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On the Euler Equation and the Transversality Condition of Overtaking Optimal Solution in Macroeconomic Dynamics

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Abstract

In this paper, we consider the traditional capital accumulation model with continuous time, and derive the Euler equation for a necessary condition and the Euler equation with the transversality condition for a sufficient condition of overtaking (or, weakly overtaking) optimality.

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Keywords: capital accumulation, overtaking optimality, Euler equation, transversality condition.

1 Introduction

In usual macroeconomic textbook that treats the continuous time Ramsey-Cass-Koopmans capital accumulation model, (e.g., Romer (2011), Blanchard and Fischer (1989), Acemoglu (2009), etc.,) both the Euler equation and the transversality condition are argued. These arguments have, however, several common defects. First, to derive the Euler equation, they assume that the optimal consumption path is differentiable in time. In their proof, however, there is no verification for such a differentiability. Second, they assume the existence of the positive discount rate for utility, while Ramsey's

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(1928) classical treatment for such a model includes no discount rate. Third, they usually assume either the boundedness of the utility function or the Inada condition for showing the existence of the optimal solution, and thus they cannot treat some models such as the logarithmic AK model. Fourth, they do not derive a solution of such a model mathematically.

In this paper, we treat the general capital accumulation model with continuous time, and show under very weak conditions that 1) the Euler equation is a necessary condition for the **overtaking** optimality, (Theorems 1-3) and 2) the Euler equation together with the transversality condition is a sufficient condition for overtaking optimality. (Theorem 4)

In section 2, we present our model, and prepare the necessary notions and definitions. All results are contained in section 3.

2 The Model and Basic Notions

2.1 The Model

Traditional optimal capital accumulation model is written as follows.

$$\begin{aligned}
\max \quad & \int_0^\infty e^{-\rho t} u(c(t))dt \\
\text{subject to. } & k(0) = \bar{k} > 0,\ k(t) \geq 0,\ c(t) \geq 0, \\
& \dot{k}(t) = f(k(t)) - c(t) \text{ a.e.}, \\
& c(\cdot) \in W,
\end{aligned}$$

where k denotes the amount of capital, c denotes the amount of private consumption, and ρ denotes the time discount rate. The set W is some functional space. The equation

$$\dot{k}(t) = f(k(t)) - c(t)$$

represents two relationships in economy. First relationship is the equality between the production and the consumption. That is,

$$g(k(t)) = c(t) + i(t),$$

where the function g is the production function and i denotes the amount of investment. Second relationship is the relationship between the capital stock and investment. That is,

$$\dot{k}(t) = i(t) - dk(t),$$

where $d \geq 0$ represents the capital wastage ratio. To connect these two equation, we have

$$\dot{k}(t) = g(k(t)) - dk(t) - c(t) \equiv f(k(t)) - c(t),$$

where $f(k) = g(k) - dk$, as desired.

However, in this paper, u is not necessarily bounded and ρ is not necessarily positive, and thus, $\int_0^\infty e^{-\rho t}u(c(t))dt$ may not be able to be defined. Therefore, we modify the above problem to the following one:

$$\begin{aligned} \max \quad & lim_{T\to\infty}\int_0^T e^{-\rho t}u(c(t))dt \\ \text{subject to. } & k(0) = \bar{k} > 0,\ k(t) \geq 0,\ c(t) \geq 0, \\ & \dot{k}(t) = f(k(t)) - c(t) \text{ a.e.}, \\ & c(\cdot) \in W, \end{aligned} \tag{1}$$

where lim means either lim sup or lim inf. We need several assumptions.

Assumption 1. u is continuous, strictly concave, and increasing function defined on $\mathbb{R}_+$ and continuously differentiable on $\mathbb{R}_{++}$.

Assumption 2. $f(0) \geq 0$ and f is continuous and concave on $\mathbb{R}_+$, and continuously differentiable on $\mathbb{R}_{++}$. Moreover, there exists $k > 0$ such that $f(k) > 0$.[1]

Assumption 3. W denotes the set of all locally integrable function on $\mathbb{R}_+$.

Note that, under Assumption 3, k must be absolutely continuous on every compact set in $\mathbb{R}_+$.

2.2 Overtaking Optimality

We have modified the traditional capital accumulation problem to problem (1). However, even in this model, some problematic cases occurs. If there exist $(k_1(t), c_1(t))$ and $(k_2(t), c_2(t))$ such that $lim_{T\to\infty}\int_0^T e^{-\rho t}u(c_i(t))dt = +\infty$, then these must be equivalent, even though $c_1(t) > c_2(t)$ for every t. This is problematic, and thus we need a criterion that can compare these processes.

First, we define the notion of admissibility.

Definition 1. A pair of functions $(k(t), c(t))$ defined on $\mathbb{R}_+$ is called **admissible** if the following properties hold.

[1]We do not assume the Inada condition for f or g.

1. $c(t)$ is integrable and $k(t)$ is absolutely continuous on any compact interval in $\mathbb{R}_+$.[2]

2. $k(t) \geq 0$ and $c(t) \geq 0$ for all t.

3. the following differential equation

$$\dot{k}(t) = f(k(t)) - c(t)$$

holds for almost all $t \in \mathbb{R}_+$.

Let A be the set of all admissible pair, and $A_{\bar{k}}$ be the set of all admissible pair such that $k(0) = \bar{k}$.

Define a binary relation $\succ^*$ such that for any $(k_1(t), c_1(t)), (k_2(t), c_2(t)) \in A_{\bar{k}}$,

$$(k_1(t), c_1(t)) \succ^* (k_2(t), c_2(t)) \Leftrightarrow lim_{T\to\infty} \int_0^T e^{-\rho t}[u(c_1(t)) - u(c_2(t))]dt > 0,$$

and define $(k_1(t), c_1(t)) \succsim^* (k_2(t), c_2(t))$ if and only if $(k_2(t), c_2(t)) \not\succ^* (k_1(t), c_1(t))$. If *lim* is $\limsup$, the maximal element of $\succsim^*$ is called the **overtaking optimal solution** of problem (1), and if *lim* is $\liminf$, the maximal element of $\succsim^*$ is called the **weak overtaking optimal solution** of problem (1). See Carlson, Haurie, and Leizarowitz (1991) for more detailed arguments.[3]

We mention that for $(k^*(t), c^*(t)) \in A_{\bar{k}}$, if

$$\int_0^\infty e^{-\rho t} u(c^*(t))dt$$

is defined and finite, then it is solution of (1) if and only if it is weakly overtaking optimal solution of (1), if and only if it is overtaking optimal solution of (1). Therefore, the notion of overtaking optimality is an extension of usual optimality.

2.3 The Euler Equation and the Transversality Condition

Let $(k(t), c(t))$ be admissible. This pair is said to satisfy the **Euler equation** on an interval I if and only if $c(t)$ is continuous, $u \circ c$ is continuously

[2] In this paper, the word 'interval' means a convex set in $\mathbb{R}$ that includes at least two different points.

[3] In economics, our overtaking optimality is sometimes called **catching-up optimality**, and in this case, our weak overtaking optimality is called overtaking optimality.

differentiable, and

$$\frac{d}{dt}(u \circ c)(t) = (\rho - f'(k(t)))(u \circ c)(t) \tag{2}$$

holds for all t.

In traditional model, it is said that $(k^*(t), c^*(t))$ is a solution only if it satisfies the Euler equation on $\mathbb{R}_+$. However, in our model, $c^*(t)$ is not necessarily continuous, and thus this statement is a little incorrect. Therefore, we extend the above definition. An admissible pair $(k(t), c(t))$ satisfies the Euler equation **a.e.** on an interval I if and only if there exists a continuous function $\tilde{c}(t)$ such that $(k(t), \tilde{c}(t))$ is also admissible and satisfies Euler equation on I. Then, we can modify the above claim to the following one: "$(k^*(t), c^*(t))$ is a solution only if it satisfies the Euler equation a.e. on $\mathbb{R}_+$".

However, in our model, $\int_0^\infty e^{-\rho t} u(c^*(t))dt$ may be not defined, and thus this statement is not obvious. Also, Euler equation is only a necessary condition of **inner solution**. This problem is partially solved in theorems 2 and 3, although the proof is very long.

Next, again let $(k(t), c(t))$ be admissible. This pair is said to satisfy the **transversality condition** if and only if

$$\lim_{t \to \infty} e^{-\rho t} u'(c(t))k(t) = 0. \tag{3}$$

It is also said that usually $(k^*(t), c^*(t))$ is a solution if and only if it satisfies the Euler equation and the transversality condition. However, the necessity of the transversality condition for optimality heavily depends on the finiteness of $\int_0^\infty e^{-\rho t} u(c(t))dt$, and thus we cannot derive such a theorem.

2.4 The Magic of Capital

Although this subsection is not necessary for later arguments, we should note an odd phenomenon on some admissible process. Suppose that $f(k) = \sqrt{k}$. Define,

$$k(t) = \frac{t^2}{16},\ c(t) = \frac{t}{8}.$$

Then, the pair $(k(t), c(t))$ is admissible, although $k(0) = 0$. Moreover, if $\rho = 1$ and $u(c) = -c^{-1/2}$, then

$$\begin{aligned} 0 \ge& \int_0^\infty e^{-\rho t} u(c(t))dt \\ \ge& -\sqrt{8}\left[\int_0^1 t^{-1/2}dt + \int_1^\infty e^{-t}dt\right] \\ =& -\sqrt{2} - \sqrt{8}e^{-1} > -\infty, \end{aligned}$$

and thus $\int_0^\infty e^{-\rho t}u(c(t))dt$ is finite even though $u(0) = -\infty$. This phenomenon is named the **magic of capital**.[4]

The magic of capital obstruct our proof throughout the proof of theorems, because the existence of such a path means that the optimality of $(k^*(t), c^*(t))$ cannot immediately excludes the existence of t such that $k^*(t) = 0$. This is quite troublesome.

3 Results

3.1 The Necessity of the Euler Equation

First, we define a notion of inner solution.

Definition 2. Suppose that $(k(t), c(t))$ is admissible. We say that this pair is **positive on** $[0, T]$ if and only if the following two statements hold.

1) $k(t) > 0$ for all $t \in [0, T]$.

2) There exists $c > 0$ such that the Lebesgue measure of the set $\Delta(c) = \{t \in [0, T] | c(t) < c\}$ is zero.

If $(k^*(t), c^*(t))$ is (weak, or strong) overtaking optimal solution, then it is called **inner solution** if and only if it is positive on $[0, T]$ for all $T > 0$.

Theorem 1. Let $T > 0$, and suppose that $(k^*(t), c^*(t))$ is a weak overtaking optimal solution of (1) and positive on $[0, T]$. Then, $c^*(t)$ is a solution of Euler equation a.e. on $[0, T]$.

Proof Suppose that $(k^*(t), c^*(t))$ satisfies the statements of theorem, and choose any continuously differentiable function $k : [0, T] \to \mathbb{R}$ with $k(0) = k(T) = 0$. Define

$$g(s) = \int_0^T e^{-\rho t}u(f(k^*(t) + sk(t)) - (\dot{k}^*(t) + s\dot{k}(t)))dt.$$

Because $(k^*(t), c^*(t))$ is a weak overtaking optimal solution of (1) and positive on $[0, T]$, g can be defined on a neighborhood of 0, and 0 is a maximal point of g. By Lebegsue's Dominated Convergence Theorem, we can show that

$$g'(0) = \int_0^T [e^{-\rho t}u'(c^*(t))f'(k^*(t))k(t) - e^{-\rho t}u'(c^*(t))\dot{k}(t)]dt.$$

[4]See Hosoya (2019).

Meanwhile, because 0 is a maximal point of g, we have $g'(0) = 0$. Therefore, using the formula of integral by parts, we have

$$\begin{aligned} 0 = g'(0) &= \int_0^T [e^{-\rho t}u'(c^*(t))f'(k^*(t))k(t) - e^{-\rho t}u'(c^*(t))\dot{k}(t)]dt \\ &= \int_0^T \left[\int_t^T e^{-\rho\tau}u'(c^*(\tau))f'(k^*(\tau))d\tau - e^{-\rho t}u'(c^*(t))\right]\dot{k}(t)dt. \end{aligned}$$

Now, for any continuous function $x(t)$ on $[0, T]$, define

$$\Lambda(x(t)) = \int_0^T \left[\int_t^T e^{-\rho\tau}u'(c^*(\tau))f'(k^*(\tau))d\tau - e^{-\rho t}u'(c^*(t))\right]x(t)dt,$$

$$x_1^*(x(t)) = \int_0^T x(t)dt.$$

Then, the kernel of Λ includes the kernel of x_1^*. It can be easily shown that there exists $a \in \mathbb{R}$ such that $\Lambda = ax_1^*$. (See theorem 5.91 of Aliprantis and Border (2006).) Therefore,

$$\int_0^T \left[\int_t^T e^{-\rho\tau}u'(c^*(\tau))f'(k^*(\tau))d\tau - e^{-\rho t}u'(c^*(t)) - a\right]x(t)dt = 0$$

for any continuous function $x(t)$ on $[0, T]$. By Riesz representation theorem, the function

$$\int_t^T e^{-\rho\tau}u'(c^*(\tau))f'(k^*(\tau))d\tau - e^{-\rho t}u'(c^*(t)) - a$$

coincides with the Radon-Nikodym derivative of zero measure, and thus it is zero for almost all $t \in [0, T]$. Therefore,

$$\int_t^T e^{-\rho\tau}u'(c^*(\tau))f'(k^*(\tau))d\tau = e^{-\rho t}u'(c^*(t)) + a \tag{4}$$

for almost all $t \in [0, T]$, and thus

$$c^*(t) = (u')^{-1}\left(e^{\rho t}\left(\int_t^{T^*} e^{-\rho\tau}u'(c^*(\tau))f'(k^*(\tau))d\tau - a\right)\right)$$

for almost all $t \in [0, T]$. Let $c^+(t)$ denote the right-hand side of above equation. Then, $c^+(t)$ is continuous and $c^*(t) = c^+(t)$ for almost all $t \in [0, T]$, and thus

$$c^+(t) = (u')^{-1}\left(e^{\rho t}\left(\int_t^{T^*} e^{-\rho\tau}u'(c^+(\tau))f'(k^*(\tau))d\tau - a\right)\right),$$

for all $t \in [0, T]$. Hence,

$$\int_t^T e^{-\rho\tau} u'(c^+(\tau)) f'(k^*(\tau)) d\tau = e^{-\rho t} u'(c^+(t)) + a$$

for all $t \in [0, T]$, and therefore, $(u' \circ c^+)$ is differentiable and,

$$\frac{d}{dt}(u' \circ c^+)(t) = [\rho - f'(k^*(t))](u' \circ c^+)(t).$$

This completes the proof. ■

Theorem 2. Suppose that f is increasing and

$$\lim_{c \to 0} u'(c) = +\infty.$$

If $(k^*(t), c^*(t))$ is a weakly overtaking optimal solution of (1) and $k^*(t) > 0$ for all $t \in [0, T]$, then $(k^*(t), c^*(t))$ is positive on $[0, T]$.

Proof. First, we shall prove the following lemmas.

Lemma 1. Consider the following differential equation:

$$\dot{x}(t) = g_i(t, x), x(t^*) = \bar{x},$$

where g_1, g_2 are real-valued functions defined on a neighborhood $U \subset \mathbb{R}^2$ of $(t^*, \bar{x})$ that is measurable in t and continuous in x, $t \mapsto g_i(t, \bar{x})$ is integrable, $g_1(t, x) \le g_2(t, x)$ for all $(t, x) \in U$, and at least one $i^* \in \{1, 2\}$, there exists $L > 0$ such that

$$|g_{i^*}(t, x_1) - g_{i^*}(t, x_2)| \le L|x_1 - x_2|,$$

for every (t, x_1, x_2) with $(t, x_1), (t, x_2) \in U$. Suppose also that there exist solutions $x_1(t), x_2(t)$ of the above equation defined on $[t^*, T]$. Then,

$$x_1(t) \le x_2(t).$$

Proof of lemma 1. We treat only the case $i^* = 2$. The rest case can be proved symmetrically.

Suppose not. Then, there exists $T^* > t^*$ such that $x_1(T^*) > x_2(T^*)$. Let $T^+ = \inf\{t < T^* | x_1(s) > x_2(s) \text{ for all } s \in [t, T^*]\}$. Because $x_1(t^*) = x_2(t^*) = \bar{x}$, we have $t^* \le T^+$ and $x_1(T^+) = x_2(T^+)$. Now, define

$$x_3(t) = x_1(T^+) + \int_{T^+}^t g_2(s, x_1(s)) ds.$$

Note that the function $t \mapsto g_2(t, x_1(t))$ is integrable for sufficiently small neighborhood of T^+.[5] Then, for every $t \in [T^+, T^*]$,

$$x_3(t) \geq x_1(T^+) + \int_{T^+}^{t} g_1(s, x_1(s))ds = x_1(t).$$

Meanwhile,

$$\begin{aligned} |x_3(t) - x_2(t)| &\leq \int_{T^+}^{t} |g_2(s, x_1(s)) - g_2(s, x_2(s))|ds \\ &\leq L \int_{T^+}^{t} (x_1(s) - x_2(s))ds \\ &\leq L(t - T^+) \max_{s \in [T^+, t]} (x_1(s) - x_2(s)). \end{aligned}$$

This implies that if $0 < t - T^+ < L^{-1}$, then there exists $s \in [T^+, t]$ such that $|x_3(s) - x_2(s)| < x_1(s) - x_2(s)$. Therefore, $x_3(s) < x_1(s)$, a contradiction. This completes the proof of lemma 1. ■

Lemma 2. Consider the following differential equation[6]

$$\dot{k}(t) = f(k(t)) - g(t), \ k(0) = \bar{k} > 0, \tag{5}$$

where $g(t)$ is defined on some $[0, T]$ with $T > 0$, and integrable. Then,

(i) There exists $T^* > 0$ such that the solution $k(t)$ of (5) is defined and positive on $[0, T^*]$. Moreover, if there exists the solution of (5) on $[0, T^*]$, then it is unique.[7]

(ii) Let $\bar{T}$ be the supremum of T^* such that there exists the solution $k_{T^*}(t)$ of (5) on $[0, T^*]$ with $k_{T^*}(t) > 0$ for all $t \in [0, T^*]$. Then, there exists the solution $k(t)$ of (5) on $[0, \bar{T}[$ with $k(t) > 0$ for all $t \in [0, \bar{T}[$, and either $\bar{T} = T$ or $\lim_{t \to \bar{T}} k(t) = 0$. Particularly, if $\bar{T} = T$, then the solution $k(t)$ of (5) can be defined on $[0, T]$.

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[5]See the corollary of proposition 8 in section 8.1 of Ioffe and Tikhomirov (1979).

[6]We call a function $k(t)$ defined on some interval I the solution of the differential equation

$$\dot{k} = h(t, k), \ k(0) = \bar{k},$$

if it is absolutely continuous, $k(0) = \bar{k}$ and $\dot{k}(t) = h(t, k(t))$ for almost all $t \in I$.

[7]This result is not obvious, because $g(t)$ is not continuous. Actually, the proof of this claim is in Ioffe and Tikhomirov (1979). However, this book is now out of print, and thus we decide to write the proof in this paper.

(iii) Let $k_1(t), k_2(t)$ be defined on $[0, T^*]$, be positive at everywhere, and $k_i(t)$ be the solution of (5) with $g(t) = g_i(t)$. Moreover, let $g_1(t) \le g_2(t)$ for almost all $t \in [0, T^*]$. Then, $k_1(t) \ge k_2(t)$ for all $t \in [0, T^*]$.

(iv) In addition to (iii), if the set of all $t \in [0, T^*]$ with $g_1(t) < g_2(t)$ has a nonzero measure, then $k_1(T^*) > k_2(T^*)$.

Proof of lemma 2. For (i), choose any $\hat{k} > \bar{k}$, and choose $\delta > 0$ such that $\bar{k} + 2\delta < \hat{k}$ and $\bar{k} - 2\delta > 0$. Let

$$U_0 = \{k \in \mathbb{R} | |k - \bar{k}| \le \delta\},$$

and choose any $\varepsilon > 0$ such that

$$\varepsilon f'(\bar{k} - 2\delta) < 1,\ \int_0^{\varepsilon} [f(\hat{k}) + |g(t)|]dt < \delta,\ \varepsilon \le T,$$

and define $T_0 = [0, \varepsilon]$. Let X be the set of all continuous function from T_0 into U_0, and for any $x(\cdot) \in X$, define

$$P(x(\cdot))(t) = \bar{k} + \int_0^t [f(x(s)) - g(s)]ds.$$

If $t \in T_0$, then

$$|P(x(\cdot))(t) - \bar{k}| \le \int_0^t [f(\hat{k}) + |g(s)|]ds < \delta,$$

and thus P is a function from X into X. Moreover,

$$\begin{aligned} |P(x_1(\cdot))(t) - P(x_2(\cdot))(t)| &\le \int_0^t |f(x_1(s)) - f(x_2(s))|ds \\ &\le f'(\bar{k} - 2\delta) \int_0^t |x_1(s) - x_2(s)|ds \\ &\le \varepsilon f'(\bar{k} - 2\delta) \|x_1(\cdot) - x_2(\cdot)\|_{\infty}, \end{aligned}$$

where

$$\|x(\cdot)\|_{\infty} = \sup_{t \in T_0} |x(t)|.$$

This implies that P is a contraction mapping with respect to the norm $\|\cdot\|_{\infty}$, and thus there uniquely exists a fixed point $k(\cdot) \in X$ of P. Clearly $k(t)$ is a solution of (5). The uniqueness of the solution can be easily shown and we omit its proof.

For (ii), choose any $t^* < \bar{T}$. Then, there exists a solution $k_{t^*}(t)$ of (5) defined on $[0, t^*]$, and $k_{t^*}(t) > 0$ for any $t \in [0, t^*]$. Define $k(t) = k_t(t)$ for $t \in [0, \bar{T}[$. Then, $k(t)$ is a positive solution defined on $[0, \bar{T}[$. Suppose that $\bar{T} < T$ and $\limsup_{t \to \bar{T}} k(t) = 3\hat{k} > 0$.[8] Then, there exists a sequence (t_n) such that $t_n \uparrow \bar{T}$ and $2\hat{k} < k(t_n) < 4\hat{k}$ for all n. Choose any $\varepsilon > 0$ such that $\bar{T} + \varepsilon \le T$ and

$$\varepsilon f'(\hat{k}) < 1, \int_{\bar{T}}^{\bar{T}+\varepsilon} (f(5\hat{k}) + |g(t)|)dt < \hat{k},$$

and choose any t_n with $t_n > \bar{T} - \varepsilon$. Define

$$T_0 = [t_n, t_n + \varepsilon],$$

$$U_0 = [k(t_n) - \hat{k}, k(t_n) + \hat{k}],$$

and let X be the set of all continuous function from T_0 into U_0, and define

$$P(x(\cdot))(t) = k(t_n) + \int_{t_n}^{t} [f(x(s)) - g(s)]ds$$

for any $x(\cdot) \in X$. Then,

$$|P(x(\cdot))(t) - k(t_n)| \le \int_{t_n}^{t} [f(5\hat{k}) + |g(s)|]ds < \hat{k},$$

and thus P is a function from X into X. Moreover, if $t \in T_0$, then

$$\begin{aligned}|P(x_1(\cdot))(t) - P(x_2(\cdot))(t)| &\le \int_{t_n}^{t} |f(x_1(s)) - f(x_2(s))|ds \\ &\le f'(\hat{k}) \int_{t_n}^{t} |x_1(s) - x_2(s)|ds \\ &\le \varepsilon f'(\hat{k}) \|x_1(\cdot) - x_2(\cdot)\|_\infty,\end{aligned}$$

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[8] Because f is concave, we have

$$f(k) - g(t) \le f'(\bar{k})(k - \bar{k}) + f(\bar{k}) + |g(t)|.$$

Therefore, by using lemma 1, we can show that

$$k(t) \le e^{f'(\bar{k})t} \left(\bar{k} + \int_0^t e^{-f'(\bar{k})s} (f(\bar{k}) + |g(s)| - f'(\bar{k})\bar{k}) ds \right),$$

and thus

$$\limsup_{t \to \bar{T}} k(t) < \infty.$$

and thus P is a contraction mapping. Hence, P has a unique fixed point $k^+(t) \in X$. This is a positive solution of (5) with initial value condition $k^+(t_n) = k(t_n)$, and thus to connect it with $k(t)$, we get a positive solution of (5) defined on $[0, t_n + \varepsilon]$, which is absurd.

Now, suppose that $\bar{T} = T$. If $\lim_{t \to T} k(t) = 0$, then we can define $k(T) = 0$ and it is a solution of (5). Otherwise, define $g(t) \equiv 0$ for $t > T$. Then, by proved above, we have there exists a positive solution defined on $[0, T]$. This completes the proof of (ii).

For (iii), we can apply lemma 1, because f is Lipschitz on $[\hat{k}, +\infty[$, where

$$\hat{k} < \inf\{\min\{k_1(t), k_2(t)\} | t \in [0, T^*]\}.$$

Finally, we will show (iv). Suppose that the set $\{t \in [0, T^*] | g_1(t) < g_2(t)\}$ has a nonzero measure. We have $k_1(t) \ge k_2(t)$ for all $t \in [0, T^*]$. Therefore,

$$\begin{aligned} k_1(T^*) &= \bar{k} + \int_0^{T^*} [f(k_1(t)) - g_1(t)]dt \\ &\ge \bar{k} + \int_0^{T^*} [f(k_2(t)) - g_1(t)]dt \\ &> \bar{k} + \int_0^{T^*} [f(k_2(t)) - g_2(t)]dt = k_2(T^*), \end{aligned}$$

as desired. This completes the proof of lemma 2. ■

Let $k^+(t)$ be the solution of

$$\dot{k} = f(k), \ k(0) = \bar{k}.$$

By (ii) of lemma 2, we have that $k^+(t)$ is defined on $\mathbb{R}_+$. By (iii) of lemma 2, we have $k^*(t) \le k^+(t)$ for all $t > 0$. If $k^*(T) = k^+(T)$ for any $T > 0$, by (iv) of lemma 2, we have $c^*(t) \equiv 0$. However, if we define $c(t) \equiv f(\bar{k}), \ k(t) \equiv \bar{k}$, then $(k(t), c(t))$ is admissible and $(k(t), c(t)) \succ^* (k^*(t), c^*(t))$, which contradicts that the optimality assumption of $(k^*(t), c^*(t))$. Therefore, for sufficiently large $T > 0$, we have $k^*(T) < k^+(T)$.

Next, we will show that if $k^*(t) > 0$ for all $t \in [0, T]$ and $0 < k^*(T) < k^+(T)$, then $k^*(t), c^*(t)$ is positive on $[0, T]$. Suppose not. Because $k^*(T) < k^+(T)$, there exists $\varepsilon > 0$ such that

$$\Delta_1 \equiv \{t \in [0, T] | c^*(t) \ge \varepsilon\}$$

has a positive measure.

Now, we introduce a lemma.

Lemma 3. Suppose that $\Delta \subset [0,T]$ is measurable, and consider the following differential equation

$$\dot{k}(t) = f(k(t)) - c^*(t) - a1_\Delta(t),\ k(0) = \bar{k},$$

where $1_\Delta(t)$ is the indicator function of Δ. Then, there exist $\delta > 0, \hat{k} > 0$, and $K_1 > 0$ independent of Δ such that for any a with $0 < a < \delta$, there exists a solution $k^a(t)$ defined on $[0,T]$ such that $k^a(t) \ge \hat{k}$ for all $t \in [0,T]$, and

$$0 \le k^*(t) - k^a(t) \le a\lambda(\Delta)K_1$$

for all $t \in [0,T]$. (Where λ denotes the Lebesgue measure.)

Proof of lemma 3. First, we show the following claim: suppose that $A > 0$ and $B(t)$ is an integrable function on $[0,T]$, and a continuous function $x(t)$ satisfies

$$x(t) \le \int_0^t [Ax(s) + B(s)]ds$$

for all $t \in [0,T]$, then

$$x(t) \le e^{At}\int_0^t e^{-As}B(s)ds$$

for all $t \in [0,T]$. Actually, define

$$x_0(t) = x(t),\ x_{n+1}(t) = \int_0^t [Ax_n(s) + B(s)]ds.$$

Then, $x_n(t)$ is nondecreasing in n. Define

$$\|x(\cdot)\|_\infty = \max_{t\in[0,T]} |x(t)|.$$

Then, for any $t \in [0,T]$,

$$|x_2(t) - x_1(t)| \le A\int_0^t |x_1(s) - x_0(s)|ds \le A\|x_1(\cdot) - x_0(\cdot)\|_\infty t,$$
$$|x_3(t) - x_2(t)| \le A\int_0^t |x_2(s) - x_1(s)|ds \le \frac{A^2}{2!}\|x_1(\cdot) - x_0(\cdot)\|_\infty t^2,$$
$$\ldots$$
$$|x_{n+1}(t) - x_n(t)| \le \frac{A^n t^n}{n!}\|x_1(\cdot) - x_0(\cdot)\|_\infty,$$

and thus, $(x_n(\cdot))$ is a Cauchy sequence of continuous functions on $[0,T]$, and thus it converges to some $x^*(t)$ uniformly. Clearly, $x^*(t)$ satisfies

$$x^*(t) = \int_0^t [Ax^*(s) + B(s)]ds,$$

which implies that

$$x^*(t) = e^{At}\int_0^t e^{-As}B(s)ds.$$

Because $x_n(t)$ is nondecreasing in n, our claim is correct.

Next, choose any $\hat{k} > 0$ such that

$$\hat{k} < \min_{t\in[0,T]} k^*(t),$$

and let $K_1 = e^{f'(\hat{k})T}$ and

$$\delta = \frac{\min_{t\in[0,T]} k^*(t) - \hat{k}}{K_1 T}.$$

Let $0 < a < \delta$, and choose any measurable set $\Delta \subset [0,T]$. Consider the following differential equation

$$\dot{k}(t) = f(k(t)) - c^*(t) - a1_\Delta(t),\ k(0) = \bar{k}.$$

Because of (i) of lemma 2, there exists a solution $k^a(t)$ defined on $[0,T^*]$ for some $T^* \le T$. Since $k^a(0) = \bar{k} > \hat{k}$, we have $k^a(t) > \hat{k}$ for sufficiently small $t > 0$. If $k^a(s) > \hat{k}$ for any $s \in [0,t]$, then

$$\begin{aligned} k^*(t) - k^a(t) &= \int_0^t [f(k^*(s)) - f(k^a(s)) + a1_\Delta(s)]ds \\ &\le \int_0^t [f'(\hat{k})(k^*(s) - k^a(s)) + a1_\Delta(s)]ds, \end{aligned}$$

and thus,

$$k^*(t) - k^a(t) \le ae^{f'(\hat{k})t}\int_{\Delta\cap[0,t]} e^{-f'(\hat{k})s}ds \le ae^{f'(\hat{k})T}\lambda(\Delta\cap[0,t]) \le aK_1 t.$$

If $k^a(t) < \hat{k}$ for some t, then for $t^* = \inf\{t \in [0,T] | k^a(t) < \hat{k}\}$, we have $k^a(t^*) = \hat{k}$ and

$$k^*(t^*) - k^a(t^*) \le aK_1 t^* < \delta K_1 T \le k^*(t^*) - \hat{k},$$

and thus $k^a(t^*) > \hat{k}$, a contradiction. Therefore, we have $k^a(t) > \hat{k}$ if it can be defined, and by (ii) of lemma 2, $k^a(t)$ can be defined on $[0, T]$. Moreover,

$$k^*(t) - k^a(t) \le a\lambda(\Delta)K_1,\ k^a(t) > \hat{k}$$

for any $t \in [0, T]$. This completes the proof of lemma 3. ■

Choose a sufficiently large $M > 0$ such that

$$u'(\varepsilon) < \frac{M}{2K_1 \max\{e^{-\rho T}, 1\}}.$$

Choose any $c > 0$ such that $u'(c) > \frac{M}{\min\{e^{-\rho T}, 1\}}$, and define

$$\Delta(c) = \{t \in [0, T] | c^*(t) \le c\}.$$

Because $(k^*(t), c^*(t))$ is not positive, we have $\Delta(c)$ has a positive measure.[9] Define

$$c_1(t) = c^*(t) + \delta_1 1_{\Delta(c)}(t),$$

where

$$0 < \delta_1 < \delta,\ u'(c + \delta_1) > \frac{M}{\min\{e^{-\rho T}, 1\}},\ \delta_1 \frac{\lambda(\Delta(c))}{\lambda(\Delta_1)} K_1 \le 1,$$

$$u'\left(\varepsilon - \delta_1 \frac{\lambda(\Delta(c))}{\lambda(\Delta_1)} K_1\right) < \frac{M}{2K_1 \max\{e^{-\rho T}, 1\}}.$$

Lemma 4. Consider the following differential equation

$$\dot{k}(t) = f(k(t)) - c_1(t) + b1_{\Delta_1}, k(0) = \bar{k}.$$

Particularly, let $k_0(t)$ be the solution with $b = 0$ defined on $[0, T]$. For any b with $0 < b \le 1$, there exists a solution $k_b(t)$ of above equation defined on $[0, T]$, $k_b(T)$ is continuous in b, and

$$k_b(T) - k_0(T) \ge b\lambda(\Delta_1).$$

Proof of lemma 4. We first note the following claim. Let $A > 0$ and $B(t)$ is integrable, and a continuous function $x(t)$ satisfies the following inequality

$$x(t) \ge \int_0^t [Ax(s) + B(s)]ds$$

[9] Note that $K_1 \ge 1$, and thus $c < \varepsilon$.

for all t. Then,

$$x(t) \geq e^{At} \int_0^t e^{-As} B(s) ds$$

for all t. This claim can be proved by almost the same logic as that proved in lemma 3, and thus we omit its proof.

Now, let $k_b(t)$ be the solution of the following differential equation

$$\dot{k}(t) = f(k(t)) - c_1(t) + b1_{\Delta_1}(t),\ k(0) = \bar{k}.$$

By lemma 2, $k_b(t) \geq k_0(t)$ and thus $k_b(t)$ is defined on $[0, T]$. By the claim in the proof of lemma 3,

$$k_b(t) - k_0(t) \leq bTK_1.$$

Let

$$\tilde{k} = TK_1 + \max_{t \in [0,T]} k^*(t).$$

Then, for any b with $0 < b \leq 1$, $k_b(t) \leq \tilde{k}$ for any $t \in [0, T]$, and

$$\begin{aligned} k_b(t) - k_0(t) &= \int_0^t [f(k_b(s)) - f(k_0(s)) + b1_{\Delta_1}(s)] ds \\ &\geq \int_0^t [f'(\tilde{k})(k_b(s) - k_0(s)) + b1_{\Delta_1}(s)] ds. \end{aligned}$$

Therefore, we have

$$k_b(t) - k_0(t) \geq b \int_{\Delta_1 \cap [0,t]} e^{f'(\tilde{k})(t-s)} ds.$$

Particularly, if $t = T$, we have

$$k_b(T) - k_0(T) \geq b \int_{\Delta_1} e^{f'(\tilde{k})(T-s)} ds \geq b\lambda(\Delta_1).$$

Meanwhile, for b_1, b_2 with $0 \leq b_1 < b_2 \leq 1$,

$$\begin{aligned} k_{b_2}(t) - k_{b_1}(t) &= \int_0^t [f(k_{b_2}(s)) - f(k_{b_1}(s)) + (b_2 - b_1)1_{\Delta_1}(s)] ds \\ &\leq \int_0^t [f'(\hat{k})(k_{b_2}(s) - k_{b_1}(s)) + (b_2 - b_1)1_{\Delta_1}(s)] dt, \end{aligned}$$

and thus

$$0 \leq k_{b_2}(T) - k_{b_1}(T) \leq (b_2 - b_1)K_1 T,$$

which implies that $k_b(T)$ is continuous in b. This completes the proof. ■

By lemma 4, we have there exists δ_2 such that $0 < \delta_2 \le 1$, $k_{\delta_2}(T) = k^*(T)$ and

$$\delta_2\lambda(\Delta_1) \le k_{\delta_2}(T) - k_0(T) = k^*(T) - k_0(T) \le \delta_1\lambda(\Delta(c))K_1.$$

Thus,

$$u'(\varepsilon - \delta_2) < \frac{M}{2K_1 \max\{e^{-\rho T}, 1\}}.$$

Now, define

$$c(t) = c_1(t) - \delta_2 1_{\Delta_1}(t),$$

and let $k(t) = k_{\delta_2}(t)$ in lemma 4 if $t \in [0, T]$ and $k(t) = k^*(t)$ otherwise. Then,

$$\begin{aligned}
&\int_0^T e^{-\rho t}u(c(t))dt \\
\ge &\int_0^T e^{-\rho t}u(c^*(t))dt + M\lambda(\Delta(c))\delta_1 - \frac{M}{2K_1}\lambda(\Delta_1)\delta_2 \\
> &\int_0^T e^{-\rho t}u(c^*(t))dt,
\end{aligned}$$

and thus $(k(t), c(t)) \succ^* (k^*(t), c^*(t))$, a contradiction. Therefore, we have if $0 < k^*(T) < k^+(T)$, $(k^*(t), c^*(t))$ is positive on $[0, T]$.

Lastly, suppose that $k^*(t) > 0$ for all $t \in [0, T]$. By argued above, we have there exists $T^* \ge T$ such that $0 < k^*(T^*) < k^+(T^*)$. Then, $(k^*(t), c^*(t))$ is positive on $[0, T^*]$, and thus it is positive on $[0, T]$. This completes the proof. ■

As a corollary of the above theorem, the following result is obtained.

Theorem 3. Suppose that $f(0) = 0$, f is increasing, and

$$\lim_{c\to 0} u'(c) = +\infty.$$

Moreover, suppose that at least one of the following statements holds.

- $(k^*(t), c^*(t))$ is a weakly overtaking optimal solution of (1) and f is Lipschitz around zero.
- $(k^*(t), c^*(t))$ is an overtaking optimal solution of (1).

Then, $(k^*(t), c^*(t))$ is an inner solution.

Proof. By theorem 2, it suffices to show that $k^*(T) > 0$ for all $T > 0$. Suppose not. Let $T^* > 0$ be the minimum of T such that $k^*(T) = 0$. Then, by theorem 1, we can assume without loss of generality that $c^*(t)$ satisfies the Euler equation

$$\frac{d}{dt}(u' \circ c^*)(t) = (\rho - f'(k^*(t)))(u' \circ c^*)(t)$$

on $[0, T^*[$. If $\lim_{k \downarrow 0} f'(k) \le \rho$, then $\frac{d}{dt}(u' \circ c^*)(t) \ge 0$ for every $t \in [0, T^*]$, and thus $c^*(t)$ is nonincreasing. If $\lim_{k \downarrow 0} f'(k) > \rho$, then $\frac{d}{dt}(u' \circ c^*)(t) < 0$ for every $t \in [0, T^*]$ such that $T^* - t$ is sufficiently small. In both cases, there exists $c^*(T^*) = \lim_{t \uparrow T^*} c^*(t)$. Therefore, we can assume that without loss of generality that $c^*(t)$ is continuous on $[0, T^*]$.

First, suppose that $k^*(t) \equiv 0$ for $t \ge T^*$. Then, we must have $c^*(t) \equiv 0$ for almost all $t \ge T^*$. If either $u(0) = -\infty$ or $\rho \le 0$, then $(k(t), c(t)) \succ^* (k^*(t), c^*(t))$, where $c(t) \equiv f(\bar{k}), k(t) \equiv \bar{k}$, a contradiction. Hence, we can assume that $\rho > 0$, and without loss of generality, $u(0) = 0$. Let $k_0(t)$ be a solution of the following differential equation defined on $[0, T]$:[10]

$$\dot{k}(t) = f(k(t)) - c^*(t) + a 1_{[0, T^*/2]},$$

where $a > 0$ is smaller than $\min\{1, \min_{t \in [0, T^*/2]} c^*(t)\}$, and define

$$c(t) = \begin{cases} c^*(t) - a & \text{if } t \in [0, T^*/2], \\ c^*(t) & \text{if } t \in]T^*/2, T^*], \\ f(k_0(T^*)) & \text{if } t > T^*, \end{cases}$$

$$k(t) = \begin{cases} k_0(t) & \text{if } t \in [0, T^*], \\ k_0(T^*) & \text{if } t > T^*. \end{cases}$$

Then, $(k(t), c(t))$ is admissible, and by the same arguments in the proof of lemma 4, we have

$$a \frac{T^*}{2} \le k(T^*).$$

[10]Note that $k_0(t)$ can be defined on $[0, T]$ by lemma 2.

Therefore,

$$\begin{aligned}
&lim_{T\to\infty}\int_0^T e^{-\rho t}[u(c(t))-u(c^*(t))]dt\\
=&\int_0^{T^*/2} e^{-\rho t}[u(c(t))-u(c^*(t))]dt+\frac{e^{-\rho T^*}}{\rho}u(f(k(T^*)))\\
\ge&\frac{e^{-\rho T^*}}{\rho}u(f(aT^*/2))-\max_{t\in[0,T^*/2]}u'(c^*(t)-a)T^*a,
\end{aligned}$$

where the right-hand side is positive for any sufficiently small $a>0$, a contradiction.

Hence, we must have $k^*(t^*)>0$ for some $t^*>T^*$.[11] First, we show that $\lim_{k\to 0}f'(k)=+\infty$. Suppose not. Then, f is Lipschitz around zero. Clearly, the zero function is a solution of the following differential equation:

$$\dot{k}(t)=f(k(t)),\ k(T^*)=0.$$

By lemma 1, we must have $k(t)\le 0$ for all t, which contradicts that $k(t^*)>0$. Therefore, there exists $t^+\in[0,T^*[$ such that $\rho-f'(k^*(t))>0$ for every $t\in[t^+,T^*[$, which implies that $c^*(t)$ is increasing in t on $[t^+,T^*[$ by the Euler equation. Moreover, by our initial assumption, $(k^*(t),c^*(t))$ must be overtaking optimal.

Let $T^+=\inf\{t<t^*|k^*(s)>0 \text{ for all } s\in[t,t^*]\}$. It can easily be shown that for every $t'>0$, $(k^*(t+t'),c^*(t+t'))$ is an overtaking optimal solution of (1) with $k(0)=k^*(t')$. Thus, if $k^*(t)$ is positive on $[\hat{t},\bar{t}]$, then by theorems 1 and 2, $c^*(t)$ must satisfies the Euler equation a.e., and thus it can be assumed to be continuous, and therefore $k^*(t)$ is continuously differentiable. Because $k^*(T^+)=0, k^*(t)>0$ for $t\in]T^+,t^*]$ and $\lim_{k\downarrow\infty}f'(k)=\infty$, $c^*(t)$ is increasing on $[T^+,t^*]$, and thus $c^*(T^+)\equiv\lim_{t\downarrow T^+}c^*(t)$ can be defined and,

$$0\le\dot{k}^*(T^+)=f(k^*(T^+))-c^*(T^+)=-c^*(T^+),$$

which implies that $c^*(T^+)=0$.

By mean value theorem, there exists t_1,t_2 such that $k^*(t_1)=k^*(t_2),\dot{k}^*(t_1)<0,\dot{k}^*(t_2)>0$, and $t_1<T^*<t_2\le t^*$. Let

$$k_1(t)=\begin{cases}k^*(t) & \text{if } 0\le t\le t_1,\\ k^*(t+t_2-t_1) & \text{otherwise,}\end{cases}$$

and

$$c_1(t)=\begin{cases}c^*(t) & \text{if } 0\le t\le t_1,\\ c^*(t+t_2-t_1) & \text{otherwise.}\end{cases}$$

[11] We mention that this case should be considered. Remember the magic of capital.

If $0 < t - t_1$ and $t - t_1$ is sufficiently small, then $k_1(t) > k^*(t)$ and $c^*(t) < c_1(t)$. If $k_1(t) > k^*(t)$ for all $t > t_1$, then define $k_2(t) = k_1(t)$ and $c_2(t) = c_1(t)$. Otherwise, let $t^+ = \inf\{t > t_1 | k_1(t) \le k^*(t)\}$ and define

$$k_2(t) = \begin{cases} k_1(t) & \text{if } 0 \le t \le t^+, \\ k^*(t) & \text{otherwise,} \end{cases}$$

and

$$c_2(t) = \begin{cases} c_1(t) & \text{if } 0 \le t \le t^+, \\ c^*(t) & \text{otherwise.} \end{cases}$$

Further, define

$$c_3(t) = \frac{c^*(t) + c_2(t)}{2},$$

and let $k_3(t)$ be a solution of the following equation:

$$\dot{k}(t) = f(k(t)) - c_3(t),\ k(0) = \bar{k}.$$

We should show that $k_3(t)$ can be defined on $\mathbb{R}_+$. Clearly $k_3(t) = k^*(t)$ on $[0, t_1]$. By (ii) of lemma 2, it suffices to show that $k_3(t) \ge \frac{k_2(t)+k^*(t)}{2}$ for $t_1 \le t \le t^+$. (Note that $k_2(t) > 0$ for $t \in [t_1, t^+]$.) Suppose not. Then, there exists $\tilde{t} \in]t_1, t^+[$ such that $k_3(t)$ can be defined on $[t_1, \tilde{t}]$ and $k_3(\tilde{t}) < \frac{k_2(\tilde{t})+k^*(\tilde{t})}{2}$. Let $\bar{t} = \inf\{t < \tilde{t} | k_3(s) < \frac{k_2(s)+k^*(s)}{2}$ for all $s \in [t, \tilde{t}]\}$. Then, $k_3(\bar{t}) = \frac{k_2(\bar{t})+k^*(\bar{t})}{2}$ and $k_3(\bar{t}) > 0$. Choose any $\hat{k} > 0$ with $\hat{k} < k_3(\bar{t})$. Then,

$$\begin{aligned} \frac{k_2(t) + k^*(t)}{2} - k_3(t) &= \int_{\bar{t}}^{t} \left[f\left(\frac{k_2(s) + k^*(s)}{2} \right) - f(k_3(s)) \right] ds \\ &\le (t - \bar{t}) f'(\hat{k}) \sup_{s \in [\bar{t}, t]} \left(\frac{k_2(s) + k^*(s)}{2} - k_3(s) \right), \end{aligned}$$

where $t - \bar{t} > 0$ is sufficiently small. If $(t - \bar{t}) f'(\hat{k}) < 1$, then there exists $s \in [\bar{t}, t]$ such that

$$\frac{k_2(s) + k^*(s)}{2} - k_3(s) < \frac{k_2(s) + k^*(s)}{2} - k_3(s),$$

a contradiction.

Therefore, we have $(k_3(t), c_3(t))$ is a well-defined admissible pair. Because $(k^*(t), c^*(t))$ is a solution of (2) and 2) holds, we have

$$\limsup_{T \to \infty} \int_0^T e^{-\rho t} [u(c_3(t)) - u(c^*(t))] dt \le 0.$$

Since $c_2(t) \neq c^*(t)$ on the set of positive measure, this implies that

$$\limsup_{T\to\infty} \int_0^T e^{-\rho t}[u(c_2(t)) - u(c^*(t))]dt < 0.$$

If $k_2(t) \equiv k_1(t)$, then let $k_4(t) \equiv k^*(t)$ and $c_4(t) \equiv c^*(t)$. Otherwise, define

$$k_4(t) = \begin{cases} k^*(t) & \text{if } 0 \le t \le t_2 + t^+ - t_1, \\ k^*(t + t_1 - t_2) & \text{otherwise,} \end{cases}$$

and

$$c_4(t) = \begin{cases} c^*(t) & \text{if } 0 \le t \le t_2 + t^+ - t_1, \\ c^*(t + t_1 - t_2) & \text{otherwise.} \end{cases}$$

Because $k^*(t_2 + t^+ - t_1) = k^*(t^+)$, we have $(k_4(t), c_4(t))$ is admissible. Moreover, for every admissible pair $(k(t), c(t))$, we have

$$\begin{aligned} & \limsup_{T\to\infty} \int_0^T e^{-\rho t}[u(c(t)) - u(c_4(t))]dt \\ \le\ & \limsup_{T\to\infty} \int_0^T e^{-\rho t}[u(c(t)) - u(c^*(t))]dt \\ & + \limsup_{T\to\infty} \int_{t_2+t^+-t_1}^T e^{-\rho t}[u(c^*(t)) - u(c_4(t))]dt \\ \le\ & e^{-\rho(t_2-t_1)} \limsup_{T\to\infty} \int_{t^+}^T e^{-\rho t}[u(c^*(t + t_2 - t_1)) - u(c^*(t))]dt \le 0, \end{aligned}$$

which implies that $(k_4(t), c_4(t))$ is also an overtaking optimal solution of (1). Define

$$k_5(t) = \begin{cases} k^*(t) & \text{if } 0 \le t \le t_2, \\ k^*(t + t_1 - t_2) & \text{otherwise,} \end{cases}$$

and

$$c_5(t) = \begin{cases} c^*(t) & \text{if } 0 \le t \le t_2, \\ c^*(t + t_1 - t_2) & \text{otherwise.} \end{cases}$$

Then,

$$\begin{aligned} & \limsup_{T\to\infty} \int_0^T e^{-\rho t}[u(c_5(t)) - u(c_4(t))]dt \\ =\ & e^{-\rho t_2} \limsup_{T\to\infty} \int_0^T e^{-\rho t}[u(c^*(t)) - u(c_2(t))]dt \\ \ge\ & e^{-\rho t_2} \liminf_{T\to\infty} \int_0^T e^{-\rho t}[u(c^*(t)) - u(c_2(t))]dt > 0, \end{aligned}$$

a contradiction. This completes the proof. ■

3.2 The Sufficiency Result

Theorem 4. Suppose that $(k^*(t), c^*(t))$ is admissible, $c^*(t)$ is a continuous function, and both the Euler equation on $\mathbb{R}_+$, and the transversality condition hold. Then, $(k^*(t), c^*(t))$ is an overtaking optimal solution of (1).

Proof. First, note that

$$L(k, \dot{k}) = u(f(k) - \dot{k})$$

is a jointly concave function. Therefore, we have

$$L(k, \dot{k}) - L(k^*, \dot{k}^*) \leq L_k(k^*, \dot{k}^*)(k - k^*) + L_{\dot{k}}(k^*, \dot{k}^*)(\dot{k} - \dot{k}^*)$$

for all $(k, \dot{k})$ and $(k^*, \dot{k}^*)$. Thus, for any admissible process $(k(t), c(t))$,

$$\begin{aligned}
&\int_0^T e^{-\rho t}[u(c(t)) - u(c^*(t))]dt \\
\leq &\int_0^T [e^{-\rho t}u'(c^*(t))f'(k^*(t))(k(t) - k^*(t)) - e^{-\rho t}u'(c^*(t))(\dot{k}(t) - \dot{k}^*(t))]dt \\
= &\int_0^T \left[\frac{d}{dt}(e^{-\rho t}u'(c^*(t))(k^*(t) - k(t)))\right] dt \\
= &\, e^{-\rho T}u'(c^*(T))(k^*(T) - k(T)) \\
\leq &\, e^{-\rho T}u'(c^*(T))k^*(T),
\end{aligned}$$

by the Euler equation. Therefore, taking $lim_{T\to\infty}$, we have the right-hand side is zero by the transversality condition, and thus

$$(k^*(t), c^*(t)) \succsim^* (k(t), c(t)),$$

which implies that $(k^*(t), c^*(t))$ is an overtaking optimal solution of (1). ■

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