2 = -1$, so $\sin\omega\tau_2 = 0$ which contradicts the second equation. Therefore there is no stability switch.[3]

Proposition 3 *If $\tau_1 = 0$, then the steady state is locally asymptotically stable with all $\tau_2 > 0$.*

Case 2: $\tau_1 > 0,\ \tau_2 > 0$

In the general case of $\tau_1 > 0$ and $\tau_2 > 0$, equation (10) can be rewritten as

$$P_0(\lambda) + P_1(\lambda)e^{-\lambda\tau_1} + P_2(\lambda)e^{-\lambda\tau_2} = 0 \tag{12}$$

with

$$P_0(\lambda) = \lambda \text{ and } P_1(\lambda) = P_2(\lambda) = -A.$$

Before looking for stability switching curves, the following conditions should be verified (Gu et al. (2005)):

(i) $\deg[P_0(\lambda)] \geq \max\{\deg[P_1(\lambda)], \deg[P_2(\lambda)]\}$.

(ii) $P_0(0) + P_1(0) + P_2(0) \neq 0$.

[3] Mathematically, we have the same result if $\tau_1 > 0$ and $\tau_2 = 0$. However this symmetric case is assumed away by assumption $\tau_1 < \tau_2$.

(iii) The polynomials $P_0(\lambda),\ P_1(\lambda)$ and $P_2(\lambda)$ do not have any common roots.

(iv) $\lim_{\lambda\to\infty}\left(\left|\dfrac{P_1(\lambda)}{P_0(\lambda)}\right|+\left|\dfrac{P_2(\lambda)}{P_0(\lambda)}\right|\right)<1.$

Equation (12) satisfies these conditions. Since deg$[P_0(\lambda)]=1$ and deg$[P_1(\lambda)]=$deg$[P_2(\lambda)]=0$, condition (i) is satisfied. Condition (ii) is satisfied as $P_0(0)+P_1(0)+P_2(0)=-2A\neq 0$. Condition (iii) is apparently satisfied as $P_0(\lambda),\ P_1(\lambda)$ and $P_2(\lambda)$ have no common roots. Condition (iv) also holds, since

$$\left(\left|\frac{P_1(\lambda)}{P_0(\lambda)}\right|+\left|\frac{P_2(\lambda)}{P_0(\lambda)}\right|\right)=-\frac{2A}{|\lambda|}\to 0 \text{ as } \lambda\to\infty.$$

Dividing equation (12) by λ, we get

$$1+a_1(\lambda)e^{-\lambda\tau_1}+a_2(\lambda)e^{-\lambda\tau_2}=0, \tag{13}$$

where new functions are

$$a_1(\lambda)=\frac{-A}{\lambda} \text{ and } a_2(\lambda)=\frac{-A}{\lambda}.$$

We examine the stability switches of the non-trivial solution of dynamic equation (9) as the delays (τ_1,τ_2) vary. The modified characteristic equation (13) must have a pair of pure conjugate imaginary roots and stability switch occurs for the corresponding critical delays. So let $\lambda=i\omega,\ \omega>0$ and substitute it into equation (13),

$$1+a_1(i\omega)e^{-i\omega\tau_1}+a_2(i\omega)e^{-i\omega\tau_2}=0 \tag{14}$$

where

$$a_1(i\omega)=a_2(i\omega)=i\frac{A}{\omega}.$$

We now solve equation (14). To this purpose, we treat the three terms in the left hand side of equation (14) as three vectors in the complex plane with the magnitudes, 1, $|a_1(i\omega)|$ and $|a_2(i\omega)|$ where the absolute values are

$$|a_1(i\omega)|=|a_2(i\omega)|=-\frac{A}{\omega}.$$

The right hand side of equation (14) is zero, implying that if we put these vectors head to tail, then they form a triangle as illustrated in Figure 1. Similar triangle can be formed under the real axis. Since the sum of lengths of the two line segments is not shorter than that of the remaining line segment in a triangle, these absolute values satisfy the following inequality conditions

$$1\leq |a_1(i\omega)|+|a_2(i\omega)|, \tag{15}$$

and

$$-1\leq |a_1(i\omega)|-|a_2(i\omega)|\leq 1 \tag{16}$$

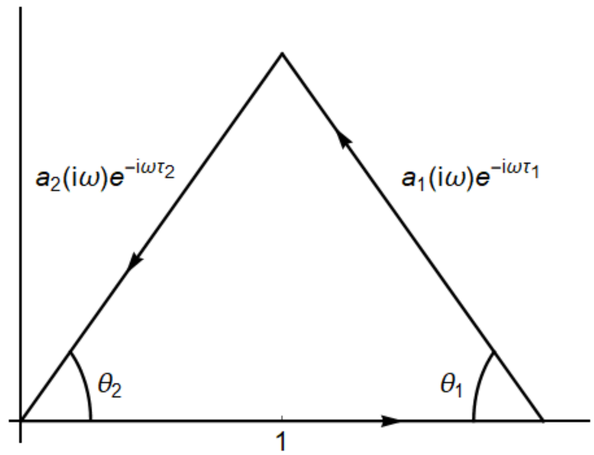

Figure 1. Triangle representation of equation (11)

Relation (16) is clearly satisfied, and (15) requires that

$$-\frac{2A}{\omega} \geq 1 \text{ or } 0 < \omega \leq -2A. \tag{17}$$

By using the cosine rule,

$$\begin{aligned} \theta_1 = \theta_2 &= \cos^{-1}\left(\frac{1 + |a_1(i\omega)|^2 - |a_2(i\omega)|^2}{2\,|a_1(i\omega)|}\right) \\ &= \cos^{-1}\left(-\frac{\omega}{2A}\right). \end{aligned} \tag{18}$$

Notice also that

$$\theta_1 = \theta_2 \in \left(0, \frac{\pi}{2}\right)$$

and

$$\arg[a_1(i\omega)] = \arg\left[a_2(i\omega)\right] = \arg\left[i\frac{A}{\omega}\right] = \frac{3\pi}{2}.$$

So the stability switching curves are given as

$$\tau_1^{\pm} = \frac{1}{\omega}\left(\frac{3\pi}{2} + (2u-1)\,\pi \pm \theta_1\right) \tag{19}$$

and

$$\tau_2^{\mp} = \frac{1}{\omega}\left(\frac{3\pi}{2} + (2v-1)\,\pi \mp \theta_2\right) \tag{20}$$

where both τ_1 and τ_2 are positive with all nonnegative integer values of u and v. Hence we have infinitely many stability switching curves. Some stability switching curves are illustrated in Figure 2, where the points (τ_1^+, τ_2^-) are shown in red color and the points (τ_1^-, τ_2^+) in blue.

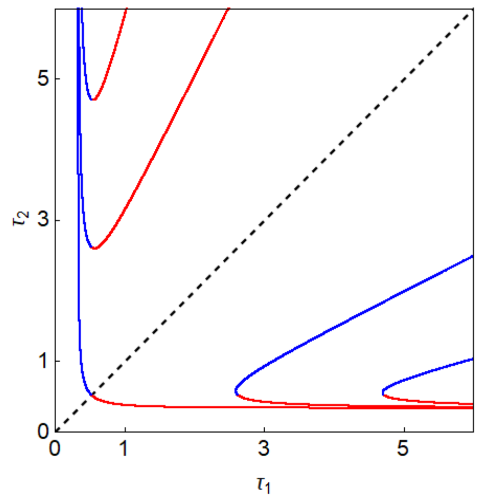

Figure 2. Stability switching curves for $u, v = 0, 1, 2$

5 Continuous time dynamics II

An alternative model can be formulated from (1), if we rewrite it as

$$x(t)-x(t-1) = -x(t-1)+\left\{x(t-\tau_1) + K\frac{\pi(x(t-\tau_1)) - \pi(x(t-\tau_2))}{x(t-\tau_1) - x(t-\tau_2)}\right\} \ (\tau_1 < \tau_2)\,. \tag{21}$$

By using $x(t) - x(t-1) = \delta\dot{x}(t)$ where $\delta > 0$ is the inertia coefficient, and replacing $t-1$ with t with new values of τ_1 and τ_2, we have

$$\delta\dot{x}(t) = -x(t) + \left\{x(t-\tau_1) + K\frac{\pi(x(t-\tau_1)) - \pi(x(t-\tau_2))}{x(t-\tau_1) - x(t-\tau_2)}\right\}. \tag{22}$$

This is a nonlinear delay differential equation, which is reduced to the difference equation (1) if $\delta = 0$. Its linearized version is the following:

$$\delta\dot{x}(t) = -x(t) + (1+A)x(t-\tau_1) + Ax(t-\tau_2)$$

with characteristic equation

$$\delta\lambda + 1 - (1+A)e^{-\lambda\tau_1} - Ae^{-\lambda\tau_2} = 0. \tag{23}$$

Without delay $\tau_1 = \tau_2 = 0$,

$$\lambda = \frac{2A}{\delta} < 0$$

and the steady state is locally asymptotically stable.

In the case of delays we get again equation (13) with

$$a_1(\lambda) = -\frac{1+A}{\delta\lambda+1} \text{ and } a_2(\lambda) = -\frac{A}{\delta\lambda+1}.$$

So

$$a_1(i\omega) = -\frac{1+A}{i\delta\omega+1} = \frac{-(1+A)+i\delta\omega(1+A)}{1+(\delta\omega)^2}$$

and

$$a_2(i\omega) = -\frac{A}{i\delta\omega+1} = \frac{-A+i\delta\omega A}{1+(\delta\omega)^2}$$

implying that

$$|a_1(i\omega)|^2 = \frac{(1+A)^2+[\delta\omega(1+A)]^2}{\left[1+(\delta\omega)^2\right]^2} = \frac{(1+A)^2}{1+(\delta\omega)^2}$$

and

$$|a_2(i\omega)|^2 = \frac{A^2+(\delta\omega A)^2}{\left[1+(\delta\omega)^2\right]^2} = \frac{A^2}{1+(\delta\omega)^2}.$$

Conditions (15) and (16) have now the forms

$$\frac{|1+A|}{\sqrt{1+(\delta\omega)^2}} + \frac{-A}{\sqrt{1+(\delta\omega)^2}} \geq 1 \tag{24}$$

and

$$-1 \leq \frac{|1+A|+A}{\sqrt{1+(\delta\omega)^2}} \leq 1 \tag{25}$$

Now we have to consider two cases:

(i) $-1 \leq A \leq 0$

In this case (24) gives

$$1 \geq \sqrt{1+(\delta\omega)^2}$$

which is impossible, so there is no stability switch.

Proposition 4 *If $-1 \leq A \leq 0$, then the steady state is locally asymptotically stable with all $\tau_1 \geq 0$ and $\tau_2 \geq 0$.*

(ii) $A < -1$

This condition is equivalent to $A + 1 < 0$, then (24) gives

$$-2A - 1 \geq \sqrt{1 + (\delta\omega)^2}$$

or

$$\omega^2 \leq \frac{4A(A+1)}{\delta^2}$$

showing that this condition holds if

$$0 < \omega \leq \frac{2}{\delta}\sqrt{A(A+1)}. \tag{26}$$

Similarly, condition (25) has the form

$$-1 \leq \frac{-A - 1 + A}{\sqrt{1 + (\delta\omega)^2}} \leq 1$$

which always holds.

Proposition 5 *If $A < -1$, then stability switch might occur with all ω values satisfying relation (26).*

Similarly to the previously discussed model, based on Figure 1, the application of the low of cosine presents,

$$\theta_1(\omega) = \cos^{-1}\left(\frac{1 + |a_1(i\omega)|^2 - |a_2(i\omega)|^2}{2\,|a_1(i\omega)|}\right) = \cos^{-1}\left(\frac{(\delta\omega)^2 + 2(1+A)}{2\,|1+A|\sqrt{1 + (\delta\omega)^2}}\right) \tag{27}$$

and

$$\theta_2(\omega) = \cos^{-1}\left(\frac{1 + |a_2(i\omega)|^2 - |a_1(i\omega)|^2}{2\,|a_2(i\omega)|}\right) = \cos^{-1}\left(\frac{(\delta\omega)^2 - 2A}{2\,|A|\sqrt{1 + (\delta\omega)^2}}\right). \tag{28}$$

Notice that $\theta_1 \in [0, \pi]$ and $\theta_2 \in [0, \pi/2]$, furthermore, the arguments of $a_1(i\omega)$ and $a_2(i\omega)$ are

$$\arg\left[a_1(i\omega)\right] = -\tan^{-1}(\delta\omega) + 2\pi$$

and

$$\arg\left[a_2(i\omega)\right] = -\tan^{-1}(\delta\omega) + 2\pi.$$

The arguments and the internal angles just obtained satisfy the following relations

$$\pm\theta_1 = \pi - \left\{\arg\left[a_1(i\omega)e^{-i\omega\tau_1}\right] + 2u\pi\right\} \text{ (u is integer)}$$

and

$$\mp\theta_2 = \pi - \left\{\arg\left[a_2(i\omega)e^{-i\omega\tau_2}\right] + 2v\pi\right\} \text{ (v is integer).}$$

Using the formula $\arg\left[a_i(i\omega)e^{-i\omega\tau_1}\right] = \arg\left[a_i(i\omega)\right] + \arg\left[e^{-i\omega\tau_i}\right]$ for $i = 1, 2$, we can solve these equations for the delays, τ_1 and τ_2,

$$\tau_1^{\pm} = \frac{1}{\omega}\left(\arg\left[a_1(i\omega)\right] + (2u-1)\pi \pm \theta_1(\omega)\right) \tag{29}$$

and

$$\tau_2^{\mp} = \frac{1}{\omega}\left(\arg\left[a_2(i\omega)\right] + (2v-1)\pi \mp \theta_2(\omega)\right) \tag{30}$$

where u and v are integers such that both τ_1 and τ_2 are nonnegative. Similarly to the previous case, there are infinitely many stability switching curves. Some curves are illustrated in Figure 3, where the points (τ_1^+, τ_2^-) are shown in red color and the points (τ_1^-, τ_2^+) in blue.

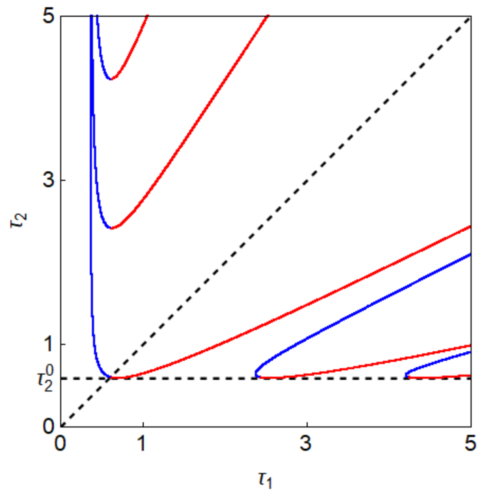

Figure 3. Stability switching curves for $u, v = 1, 2, 3$

Finally some special cases are discussed. We have already mentioned that the steady state is locally asymptotically stable if $\tau_1 = \tau_2 = 0$. Assume next that $\tau_1 = 0$ and $\tau_2 > 0$. The characteristic equation becomes

$$\delta\lambda + 1 - (1 + A) - Ae^{-\lambda\tau_2} = 0. \tag{31}$$

At a stability switch, $\lambda = i\omega$, so

$$i\delta\omega + 1 - (1 + A) - A\left(\cos\omega\tau_2 - i\sin\omega\tau_2\right) = 0.$$

Separation of the real and imaginary parts yields

$$-A - A\cos\omega\tau_2 = 0,$$

$$\delta\omega + A\sin\omega\tau_2 = 0,$$

where the first equation gives $\cos\omega\tau_2 = -1$, implying $\sin\omega\tau_2 = 0$. It contradicts the second equation.[4]

Proposition 6 *If $\tau_1 = 0$, then the steady state is locally asymptotically stable for all $\tau_2 \geq 0$.*

Assume finally that $\tau_1 = \tau_2 = \tau > 0$. Similarly to the previous case it is easy to see that there is no stability switch. Then a one-delay characteristic equation is obtained:

$$\delta\lambda + 1 - (1+2A)e^{-\lambda\tau} = 0. \tag{32}$$

Needless to say, the steady state is locally asymptotically stable at $\tau = 0$ (that is, when there are no delays). At any stability switch, $\lambda = i\omega$ ($\omega > 0$). So

$$i\delta\omega + 1 - (1+2A)(\cos\omega\tau - i\sin\omega\tau) = 0.$$

By separating the real and imaginary parts,

$$1 - (1+2A)\cos\omega\tau = 0,$$

$$\delta\omega + (1+2A)\sin\omega\tau = 0,$$

so

$$\omega^2 = \frac{4A(A+1)}{\delta^2} > 0$$

and therefore

$$\omega = \frac{2}{\delta}\sqrt{A(A+1)} \tag{33}$$

and the critical values of the delay are

$$\tau_n = \frac{1}{\omega}\left[\cos^{-1}\left(\frac{1}{1+2A}\right) \pm 2n\pi\right] \text{ for } n = 0, 1, 2, ... \tag{34}$$

The direction of the stability switch is obtained by selecting τ as the bifurcation parameter and considering λ as function of τ: $\lambda = \lambda(\tau)$. Implicitly differentiating (32) with respect to τ yields

$$\delta\lambda' - (1+2A)e^{-\lambda\tau}\left(-\lambda'\tau - \lambda\right) = 0$$

so

$$\lambda' = \frac{-(1+2A)e^{-\lambda\tau}\lambda}{\delta + (1+2A)e^{-\lambda\tau}\tau} = \frac{-\lambda(\delta\lambda+1)}{(\delta\lambda+1)\tau + \delta} = \frac{-\lambda^2\delta - \lambda}{\delta + \tau + \delta\tau\lambda}$$

[4] Mathematically, we can examine the case of $\tau_1 > 0$ and $\tau_2 = 0$ in which the characteristic equation is the following:

$$\delta\lambda + 1 - (1+A)e^{-\lambda\tau_1} - A = 0$$

Similarly to the previous case it is easy to see that there is no stability switch. However this case violates the assumption $\tau_1 \leq \tau_2$.

where equation (32) is used. At $\lambda = i\omega$,

$$\mathrm{Re}\left[\lambda'\right] = \mathrm{Re}\left[\frac{\omega^2\delta - i\omega}{\delta + \tau + i\delta\tau\omega} \cdot \frac{\delta + \tau - i\delta\tau\omega}{\delta + \tau - i\delta\tau\omega}\right] = \frac{(\omega\delta)^2}{(\delta + \tau)^2 + (\delta\tau\omega)^2} > 0$$

so the real part of λ' is positive. Therefore at each critical value at least one pair of eigenvalues changes the sign of its real part from negative to positive, so stability is lost at τ_0 and stability cannot be regained with larger values of τ.

Proposition 7 *If $\tau_1 = \tau_2 = \tau$, then the steady state is locally asymptotically stable for $\tau < \tau_0$, destabilized via Hopf bifurcation at $\tau = \tau_0$ and stability cannot be regained with larger values of τ.*

Figure 4 shows τ_0 as function of δ with fixed value of $A = -3/2$. Clearly with any value of A the graph is always an increasing linear function. The stability region in the (δ, τ) space is under this line.

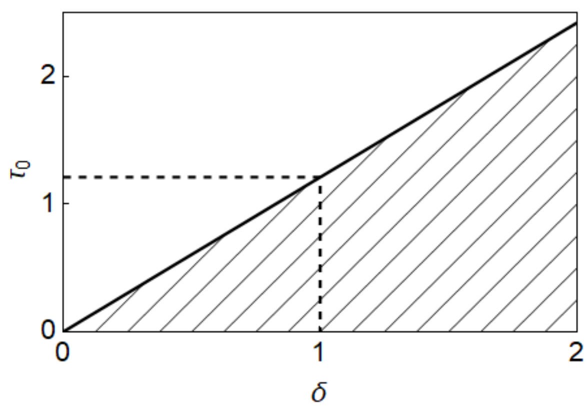

Figure 4. τ_0 as function of δ

Figure 5 shows τ_0 as function of A with fixed $\delta = 1$. The stability region in the (A, τ) space is under this curve.

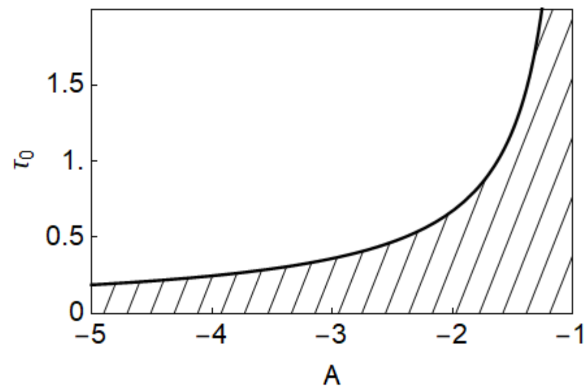

Figure 5. τ_0 as function of $A < -1$

6 Concluding Remarks

In this paper gradient adjustment process was introduced in a monopoly, when the firm does not have an analytic form of its profit function, but is able to observe the actual profit at any time. The marginal profit was approximated with a simple finite difference formula based on two past profit observations leading to dynamic models with two time delays. Assuming discrete time scales first, a general stability condition was derived and applied to special cases giving the same special stability conditions which are known from the literature. Two different continuous dynamics were then introduced and analyzed. The first was a simple modification of the discrete model, the other included an inertia coefficient with the derivative of the output trajectory. Without delays both systems are locally asymptotically stable, and stability can be lost with increasing positive values of the delays. The stability switching curves were derived and illustrated in both cases.

In approximating the marginal profit a very simple differentiation formula was used. However more sophisticated formulas could provide better approximations with increased numbers of delays. It is an interesting task to see how the more sophisticated differentiation formulas alter the stability conditions.

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