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Unawareness and Reverse Symmetry: Aumann Structure with Complete Lattice

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Abstract

This paper models Aumann structures with complete lattices, and discusses unawareness. In multi-attribute state space models, although previous studies discussing unawareness assume that the family of spaces with a complete lattice, e.g., Heifetz et al. (2006) and Li (2009), there is no model that a standard state space is a complete lattice. Without formulating the family of spaces, this paper models a state space with a complete lattice by a set-theoretical and a constructive approach. However, in our models, although almost properties of the unawareness operator hold as previous studies, interestingly, Symmetry with non-trivial unawareness does not hold. This paper proposes a novel property, Reverse Symmetry, shows that non-trivial unawareness holds if and only if Reverse symmetry holds in our model, and suggests the implications.

Keywords: State Space, Aumann Structure, Unawareness, Complete Lattice, Constructive Approach, Overloaded Operator.

JEL Codes: C70, C72, D80, D83.

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I Introduction

Unawareness was proposed by Fagin and Halpern (1988) as a higher-order unknown. However, in partitional standard state space models (or standard Aumann structure models), if the knowledge operator satisfies Necessitation ($K(\Omega) = \Omega$), and the unawareness operator satisfies Plausibility ($U(E) = \neg K(E) \cap \neg K \neg K(E)$), KU Introspection ($KU(E) = \emptyset$)and AU Introspection ($U(E) = UU(E)$), non-trivial unawareness ($U(E) \neq \emptyset$) cannot be modeled (Dekel et al. 1998).

To avoid this issue, there are two approaches discussing unawareness: one is unawareness structures proposed by Heifetz et al. (2006) and Li (2009). In their models, unawareness indicates the lack of conception. The other indicates non-partitional standard state space models (e.g., Modica and Rustichini 1994; 1999, Geanakoplos 1989). State spaces in unawareness structures are complete lattices, while those in the standard state space models are not. Roughly, a standard state space is flat, while a state space with unawareness is non-flat.[1] [2]

In Heifetz et al. (2006) and Li (2009), the family of disjoint state spaces is a

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[1] Recently, Fukuda (2020) provides generalized state space models which nests both unawareness structures and non-partitional standard state space models.

[2] Schipper (2014) provides a historical survey about unawareness. Schipper (2015) provides a mathematical survey about it in modal logics.

complete lattice. However, there is no model that a standard state space itself is a complete lattice. This paper models the state spaces, called *constructive state spaces*, by a set-theoretical and constructive approach. Our state space is equivalent to the space in Heifetz et al. (2006) and Li (2009).

In our approach, a standard operator is not convenient, because the operator cannot define a state related with ϕ in the state space of Heifetz et al. (2006) or Li (2009). To avoid this issue, we must formulate the *overloaded operator*. Our operator has multiple arities. Although our state space is similar to that of Heifetz et al. (2008), discussions and assumptions about unawareness are different between us and them.

In our state space, any (subjective) state space is the subset of the constructive state space. The meaning of some state that belongs to some state space is not same as the meaning of the state that belongs to the different space, as some attribute is included in the former, which may not be included in the latter. The meaning of a state depends on the state that state space belongs to. Thus, the meaning of state is decided in relation to the other states on a given state space.

This paper models constructive Aumann structures with constructive state spaces and defines the possibility correspondence, knowledge operator, and awareness/unawareness operator on the state space. Almost the same properties of the knowledge and awareness/unawareness operators were employed in previous studies.

However, interestingly, Symmetry ($U(E) = U(\neg E)$) crashes non-trivial unawareness, although previous studies prove the property (e.g., Heifetz et al 2006;

2013a, Li 2009, and Fukuda 2020) or assume it (e.g., Modica and Rustichini 1994; 1999, Halpern 2001, Heifetz et al. 2008, and Sadzik 2021). We call the property that Symmetry does not hold *Reverse Symmetry*. Reverse Symmetry has two implications. One is that we must not discuss the event that the agent can perceive and the negation that she cannot perceive using the same approach. The other is that S5 in modal logics may not be necessary in discussing unawareness. Modica and Rustichini (1994) assume Symmetry and show that S4 with Symmetry equals to S5. In contrast, because this paper shows Reverse Symmetry in our model, we must consider S4 with Reverse Symmetry.

A constructive Aumann structure induces generalization of the main theorems in Dekel et al. (1998) and Chen et al. (2012). Because several properties of the knowledge and the awareness/unawareness operators are the same in previous studies, our models are intermediate between the standard state space and unawareness structure.

This paper is organized as follows. The following section models state spaces in Heifetz et al. (2006) and Li (2009). Section III models a constructive state space and compares the state space with the one in Heifetz et al. and Li. In Section IV, we model a constructive Aumann structure, and define and discuss the possibility correspondence, the knowledge operator, and awareness/unawareness operator. The last section presents the concluding remarks.

II State Space in Previous Studies about Unawareness

This section formulates state spaces defined by Heifetz et al. (2006) and Li (2009).[3]

2-1 *State Space in Heifetz et al. (2006)*

First, we formulate state spaces proposed by Heifetz et al. (2006), and call the state spaces HMS-state spaces. Let $\mathcal{S} = \{S_\lambda\}_{\lambda \in \Lambda}$ be a complete lattice of disjoint state spaces, and let $\preccurlyeq$ be a partial order on $\mathcal{S}$. For any $S, S' \in \mathcal{S}$, $S \succcurlyeq S'$ is interpreted as "S is more expressive that S'." Then, there exists a surjective projection $R^S_{S'}: S \longrightarrow S'$, that is, for any $\omega \in S$, $R^S_{S'}(\omega) \in S'$. An HMS-state space is denoted by $\Sigma = \bigcup_{\lambda \in \Lambda} S_\lambda$. A standard state space is not Σ, but each state spate on $\mathcal{S}$. Hence, an HMS-state space is the union of disjoint standard state spaces.

Example 1 Suppose that $\mathcal{S} = \left\{S_{\{x,y\}}, S_{\{x\}}, S_{\{y\}}, S_{\{\phi\}}\right\}$ is a complete lattice of disjoint spaces, and let $S_{\{x,y\}} = \{xy, x\neg y, \neg xy, \neg x\neg y\}$, $S_{\{x\}} = \{x, \neg x\}$, $S_{\{y\}} = \{y, \neg y\}$ and $S_{\{\phi\}} = \{\phi\}$. For example, x indicates that x is true, while $\neg x$ means that x is false. Given two different spaces $S_{\{x,y\}}, S_{\{x\}}$, $R^{S_{\{x,y\}}}_{S_{\{x\}}}(xy) = R^{S_{\{x,y\}}}_{S_{\{x\}}}(x\neg y) = x$, and $R^{S_{\{x,y\}}}_{S_{\{x\}}}(\neg xy) = R^{S_{\{x,y\}}}_{S_{\{x\}}}(\neg x\neg y) = \neg x$. Then, an agent who can perceive only $S_{\{x\}}$ cannot perceive y. The agent perceives x for xy or $x\neg y$, while she perceives

[3] This paper does not discuss unawareness structures. Related literatures discussing their models are Heifetz et al. (2006; 2008; 2013a), Li (2009), Heinsalu (2012), Galanis (2013; 2018), Schipper (2014; 2015) and Fukuda (2020).

$\neg x$ for $\neg xy$ or $\neg x \neg y$. In the example, the HMS-state space is $\Sigma = S_{\{x,y\}} \cup S_{\{x\}} \cup S_{\{y\}} \cup S_{\{\phi\}}$, while the standard state space is each element of $\mathcal{S}$, that is, $S_{\{x,y\}}$, $S_{\{x\}}$, $S_{\{y\}}$, and $S_{\{\phi\}}$. The example is depicted in Figure 1. ■

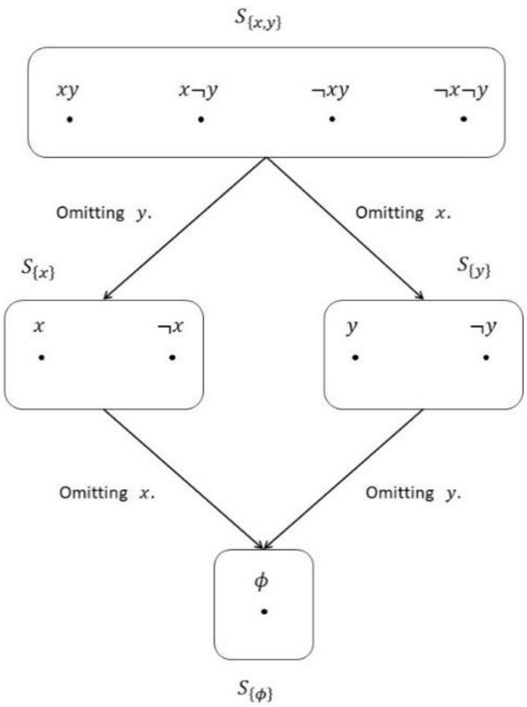

Fig. 1: HMS-state space.

2-2 State Space in Li (2009)

Subsequently, we formulate state spaces proposed by Li (2009) and call the spaces Li-state spaces. Let Q^* be the set of questions. $A_q = \{a_q, \neg a_q\}$ is the set of answers about $q \in Q^*$. Here, a Cartesian product $\prod_{q \in Q^*} A_q$ is the objective state space, and $\prod_{q \in Q} A_q$ is a subjective state space, where $Q \subseteq Q^*$. If $Q = \emptyset$, let $\prod_{q \in Q} A_q = \{\phi\}$. It is evident that two different spaces $A, A' \in \left\{\prod_{q\in Q} A_q \middle| Q \subseteq Q^*\right\}$ are disjoint. For any $Q, Q' \in 2^{Q^*} \setminus \{\emptyset\}$, such that $Q' \subseteq Q \subseteq Q^*$, there is a surjective projection $\pi^{Q}_{Q'}: \prod_{q\in Q} A_q \longrightarrow \prod_{q\in Q'} A_q$. Hence, for any $\omega \in \prod_{q\in Q} A_q$, $\pi^{Q}_{Q'}(\omega) \in \prod_{q\in Q'} A_q$. Denote by $\mathcal{A} = \bigcup_{Q\subseteq Q^*} \prod_{q\in Q} A_q$ a Li-state space. A standard state space is not $\mathcal{A}$, but each element on $\mathcal{A}$. Thus, a Li-state space is the union all disjoint spaces on $\mathcal{A}$.

Example 2 Suppose that $Q^* = \{q(x), q(y)\}$ is the set of questions. Here, $q(x)$ is a question about an attribute x, and $q(y)$ is a question about an attribute y. Then, the sets of answers for each question are $A_{q(x)} = \{a_{q(x)}, \neg a_{q(x)}\}$ and $A_{q(y)} = \{a_{q(y)}, \neg a_{q(y)}\}$. Given x, $a_{q(x)}$ is interpreted as "the answer for $q(x)$ is *yes*," while $\neg a_{q(x)}$ is interpreted as "the answer for $q(x)$ is *no*." The objective state space is $A_{q(x)} \times A_{q(y)}$, while subjective state spaces are $A_{q(x)}$, $A_{q(y)}$, and $A_{q(\phi)}$. Given $\{q(x)\} \subseteq Q^*$, there is a surjective projection $\pi^{Q^*}_{\{q(x)\}}: A_{q(x)} \times A_{q(y)} \longrightarrow A_{q(x)}$. Then, $\pi^{Q^*}_{\{q(x)\}}(a_{q(x)}, a_{q(y)}) = \pi^{Q^*}_{\{q(x)\}}(a_{q(x)}, \neg a_{q(y)}) = a_{q(x)}$ and $\pi^{Q^*}_{\{q(x)\}}(\neg a_{q(x)}, a_{q(y)}) = \pi^{Q^*}_{\{q(x)\}}(\neg a_{q(x)}, \neg a_{q(y)}) = \neg a_{q(x)}$. An agent who can perceive only an attribute x perceives a state $a_{q(x)}$ for $(a_{q(x)}, a_{q(y)})$ and $(a_{q(x)}, \neg a_{q(y)})$, the agent perceives a state $\neg a_{q(x)}$ for $(\neg a_{q(x)}, a_{q(y)})$ or $(\neg a_{q(x)}, \neg a_{q(y)})$. Then, a Li-state space is $\mathcal{A} = A_{q(x)} \times A_{q(y)} \cup A_{q(x)} \cup A_{q(y)} \cup A_{q(\phi)}$, while standard state spaces are $A_{q(x)} \times A_{q(y)}, A_{q(x)}, A_{q(y)}$, and $A_{q(\phi)}$. The example is shown in Figure 2. ■

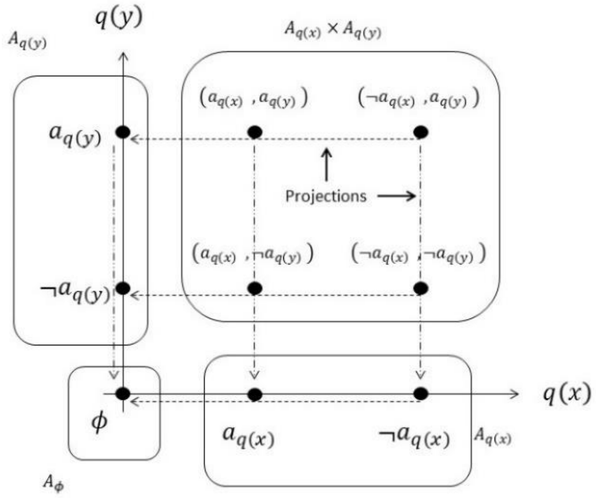

Fig. 2: Li-state space.

III Constructive State Space

Standard state space models do not assume that they are semi-lattices, even if the spaces have multi-attribute properties, e.g., dice.[4] In contrast, state spaces in unawareness structures are complete lattices. However, because each element of the family of spaces in their models is a standard state space, each state space is not a semi-lattice. This section shows that standard state space is a semi-lattice (or a complete lattice). Because our formulating approach is similar to those of Heifetz et al. (2008) and Li (2009), as a constructive approach, we call the space a constructive state space.

[4] We can represent multi-attribute for dices. For example, for “1,” we represent “1, but not 2, not 3, not 4, not 5, and not 6.”

3-1 Overloaded Function

First, we define *functions overloading*. Given two sets X and Y and for any $k = 0, 1, \cdots, n$, X_k is defined as follows:

$$X_k = \begin{cases} \emptyset & \text{if } k = 0; \\ \times_k X & \text{otherwise.} \end{cases}$$

Definition of overloaded functions: A function f is overloaded by $n+1$-tuple arities $(0, 1, \cdots, n)$ if f is $n+1$ mappings as follows:

$$f\colon \bigcup_{k=0}^{n} X_k \to Y.$$

This paper assumes that overloading is applicable to operators.

3-2 Overloaded Operator and Constructive State Space

This section models state spaces by using a constructive approach. Let P be the set of basic propositions or conceptions. Given an *overloaded operator with 3-tuple arities* $(0, 1, 2)$, $\vee$, and the following conditions satisfied by the operator:

C1 For any $p \in P$, $p \vee \, = \, \vee p = p \vee p = p$.

C2 $\vee \, = \phi$.

C3 For any $p, p' \in P$, $p \vee p' = p' \vee p$.

C4 For any $p, p', p'' \in P$, $p \vee (p' \vee p'') = (p \vee p') \vee p''$.

C1 means that p can be led by itself when the arity of $\vee$ is not only 2 but also 1. C2 is a technical condition. When the arity is 0, $\vee$ leads ϕ. The ϕ is interpreted as "every proposition is not true." C3 means that $\vee$ satisfies a commutative law, and C4 means that $\vee$ satisfies an absorption law.

Here, for any subset of basic propositions $X \subseteq P$, where P may be an empty set, $\bigvee_{p \in X} p$ is a state. Let $\Omega = \left\{\bigvee_{p \in X} p \,\middle|\, X \subseteq P\right\}$ be the objective state space. Further, for any $X \subseteq P$, let $\Omega_X = \left\{\bigvee_{p \in Y} p \,\middle|\, Y \subseteq X\right\}$ be a subjective state space. For any $X, Y \in 2^P \setminus \{\emptyset\}$ such that $Y \subseteq X \subsetneq P$, $\bigvee_{p \in Y} p \in \Omega$ and $\bigvee_{p \in Y} p \in \Omega_X$ hold evidently. However, attributes between $\bigvee_{p \in Y} p$ in Ω and $\bigvee_{p \in Y} p$ in Ω_X are different. In Ω, $\bigvee_{p \in Y} p$ includes that any attribute $p' \in P \setminus Y$ does not hold. In contrast, in Ω_X, $\bigvee_{p \in Y} p$ includes hat any $p' \in P \setminus Y$ does not hold, but it does not include a means that any $p'' \in P \setminus X$ holds or not. $\bigvee_{p \in Y} p$ in Ω_X is related with any element in the Ω_X, but it is not related with every element in $\Omega \setminus \Omega_X$. Then, every $\bigvee_{p \in Y} p$ in Ω_X does not have any attribute $p'' \in P \setminus X$.

Let us define projections. For any basic proposition sets $X, Y \subseteq P$, there is a projection $r_Y^X \colon \Omega_X \longrightarrow \Omega_Y$. This may not be surjective. Hence, for any $\bigvee_{p \in Z:\, Z \subseteq X} p \in$

Ω_X, $r_Y^X(\bigvee_{p \in Z:\, Z \subseteq X} p) = \bigvee_{p \in Z \cap Y:\, Z \subseteq X} p \in \Omega_Y$. If $Y \subseteq X \subseteq P$, then $r_Y^X \circ r_X^P = r_Y^P$. Below, let $\bigvee_{p \in Z:\, Z \subseteq P} p = \omega$ and for any $\omega \in \Omega$ and $X \subseteq P$, let $r_X^P(\omega) = \omega_X$. For any $X \subseteq P$, r_X^X is the identity, that is, for any $\omega \in \Omega_X$, $r_X^X(\omega) = \omega$.

The objective state space Ω is a complete lattice. Although it is a standard state space with a complete lattice, let us call Ω the *constructive state space*.

Remark 1 For any subsets X, Y such that $Y \subseteq X \subseteq P$, $\Omega_Y \subseteq \Omega_X$.

Remark 1 means that our (subjective) state spaces are subsets on the objective state space, unlike in Heifetz et al. (2006) and Li (2009). The feature differs from unawareness structures, and the feature is the same to (non-partitional) standard state space models. Moreover, different state spaces have an intersection, and all intersections must have ϕ.

Our formulation is similar to that of Heifetz et al. (2008). Our formulations are set-theoretical approaches, while their formulations are logics approaches. However, there is a crucial difference between our discussion and their discussion about unawareness. Heifetz et al. assume that the awareness/unawareness operator satisfies Symmetry. In contrast, we show that the operator does not satisfy Symmetry with Non-triviality. Because results between this paper and Heifetz et al. are different even if both state spaces are same formulations, we assert that our framework is different to theirs.

Example 3 Let $P = \{x\,, y\}$ be the set of basic propositions, and let $\Omega = \{x \vee y\,, x\,, y\,, \phi\}$, $\Omega_{\{x\}} = \{x\,, \phi\}$, $\Omega_{\{y\}} = \{y\,, \phi\}$, and $\Omega_{\{\phi\}} = \{\phi\}$ be state spaces. Each state space is a subset of Ω. Because the projection must not be surjective, given two sets $\{x\}$ and $\{y\}$, $r_{\{y\}}^{\{x\}}(x) = r_{\{y\}}^{\{x\}}(\phi) = \phi$. It is evident that the intersection has ϕ. Then, Ω is a constructive state space. The example is depicted in Figure 3.

Here, let us focus on ϕ. If $\phi \in \Omega$, a state ϕ indicates that it does not represent $x \vee y$, x or y. In contrast, if $\phi \in \Omega_X$, ϕ means only that it does not represent x, but it does not mean that it represents or not $x \vee y$ and y. ϕ does not imply a conception y. That is, if any two state spaces are different, then same state does not have same attribute between them. ■

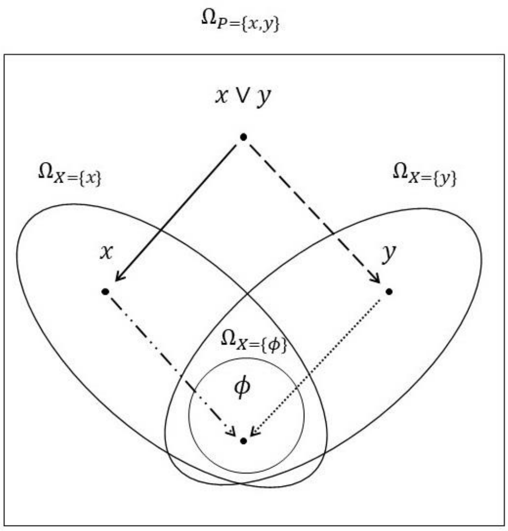

Fig. 3: Constructive Aumann-state space.

3-3 Relationships with Other State Spaces

Our constructive state spaces are related with HMS-state spaces and Li-state spaces by the following lemmas.

Lemma 1 The following are equivalent:

1. A constructive state space can be constructed.
2. An HMS-state space can be constructed.

Proof. (1⟹2) Any constructive state space Ω has the set of basic propositions P and for any subset $X \subseteq P$, there is $\Omega_X = \left\{\bigvee_{p \in Y} p \,\middle|\, Y \subseteq X\right\}$. Here, let us define the family of disjoint sets $\mathcal{S}$ and bijective mapping $f: \{\Omega_X | X \subseteq P\} \longrightarrow \mathcal{S}$. Then, for any $X, Y \subseteq P$, if $X \neq Y$, then $f(\Omega_X) \cap f(\Omega_Y) = \emptyset$. Let $\preccurlyeq$ be a partial order on $\mathcal{S}$ and be defined as follows: if $X \subseteq Y$, then $f(\Omega_X) \preccurlyeq f(\Omega_Y)$. Then, suppose that there exists a surjective projection $r_{\Omega_X}^{\Omega_Y}: \Omega_Y \longrightarrow \Omega_X$. Then, $\mathcal{S} = \{f(\Omega_X) | X \subseteq P\}$ is a complete lattice, and $\Sigma = \bigcup_{X \subseteq P} f(\Omega_X)$ is an HMS-state space.

(2⟹1) Any HMS-state space Σ has a complete lattice with disjoint spaces $\mathcal{S} = \{S_\lambda\}_{\lambda \in \Lambda}$. Here, let us define $\mathcal{S}^{min} = \left\{S \in \mathcal{S} \,\middle|\, S \textit{ is minimal element on } \mathcal{S} \setminus \{\emptyset\}\right\}$, let P be some set and be given a bijective mapping $\hat{f}: \mathcal{S}^{min} \longrightarrow P$. Then, overloaded operator with 3-tuple arities $(0, 1, 2)$, $\vee$, satisfies the following.

- For any $S \in \mathcal{S}^{min}$, $\hat{f}(S) \vee = \vee \hat{f}(S) = \hat{f}(S) \vee \hat{f}(S) = \hat{f}(S)$.
- $\vee = \emptyset$.

- For any $S, S' \in \mathcal{S}^{min}$, $\hat{f}(S) \vee \hat{f}(S') = \hat{f}(S') \vee \hat{f}(S)$.
- For any $S, S', S'' \in \mathcal{S}^{min}$, $\hat{f}(S) \vee \left(\hat{f}(S') \vee \hat{f}(S'')\right) = \left(\hat{f}(S) \vee \hat{f}(S')\right) \vee \hat{f}(S'')$.

Then, for any $\mathcal{X} \subseteq \mathcal{S}^{min}$, $\Omega_{\mathcal{X}} = \left\{\bigvee_{S \in \mathcal{X}} \hat{f}(S) \,\middle|\, \mathcal{X} \subseteq \mathcal{S}^{min}\right\}$. Let us define that for any $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{S}^{min}$, if we define a projection $r_{\mathcal{Y}}^{\mathcal{X}}: \Omega_{\mathcal{X}} \longrightarrow \Omega_{\mathcal{Y}}$, then for any $\bigvee_{S \in \mathcal{Z}:\, \mathcal{Z} \subseteq \mathcal{X}} S$, $r_{\mathcal{Y}}^{\mathcal{X}}(\bigvee_{S \in \mathcal{Z}:\, \mathcal{Z} \subseteq \mathcal{X}} S) = \bigvee_{S \in \mathcal{Z} \cap \mathcal{Y}:\, \mathcal{Z} \subseteq \mathcal{X}} S$. Then, $\left\{\hat{f}(S) \middle| S \in \mathcal{S}^{min}\right\}$ is the set of basic propositions, and $\Omega = \left\{\bigvee_{S \in \mathcal{X}} \hat{f}(S) \,\middle|\, \mathcal{X} \subseteq \mathcal{S}^{min}\right\}$ is a constructive state space. ■

Lemma 2 The following are equivalent:

1. A constructive state space can be constructed.
2. A Li-state space can be constructed.

Proof. (1⟹2) Any constructive state space Ω has the set of basic propositions P, and for any $X \subseteq P$, there is $\Omega_X = \left\{\bigvee_{p \in Y} p \,\middle|\, Y \subseteq X\right\}$. Here, given some set Q^* and a bijection mapping $g: P \longrightarrow Q^*$. Then, for any $p, p' \in P$, $g(p) \neq g(p')$. Moreover, for any $p \in P$, let us define $A_{g(p)} = \left\{a_{g(p)}, \neg a_{g(p)}\right\}$ and for any $X \subseteq P$, given $\prod_{p \in X} A_{g(p)}$. Note that, if $X = \emptyset$, we denote by $\prod_{p \in X} A_{g(p)} = \{\phi\}$. Here, for any $X, Y \subseteq P$, such that $X \neq Y$, $\prod_{p \in X} A_{g(p)} \neq \prod_{p \in Y} A_{g(p)}$ is obvious. When $Y \subseteq X$, suppose that there is a surjective projection $r_{\mathrm{Y}}^{\mathrm{X}}: \prod_{p \in X} A_{g(p)} \longrightarrow \prod_{p \in Y} A_{g(p)}$. Then, $\{g(p) | p \in P\}$ is the set of questions, and $\mathcal{A} = \bigcup_{X \subseteq P} \prod_{p \in X} A_{g(p)}$ is a Li-state space.

(2⟹1) Any Li-state space $\mathcal{A}$ has the set of questions Q^*. Here, given some set P a bijection mapping $\hat{g}: Q^* \longrightarrow P$, then, for any $q, q' \in Q^*$, if $q \neq q'$, $\hat{g}(q) \neq \hat{g}(q')$.

Let us define an overloaded operator $\vee$ as follows.

- For any $q \in Q^*$, $\hat{g}(q) \vee = \vee\, \hat{g}(q) = \hat{g}(q) \vee \hat{g}(q) = \hat{g}(q)$.
- $\vee = \phi$.
- For any $q, q' \in Q^*$, $\hat{g}(q) \vee \hat{g}(q') = \hat{g}(q') \vee \hat{g}(q)$.
- For any $q, q', q'' \in Q^*$, $\hat{g}(q) \vee \left(\hat{g}(q') \vee \hat{g}(q'')\right) = \left(\hat{g}(q) \vee \hat{g}(q')\right) \vee \hat{g}(q'')$.

Then, for any $Q \subseteq Q^*$, let $\Omega_Q = \left\{\bigvee_{q \in Q} \hat{g}(q) \,\middle|\, Q \subseteq Q^*\right\}$. For any $Q, Q' \subseteq Q^*$, given a projection $r^Q_{Q'}: \Omega_Q \longrightarrow \Omega_{Q'}$ and for any $\bigvee_{q \in Q'': Q'' \subseteq Q} \hat{g}(q) \in \Omega_Q$, let us define $r^Q_{Q'}\left(\bigvee_{q \in Q'': Q'' \subseteq Q} \hat{g}(q)\right) = \bigvee_{q \in Q'' \cap Q': Q'' \subseteq Q} \hat{g}(q)$. Then, $\{\hat{g}(q) | q \in Q^*\}$ is the set of basic propositions and, $\Omega = \left\{\bigvee_{q \in Q} \hat{g}(q) \,\middle|\, Q \subseteq Q^*\right\}$ is a constructive state space. ■

The lemmas indicate that any constructive state space can construct HMS-state space and Li-state space, and vice versa.

Proposition 1 The following are equivalent.

1. A constructive state space can be constructed.
2. HMS-state space can be constructed.
3. Li-state space can be constructed.

Let us consider constructive state spaces related with HMS-state spaces. We compare Example 2 with 3. Their relations are the following:

$$S_{\{x,y\}} \iff \Omega$$

$$S_{\{x\}} \Leftrightarrow \Omega_{\{x\}}$$

$$S_{\{y\}} \Leftrightarrow \Omega_{\{y\}}$$

$$S_{\{\phi\}} \Leftrightarrow \Omega_{\{\phi\}}$$

When $S_{\{x,y\}}$ is compared with Ω, states between the spaces are represented as follows:

$$xy \Leftrightarrow x \vee y$$

$$x\neg y \Leftrightarrow x$$

$$\neg xy \Leftrightarrow y$$

$$\neg x\neg y \Leftrightarrow \phi$$

In contrast, when $S_{\{x\}}$ is compared with $\Omega_{\{x\}}$, states between the spaces are represented as follows:

$$x \Leftrightarrow x$$

$$\neg x \Leftrightarrow \phi$$

By these comparisons, x in Ω and x in $\Omega_{\{x\}}$ have different implication and ϕ in Ω and ϕ in $\Omega_{\{x\}}$ are different as well. $\Omega_{\{x\}}$ is the lack of y.

Moreover, the following are a relationship between Ω and Σ:

$$xy \Leftrightarrow x \vee y$$

$$x \Leftrightarrow x$$

$$y \Leftrightarrow y$$

$$\phi \Leftrightarrow \phi$$

This means that each element of Ω is related with each element without negations of Σ. That is, not only Ω is related with $S_{\{x,y\}} \subseteq \Sigma$, but also Ω is related with

$\{xy, x, y, \phi\} \subseteq \Sigma$. Hence, Ω has a dual structure for Σ.

IV Constructive Aumann Structure

This section models *constructive Aumann structures* based on constructive state spaces. We focus on only a single agent, formulate possibility correspondences on constructive state spaces, and knowledge operators and awareness/unawareness operators on constructive Aumann structures, and discuss their properties. Finally, we provide generalization of main theorems proposed by Dekel et al. (1998) and Chen et al. (2012).

4-1 Possibility Correspondence

Possibility correspondences in standard Aumann structures are only defined in state spaces. In contrast, because our state spaces are semi lattices, a domain of possibility correspondences is not only the state space, but also the power set of basic propositions. Let $\langle P, \Omega, \Pi \rangle$ be the constructive Aumann structure, where Ω is constructed by P. Then, $\Pi: \Omega \times 2^{\mathrm{P}} \longrightarrow 2^{\Omega} \setminus \{\emptyset\}$ is the possibility correspondence. Suppose that an agent can perceive every basic proposition in the subset of basic propositions $X \subseteq P$, but not in $Y \subseteq P \setminus X$. Then, for any $\omega \in \Omega$, $\Pi(\omega, X) \subseteq \Omega_X$. Let us assume that the possibility correspondence satisfies the following properties.

1. Subjective Nondelusion: For any $\omega \in \Omega$, and any $X \subseteq P$, $\omega_X \in \Pi(\omega, X)$.
2. Stationarity: For any $\omega, \omega' \in \Omega$ and any $X \subseteq P$, if $\omega' \in \Pi(\omega, X)$, then $\Pi(\omega', X) = \Pi(\omega, X)$.

Example 3 (Continued.) Let $\omega_1 = x \vee y$, $\omega_2 = x$, $\omega_3 = y$ and $\omega_4 = \phi$. Suppose that an agent can perceive the basic proposition set $X = \{x\}$. By Subjective Nondelusion, $\omega_2 \in \Pi(\omega_1, X)$, $\omega_2 \in \Pi(\omega_2, X)$, $\omega_4 \in \Pi(\omega_3, X)$, and $\omega_4 \in \Pi(\omega_4, X)$. By Stationarity, $\Pi(\omega_1, X) = \Pi(\omega_2, X)$ and $\Pi(\omega_3, X) = \Pi(\omega_4, X)$. Note that whether $\Pi(\omega_1, X) = \Pi(\omega_3, X)$ or not may depend on how Π is formulated. ■

Subjective Nondelusion and Stationarity are the analogues of the partitional information function in a standard Aumann structure. When X is a proper subset of P, Π is evidently not partitional on Ω. However, it may be partitional on Ω_X.

Definition 1 (Partial Partition) Given any $X \subseteq P$. $\Pi: \Omega \times 2^{\mathrm{P}} \longrightarrow 2^{\Omega} \setminus \{\emptyset\}$ is *partially partitional* on Ω_X if there exists $\mathcal{P} = \{P_\lambda\}_{\lambda \in \Lambda}$ such that:

1. $\bigcup_{\lambda \in \Lambda} P_\lambda = \Omega_X$;
2. For any $\omega \in \Omega$, there exists P_λ such that $\omega_X \in P_\lambda$ and $\Pi(\omega, X) = P_\lambda$; and
3. For any $P_\lambda, P_{\lambda'} \in \mathcal{P}$, if $P_\lambda \neq P_{\lambda'}$, then $P_\lambda \cap P_{\lambda'} = \emptyset$.

A partial partition is the analog of the partition in a standard Aumann structure.

We can induce the following proposition.

Proposition 2 Given any $X \subseteq P$. Π is partially partitional on Ω_X if and only if Π satisfies Subjective Nondelusion and Stationarity.

Proof. ($\Longrightarrow$) Suppose that the possibility correspondence Π is partially partitional on Ω_X. Then, by Condition 1 in Definition 1, $\bigcup_{\lambda \in \Lambda} P_\lambda = \Omega_X$ and by Condition 2 in Definition 1, for any $\omega \in \Omega$, because there exists P_λ with $\omega_X \in P_\lambda$ such that $\Pi(\omega, X) = P_\lambda$, $\omega_X \in \Pi(\omega, X)$. That is, Π satisfies Subjective Nondelusion. Moreover, by Condition 3 in Definition 1, for any $P_\lambda, P_{\lambda'} \in \mathcal{P}$, if $P_\lambda \neq P_{\lambda'}$, then $P_\lambda \cap P_{\lambda'} = \emptyset$. This satisfies that for any $\omega, \omega' \in \Omega$ if $\Pi(\omega, X) \neq \Pi(\omega', X)$, then $\Pi(\omega, X) \cap \Pi(\omega', X) = \emptyset$, that is, $\omega' \notin \Pi(\omega, X)$. The contraposition is that if $\omega' \in \Pi(\omega, X)$, then $\Pi(\omega', X) = \Pi(\omega, X)$. Hence, Π satisfies Stationarity.

($\Longleftarrow$) Suppose that Π satisfies Subjective Nondelusion and Stationarity. Given P_λ with $\Pi(\omega, X) = P_\lambda$ for some $\omega \in \Omega$. By Subjective Nondelusion and the assumption of projection, for any $\omega \in \Omega_X$, $\omega \in \Pi(\omega, X)$. Therefore, $\bigcup_{\omega \in \Omega_X} \Pi(\omega, X) = \bigcup_{\lambda :\, P_\lambda = \Pi(\omega, X)} P_\lambda = \Omega_X$ is obvious, that is, Condition 1 in Definition 1 holds. By Subjective Nondelusion and the definition of P_λ, Condition 1 in Definition 1 holds. By Stationarity, for any $\omega, \omega' \in \Omega$, if $\Pi(\omega, X) \neq \Pi(\omega', X)$, then $\omega' \notin \Pi(\omega, X)$. That is, $\Pi(\omega, X) \cap \Pi(\omega', X) = \emptyset$. By the definition of P_λ, if $P_\lambda \neq P'_\lambda$, then $P_\lambda \cap P'_\lambda = \emptyset$. That is, Condition 3 in Definition 1 holds. Therefore, Π is partially partitional on Ω_X. ■

Four of the five assumptions of the possibility correspondence proposed Heifetz et al. (2006), Confinedness, Generalized Reflexivity, Projections Preserve Awareness (PPA) and Projections Preserve Knowledge (PPK) can be induced from Subjective Nondelusion and Stationarity in our model. Given $E \subseteq \Omega$, for any $X \subseteq P$, let $E_X = \{\omega_X \in \Omega | \omega \in E\}$ and let $E^X = \left\{\omega' \in \Omega \middle| \forall \omega \in E \quad \omega' = \omega \vee_{p \in Z : Z \subseteq X} p\right\}$. Then, the above properties are formulated and shown as follows.

Remark 2 If a possibility correspondence Π satisfies Subjective Nondelusion and Stationarity, then it satisfies the followings.

1. Confinedness: For any $\omega \in \Omega_X$ and any $X \subseteq P$, $\Pi(\omega, X) \subseteq \Omega_X$.
2. Generalized Reflexivity: For any $\omega \in \Omega$ and $X \subseteq P$, $\omega \in \left(\Pi(\omega, X)\right)^P$.
3. Projections Preserve Awareness: For any $\omega \in \Omega$ and $X \subseteq P$, if $\omega \in \Pi(\omega, X)$, then $\omega_X \in \Pi(\omega_X, X)$.
4. Projections Preserve Knowledge: For any $\omega \in \Omega$ and $X, Y \subseteq P$, if $\Pi(\omega, X) \subseteq \Omega_Y$, then $\left(\Pi(\omega, X)\right)_Y = \Pi(\omega_Y, X)$.

Proof. (Property 1) By Subjective Nondelusion, $\omega_X \in \Pi(\omega, X)$. By Stationarity, if $\omega' \in \Pi(\omega, X)$, then $\Pi(\omega', X) = \Pi(\omega, X)$. That is, $\omega'_X = \omega'$. Therefore, for any $\omega' \in \Pi(\omega, X)$, $\omega' \in \Omega_X$. Hence, $\Pi(\omega, X) \subseteq \Omega_X$.

(Property 2) Given any $\omega \in \Omega$ and $X \subseteq P$. Then, $\left(\Pi(\omega, X)\right)^P =$

$\{\omega'' \in \Omega \mid \forall \omega' \in \Pi(\omega, X) \quad \omega'' = \omega' \vee_{p \in Z: Z \subseteq X} p\}$. By Subjective Nondelusion, $\omega_X \in \Pi(\omega, X)$ and there exists $Z \subseteq X$ with $\omega = \omega_X \vee_{p \in Z} p$. Hence, $\omega \in (\Pi(\omega, X))^P$.

(Property 3) It is obvious by Subjective Nondelusion.

(Property 4) Given $\omega \in \Omega$, $X, Y \subseteq P$ and $\Pi(\omega, X) \subseteq \Omega_Y$. For any $\omega' \in \Pi(\omega, X)$, because $\omega' \in \Omega_Y$, $r_Y^X(\omega') = \omega'$. That is, $(\Pi(\omega, X))_Y = \Pi(\omega, X)$. Hence, by Subjective Nondelusion and Stationarity, $\Pi(\omega_Y, X) = \Pi(\omega_X, X) = \Pi(\omega, X)$. ■

Heifetz et al. (2006) describes the following remark.

Remark 3 (Heifetz et al. 2006) The possibility correspondence Π satisfies the following properties.

A) Generalized Reflexivity implies Nondelusion.

B) Confinedness and PPK implies PPA.

Proof. (A) Suppose that Π satisfies Generalized Reflexivity. Given any $\omega \in \Omega$ and $X \subseteq P$ with $\omega \in (\Pi(\omega, X))^P = \{\omega'' \in \Omega \mid \forall \omega' \in \Pi(\omega, X) \quad \omega'' = \omega' \vee_{p \in Z: Z \subseteq X} p\}$. Then, there must exist $\omega' \in \Pi(\omega, X)$ such that $\omega = \omega' \vee_{p \in Z: Z \subseteq X} p$. That is, $r_X^Y(\omega) = r_X^Y(\omega' \vee_{p \in Z: Z \subseteq P} p) = \omega'$. Hence, $\omega_X \in \Pi(\omega, X)$.

(B) Suppose that Π satisfies Confinedness and PPK. That is, for any $\omega \in \Omega_X$ and any $X \subseteq P$, $\Pi(\omega, X) \subseteq \Omega_X$, and for any $\omega \in \Omega$ and $X, Y \subseteq P$, if $\Pi(\omega, X) \subseteq \Omega_Y$, then $(\Pi(\omega, X))_Y = \Pi(\omega_Y, X)$. By a proof in the property 4 of Remark 2, for any $\omega \in$

Ω, $\left(\Pi(\omega\,, X)\right)_Y = \Pi(\omega\,, X)$. Let $Y = X$. Suppose that $\omega \in \Pi(\omega\,, X)$. Then, $\omega \in \Pi(\omega_X\,, X)$. By a definition of projections, because $\omega_X = \omega$, $\omega_X \in \Pi(\omega_X\,, X)$. ■

Heifetz et al. (2006) assumes Projections Preserve Ignorance (PPI): For any $\omega \in \Omega$ and $X, Y \subseteq P$, $\left(\Pi(\omega\,, X)\right)^P \subseteq \left(\Pi(\omega_Y\,, X)\right)^P$. The property cannot be induced from Subjective Nondelusion and Stationarity of the possibility correspondence.

Example 4 Given $P = \{x, y, z\}$. Then, $\Omega = \{x \vee y \vee z, x \vee y, y \vee z, z \vee x, x, y, z, \phi\}$. Suppose that an agent can perceive all basic propositions, that is, $X = P$, and that $\Pi(x \vee y \vee z\,, P) = \Pi(x \vee y\,, P) = \{x \vee y \vee z, x \vee y\}$, $\Pi(y \vee z\,, P) = \Pi(z \vee x\,, P) = \Pi(x\,, P) = \{y \vee z, z \vee x, x\}$, $\Pi(y\,, P) = \Pi(z\,, P) = \{y, z\}$, and $\Pi(\phi\,, P) = \{\phi\}$. The partitions are shown in Figure 4. Then, the possibility correspondence satisfies Subjective Nondelusion and Stationarity. Let $Y = \{y\}$ and $\omega = x \vee y$. Then, $\left(\Pi(\omega\,, P)\right)^P = \{x \vee y \vee z, x \vee y, y \vee z, z \vee x, x\}$, while $\left(\Pi(\omega_Y\,, P)\right)^P = \left(\Pi(\omega_Y\,, P)\right)^P = \{x \vee y \vee z, x \vee y, y \vee z, z \vee x\}$. Then, it is obvious that $\left(\Pi(\omega\,, X)\right)^P \nsubseteq \left(\Pi(\omega_Y\,, X)\right)^P$ because there exists some element $x \in \left(\Pi(\omega\,, P)\right)^P \setminus \left(\Pi(\omega_Y\,, P)\right)^P$. ■

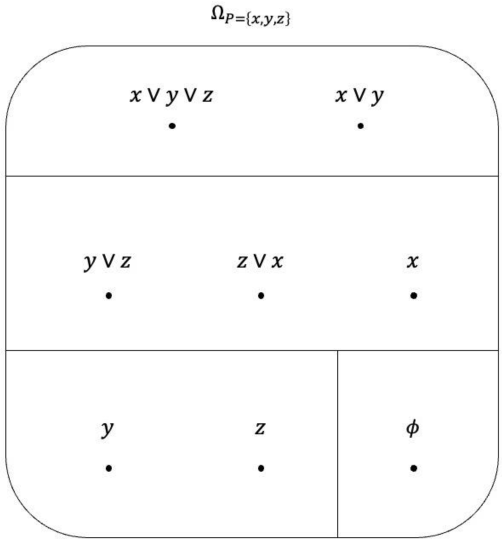

Fig. 4: Projections Preserve Ignorance is not satisfied.

Some previous studies refer to PPI under interactive situations, e.g., Heifetz et al. (2006; 2008) and Galanis (2013; 2018). In contrast, because our model is a single agent model, the assumption would not be necessary. This paper does not assume the property. [5]

4-2 Knowledge Operator

Let us define a knowledge operator. Let an event E be the subset of Ω. When an agent can perceive the subset of basic propositions $X \subseteq P$, The knowledge operator

[5] Constructive Aumann structures are non-partitional standard possibility correspondence models with multi attribute. In non-partitional standard possibility correspondence models, it does not seem to assume PPI. This paper conjectures that PPI is not necessary in constructive Aumann structure with multi agents too.

$K_X: 2^\Omega \longrightarrow 2^\Omega$ is defined as follows: $K_X(E) = \{\omega \in \Omega | \Pi(\omega, X) \subseteq E\}$ if $E \subseteq \Omega_X$; and $K_X(E) = \emptyset$ otherwise. $K_X(E)$ is interpreted as "An agent who can perceive X knows the event E." If $K_X(E) = \emptyset$, it is false that the agent knows E.

Example 3 (Continued.) Suppose that $X = \{x\}$ and that $\Pi(\omega_1, X) = \{\omega_2\}$, $\Pi(\omega_2, X) = \{\omega_2\}$, $\Pi(\omega_3, X) = \{\omega_4\}$, and $\Pi(\omega_4, X) = \{\omega_4\}$. Let $E_1 = \{\omega_2\}$. Then, $E_1 \subseteq \Omega_X$ and $\Pi(\omega_2, X) \subseteq E_1$. Therefore, $K_X(E_1) = \{\omega_2\}$, hence, the agent knows E_1. Let $E_2 = \{\omega_1, \omega_2\}$. Then, because $E_2 \nsubseteq \Omega_X$, $K_X(E_2) = \emptyset$. This means that it is false that the agent knows E_2. ■

$E \subseteq \Omega_X$ in the definition of the knowledge operator is important. When $X \neq P$, $\Pi(\omega, X) \subseteq \Omega$ is evident. Therefore, if $K_X(E) = \{\omega \in \Omega | \Pi(\omega, X) \subseteq E\}$ for every E, then $K_X(\Omega)$ is not empty, and it allows that the agent knows Ω.[6] Notably, $K_X(E)$ may be empty for some $E \subseteq \Omega_X$.

Remark 4 Given $E \subseteq \Omega_X$, the following are equivalent.

1. For any $\omega \in \Omega$, $\Pi(\omega, X) \nsubseteq E$.
2. $K_X(E) = \emptyset$.

Example 3 (Continued.) Let $E_3 = \{\omega_4\}$. Then, because $\Pi(\omega_2, X) \nsubseteq E_3$,

[6] In the case, Monotonicity is satisfied even if $X \neq P$.

$K_X(E_3) = \emptyset$. That is, at ω_2, it is false that the agent knows E_3. ■

It is evident that $K_X(E)$ is an event on Ω_X.

Proposition 2 (Heifetz et al. 2006) For any $E \subseteq \Omega$, $K_X(E) \subseteq \Omega_X$.

Proof. Given any $E \subseteq \Omega$ and $\omega \in K_X(E)$. By the definition of knowledge operator and Subjective Nondelusion, $\omega \in \Pi(\omega, X) \subseteq E$. Then, by Confinedness, as $\Pi(\omega, X) \subseteq \Omega_X$, $\omega \in \Omega_X$. Therefore, $K_X(E) \subseteq \Omega_X$. ■

Let $\neg K_X(E) = \Omega \setminus K_X(E)$ be the negation of $K_X(E)$. It interpreted as "An agent who can perceive only X does not know the event E." Here, we can show the generalization of properties of knowledge operators in Hefets et al. (2006).

Proposition 3 A knowledge operator K_X has the following properties.

K1 (Necessitation) $X = P$ if and only if $K_X(\Omega) = \Omega$.

K2 (Monotonicity) $X = P$ if and only if $E \subseteq F \Longrightarrow K_X(E) \subseteq K_X(F)$.

K3 (Conjunction) $\forall\lambda \in \Lambda \quad E_\lambda \subseteq \Omega_X$ or $\forall\lambda \in \Lambda \quad E_\lambda \nsubseteq \Omega_X \implies K_X(\bigcap_{\lambda\in\Lambda} E_\lambda) = \bigcap_{\lambda\in\Lambda} K_X(E_\lambda)$.

K4 (Truth) $K_X(E) \subseteq E$.

K5 (Positive Introspection) $K_X(E) = K_X K_X(E)$.

K6 (Negative Introspection) $X = P$ if and only if $\neg K_X(E) \subseteq K_X \neg K_X(E)$.

Proof. (K1) ($\Longrightarrow$) When $X = P$, by Nondelusion, for any $\omega \in \Omega$, $\omega \in \Pi(\omega, P) \subseteq \Omega$. That is, $\Omega \subseteq K_P(\Omega)$. Moreover, by Proposition 2, because $K_P(E) \subseteq \Omega$, $K_P(\Omega) = \Omega$.

($\Longleftarrow$) Suppose that $K_X(\Omega) = \Omega$. Assume that $X \neq P$. Then, $\Omega_X \subsetneq \Omega$. However, by the definition of the knowledge operator, $K_X(\Omega) = \emptyset$. This is a contradiction. Therefore, $X = P$.

(K2) ($\Longrightarrow$) When $X = P$, $K_P(E) = \{\omega \in \Omega | \Pi(\omega, P) \subseteq E\} \subseteq \{\omega \in \Omega | \Pi(\omega, P) \subseteq F\} = K_P(F)$.

($\Longleftarrow$) Suppose that $E \subseteq F \Longrightarrow K_X(E) \subseteq K_X(F)$. Assume that $X \neq P$. Then, $\Omega_X \subsetneq \Omega$ and $K_X(\Omega) = \emptyset$. For any $\emptyset \neq E \subseteq \Omega_X$, because $K_X(E) \supsetneq K_X(\Omega)$, this is a contradiction. Therefore, $X = P$.

(K3) Given any $\lambda \in \Lambda$, suppose that $E_\lambda \subseteq \Omega_X$. Given any $\omega \in K_X(\bigcap_{\lambda \in \Lambda} E_\lambda)$. Then, $\Pi(\omega, P) \subseteq \bigcap_{\lambda \in \Lambda} E_\lambda$. This means that for any $\lambda \in \Lambda$, $\Pi(\omega, P) \subseteq E_\lambda$. That is, for any $\lambda \in \Lambda$, because $\omega \in K_X(E_\lambda)$, $\omega \in \bigcap_{\lambda \in \Lambda} K_X(E_\lambda)$. For any $\lambda \in \Lambda$, suppose that $E_\lambda \nsubseteq \Omega_X$. Then, $K_X(E_\lambda) = \emptyset$. That is, $K_X(\bigcap_{\lambda \in \Lambda} E_\lambda) = \bigcap_{\lambda \in \Lambda} K_X(E_\lambda) = \emptyset$.

(K4) Given any $\omega' \in K_X(E)$, $\omega' \in \Pi(\omega, P) \subseteq E$. Therefore, $K_X(E) \subseteq E$.

(K5) By K4, $K_X K_X(E) \subseteq K_X$. Given any $\omega \in K_X(E)$, $\Pi(\omega, P) \subseteq E$. Here, for any $\omega' \in \Pi(\omega, P)$, $\Pi(\omega', P) \subseteq E$. Thus, $\omega' \in K_X(E)$. Hence, because $\Pi(\omega, P) \subseteq K_X(E)$, $\omega' \in K_X K_X(E)$. Thus, $K_X(E) \subseteq K_X K_X(E)$. Hence $K_X(E) = K_X K_X(E)$.

(K6) ($\Longrightarrow$) Assume that $X = P$. Given any $\omega \in \neg K_P(E)$, $\omega \notin K_P(E)$. Thus,

$\Pi(\omega, P) \nsubseteq E$. Given $\omega' \in \Pi(\omega, P)$, by Stationarity, because $\Pi(\omega, P) = \Pi(\omega', P)$, $\omega' \in \neg K_P(E)$. That is, $\Pi(\omega, P) \subseteq \neg K_P(E)$. Therefore, $\neg K_X(E) \subseteq K_X \neg K_X(E)$.

($\Longleftarrow$) Suppose that $\neg K_X(E) \subseteq K_X \neg K_X(E)$ and $X \neq P$. Then, $K_X(E) \subseteq \Omega_X \subsetneq \Omega$. Because $\neg K_X(E) \nsubseteq \Omega_X$, this must be $K_X \neg K_X(E) = \emptyset$. This is a contradiction. Therefore, $X = P$. ■

In our model, the knowledge operator satisfies Necessitation, Monotonicity and Negative Introspection if and only if the agent can perceive all basic propositions in P.

Remark 5 $K_X(\Omega_X) = \Omega_X$.

Although the remark is obvious, the agent who can perceive X believes that she faces the Aumann structure with the Ω_X. Thus, if we define some correspondence on only Ω_X, we can define the standard knowledge operator on Ω_X.

Finally, we show the following proposition proposed by Heifetz et al. (2006).

Proposition 4 (HMS 2006) $\neg K_X(E) \cap \neg K_X \neg K_X(E) \subseteq \neg K_X \neg K_X \neg K_X(E)$.

Proof. See each property 2 in Proposition 5-7 in below. ■

4.3 Awareness/Unawareness Operator

In this section, we define the unawareness operator. Suppose that an agent can perceive X. Then, the unawareness operator is defined as $U_X(E) = \neg K_X(E) \cap \neg K_X \neg K_X(E)$, while the awareness operator is defined as $A_X(E) = \neg U_X(E) = K_X(E) \cup K_X \neg K_X(E)$.

Example 3 (Continued.) For $E_1 = \{\omega_2\}$, because $\neg K_X(E_1) = \{\omega_1, \omega_3, \omega_4\}$ and $\neg K_X \neg K_X(E_1) = \emptyset$, $U_X(E_1) = \emptyset$. Therefore, $A_X(E_1) = \{\omega_2, \omega_4\}$. In contrast, for $E_2 = \{\omega_1, \omega_2\}$, because $\neg K_X(E_2) = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $\neg K_X \neg K_X(E_2) = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $U_X(E_2) = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $A_X(E_1) = \emptyset$. ■

Before we discuss properties of the knowledge and awareness/unawareness operators, we must consider three cases: the agent can perceive all basic propositions; the agent cannot perceive some non-empty subset of the basic proposition set, and an event is the subset of the state space which she can perceive; or the agent cannot perceive some non-empty subset of the basic proposition set, and an event is not the subset of the state space which she can perceive.

Let us show the following lemmas, prior to showing their properties.

Lemma 3 (Heifetz et al. 2006) $E, F \subseteq \Omega_X \implies K_X(E \cup K_X(F)) = K_X(E) \cup K_X(F)$.

Proof. First, given any $\omega \in K_X(E \cup K_X(F))$. Then, $\Pi(\omega, X) \subseteq E \cup K_X(F)$. This means that $\Pi(\omega, X) \subseteq E$ or $\Pi(\omega, X) \subseteq K_X(F)$. Hence, by K5, because $K_X(F) = K_X K_X(F)$, $K_X(E) \cup K_X K_X(F) = K_X(E) \cup K_X(F)$, and $\omega \in K_X(E) \cup K_X(F)$. That is, $K_X\big(E \cup K_X(F)\big) \subseteq K_X(E) \cup K_X(F)$. Next, given any $\omega \in K_X(E) \cup K_X(F)$. By K5, $K_X(E) \cup K_X(F) = K_X(E) \cup K_X K_X(F)$. Then, $\Pi(\omega, X) \subseteq E$ or $\Pi(\omega, X) \subseteq K_X(F)$. This means that $\Pi(\omega, X) \subseteq E \cup K_X(F)$. Therefore, because $\omega \in K_X\big(E \cup K_X(F)\big)$, $K_X(E) \cup K_X(F) \subseteq K_X\big(E \cup K_X(F)\big)$. Thus, $K_X\big(E \cup K_X(F)\big) = K_X(E) \cup K_X(F)$. ■

Lemma 4 An awareness operator has the following properties.

1. (Triviality) If $X = P$, then $A_X(E) = \Omega$.
2. (Non-triviality) If $X \neq P$ and $E \subseteq \Omega_X$, then $A_X(E) = K_X(E)$.
3. (Non-triviality) If $X \neq P$ and $E \nsubseteq \Omega_X$, then $A_X(E) = \emptyset$.

Proof. (1) Suppose that $X = P$. Then, $A_P(E) = K_P(E) \cup K_P \neg K_P(E)$. By K5, $K_P(E) \cup K_P \neg K_P(E) = K_P K_P(E) \cup K_P \neg K_P(E)$. By Lemma 3, $K_P K_P(E) \cup K_P \neg K_P(E) = K_P\big(K_P(E) \cup \neg K_P(E)\big) = K_P(\Omega) = \Omega$. Therefore, $A_P(E) = \Omega$.

(2) Suppose that $X \neq P$ and $E \subseteq \Omega_X$. Then, by Proposition 2, because $K_X(E) \subseteq \Omega_X$, $\neg K_X(E) \nsubseteq \Omega_X$. Therefore, $K_X \neg K_X(E) = K_X(\emptyset) = \emptyset$. Thus, $A_X(E) = K_X(E) \cup K_X \neg K_X(E) = K_X(E)$.

(3) Suppose that $X \neq P$ and $E \nsubseteq \Omega_X$. Then, by the definition of the knowledge operator, $K_X(E) = \emptyset$. Then, $\neg K_X(E) = \Omega$ and $K_X(\Omega) = \emptyset$. Therefore, $A_X(E) = K_X(E) \cup K_X \neg K_X(E) = \emptyset$. ■

Properties of the knowledge and awareness/unawareness operators is the following.

Proposition 5 When $X = P$, the following properties of knowledge and awareness/unawareness are obtained:

① KU Introspection: $K_X U_X(E) = \emptyset$.

② AU Introspection: $U_X(E) = U_X U_X(E)$.

③ Weak Necessitation: $A_X(E) = K_X(\Omega_X)$.

④ Strong Plausibility: $U_X(E) = \bigcap_{n=1}^{\infty}(\neg K_X)^n(E)$.

⑤ Weak Negative Introspection: $\neg K_X(E) \cap A_X \neg K_X(E) = K_X \neg K_X(E)$.

⑥ Symmetry: $A_X(\neg E) = A_X(E)$.

⑦ A-Conjunction: $A_X(\bigcap_{\lambda \in \Lambda} E_\lambda) = \bigcap_{\lambda \in \Lambda} A_X(E_\lambda)$.

⑧ AK-Self Reflection: $A_X K_X(E) = A_X(E)$.

⑨ AA-Self Reflection: $A_X A_X(E) = A_X(E)$.

⑩ A-Introspection: $K_X A_X(E) = A_X(E)$.

Proof. Suppose that $X = P$. By condition 1 in Lemma 4, $A_X(E) = \Omega$. Therefore, $U_X(E) = \emptyset$.

1) $K_X U_X(E) = K_X\big(\neg K_X(E) \cap \neg K_X \neg K_X(E)\big) = K_X \neg K_X(E) \cap K_X \neg K_X \neg K_X(E) \subseteq K_X \neg K_X(E) \cap \neg K_X \neg K_X(E) = \emptyset$.

2) By condition 1 in Lemma 4, because $A_X(E) = \Omega$ and $U_X(E) = \neg A_X(E) = \emptyset$,

$A_X U_X(E) = A_X(\emptyset) = K_X(\emptyset) \cup K_X \neg K_X(\emptyset) = \emptyset \cup K_X(\Omega) = \Omega$. Therefore, $A_X(E) = A_X U_X(E)$ and $U_X(E) = U_X U_X(E)$.

3) By $X = P$, $K_X(\Omega) = \Omega$. By Lemma 4, because $A_X(E) = \Omega$, $A_X(E) = K_X(\Omega)$.

4) By Lemma 4, $U_X(E) = \neg A_X(E) = \emptyset$. By Lemma 4 and AU Introspection, $U_X(E) = U_X U_X(E) = \emptyset$ and $U_X U_X U_X(E) = U_X(\emptyset)$. Then, as $A_X(\emptyset) = \Omega$, $U_X U_X U_X(E) = U_X(\emptyset) = \emptyset$. By repeating it, $U_X(E) = \bigcap_{n=1}^{\infty} (\neg K_X)^n (E)$.

5) $A_X \neg K_X(E) = K_X \neg K_X(E) \cup K_X \neg K_X \neg K_X(E) = K_X K_X \neg K_X(E) \cup K_X \neg K_X \neg K_X(E) = K_X\big(K_X \neg K_X(E) \cup \neg K_X \neg K_X(E)\big) = K_X(\Omega) = \Omega$. Therefore, $\neg K_X(E) \cap A_X \neg K_X(E) = \neg K_X(E) \cap \Omega = \neg K_X(E)$. By $X = P$, the knowledge operator satisfies K6, i.e., $\neg K_X(E) \subseteq K_X \neg K_X(E)$. Moreover, by K4, $K_X \neg K_X(E) \subseteq \neg K_X(E)$. Therefore, as $\neg K_X(E) = K_X \neg K_X(E)$, $\neg K_X(E) \cap A_X \neg K_X(E) = K_X \neg K_X(E)$.

6) Given $E \subseteq \Omega$, because $A_X(E) = \Omega$, $A_X(\neg E) = \Omega$. Hence, $A_X(\neg E) = A_X(E)$.

7) For any $E \subseteq \Omega$ and any $\lambda \in \Lambda$, as $A_X(E) = \Omega$, $A_X(\bigcap_{\lambda \in \Lambda} E_\lambda) = \Omega$ and $A_X(E_\lambda) = \Omega$, $\bigcap_{\lambda \in \Lambda} A_X(E_\lambda) = \Omega$. Therefore, $A_X(\bigcap_{\lambda \in \Lambda} E_\lambda) = \bigcap_{\lambda \in \Lambda} A_X(E_\lambda)$.

8) Given $E \subseteq \Omega$, because $A_X(E) = \Omega$, $A_X K_X(E) = \Omega$. Thus, $A_X K_X(E) = A_X(E)$.

9) Given $E \subseteq \Omega$, because $A_X(E) = \Omega$, $A_X A_X(E) = \Omega$. Thus, $A_X A_X(E) = A_X(E)$.

10) For any $E \subseteq \Omega$, because $A_X(E) = \Omega$, $K_X A_X(E) = K_X(\Omega)$. By K1, $K_X(\Omega) = \Omega$. Therefore, $K_X A_X(E) = A_X(E)$. ■

Proposition 6 When $X \neq P$ and $E \subseteq \Omega_X$, the following properties of knowledge and awareness/unawareness are obtained:

① KU Introspection: $K_X U_X(E) = \emptyset$.

② AU Introspection: $U_X(E) \subseteq U_X U_X(E)$.

③ Weak Necessitation: $A_X(E) \subseteq K_X(\Omega_X)$.

④ Strong Plausibility: $U_X(E) \subseteq \bigcap_{n=1}^{\infty}(\neg K_X)^n\,(E)$.

⑤ Weak Negative Introspection: $\neg K_X(E) \cap A_X \neg K_X(E) = K_X \neg K_X(E)$.

⑥ Reverse Symmetry: $A_X(\neg E) \subseteq A_X(E)$.

⑦ A-Conjunction: $A_X(\bigcap_{\lambda \in \Lambda} E_\lambda) = \bigcap_{\lambda \in \Lambda} A_X(E_\lambda)$.

⑧ AK-Self Reflection: $A_X K_X(E) = A_X(E)$.

⑨ AA-Self Reflection: $A_X A_X(E) = A_X(E)$.

⑩ A-Introspection: $K_X A_X(E) = A_X(E)$.

Proof. Suppose that $X \neq P$ and that for any $E \subseteq \Omega$, $E \subseteq \Omega_X$. Then, by Condition 2 in Lemma 4, $A_X(E) = K_X(E)$. Therefore, $U_X(E) = \neg K_X(E)$.

1) By $A_X(E) = K_X(E)$, $U_X(E) = \neg K_X(E)$. By Proposition 2, because $K_X(E) \subseteq \Omega_X$, $\neg K_X(E) \nsubseteq \Omega_X$. Therefore, $K_X\big(\neg K_X(E)\big) = \emptyset$.

2) $U_X(E) = \neg K_X(E) \subseteq \Omega$. $U_X U_X(E) = \neg K_X U_X(E)$. By KU Introspection, because $K_X U_X(E) = \emptyset$, $\neg K_X U_X(E) = \Omega$. Therefore, $U_X(E) \subseteq U_X U_X(E)$.

3) As $A_X(E) = K_X(E)$ by Condition 2 in Lemma 4 and $K_X(\Omega_X) = \Omega_X$ by Remark

5, $A_X(E) \subseteq K_X(\Omega_X)$.

4) By AU Introspection, $U_X U_X(E) = \neg K_X U_X(E) = \Omega$. $U_X U_X U_X(E) = U_X \neg K_X U_X(E) = U_X(\Omega) = \neg K_X(\Omega)$. By the definition of the knowledge operator, because $K_X(\Omega) = \emptyset$, $U_X(\Omega) = \Omega$. By repetition, $\bigcap_{n=1}^{\infty} (\neg K_X)^n (E) = \Omega$. Therefore, $U_X(E) \subseteq \bigcap_{n=1}^{\infty} (\neg K_X)^n (E)$.

5) By Condition 2 in Lemma 4, $A_X \neg K_X(E) = K_X \neg K_X(E)$. By K4, because $K_X(E) \subseteq E \subseteq \Omega_X$, $\neg K_X(E) \nsubseteq \Omega_X$. Therefore, $K_X \neg K_X(E) = \emptyset$. Thus, $\neg K_X(E) \cap A_X \neg K_X(E) = \neg K_X(E) \cap \emptyset = \emptyset = K_X \neg K_X(E)$.

6) $A_X(\neg E) = K_X(\neg E) \cup K_X \neg K_X(\neg E)$. By $E \subseteq \Omega_X$, $\neg E \nsubseteq \Omega_X$. Therefore, $K_X(\neg E) = \emptyset$. By $\neg K_X(\neg E) = \Omega$, $K_X \neg K_X(\neg E) = \emptyset$. Therefore, $A_X(\neg E) = \emptyset$. By Condition 2 in Lemma 4, because $A_X(E) = K_X(E)$, $A_X(\neg E) \subseteq A_X(E)$.

7) For any $E \subseteq \Omega_X$, because $A_X(E) = K_X(E)$, $A_X(\bigcap_{\lambda \in \Lambda} E_\lambda) = K_X(\bigcap_{\lambda \in \Lambda} E_\lambda)$. Moreover, for any $\lambda \in \Lambda$, because $A_X(E_\lambda) = K_X(E_\lambda)$, $\bigcap_{\lambda \in \Lambda} A_X(E_\lambda) = \bigcap_{\lambda \in \Lambda} K_X(E_\lambda)$. Therefore, by K3, $A_X(\bigcap_{\lambda \in \Lambda} E_\lambda) = \bigcap_{\lambda \in \Lambda} A_X(E_\lambda)$.

8) For any $E \subseteq \Omega_X$, by K4, $K_X(E) \subseteq E$. Therefore, by $A_X K_X(E) = K_X K_X(E)$, $K_X K_X(E) = K_X(E)$. Thus, $A_X K_X(E) = A_X(E)$.

9) For any $E \subseteq \Omega_X$, by K4, as $K_X(E) \subseteq E$. $A_X(E) = K_X(E)$, $A_X A_X(E) = A_X K_X(E) = K_X K_X(E) = K_X(E)$. Therefore, $A_X A_X(E) = A_X(E)$.

10) For any $E \subseteq \Omega_X$, because $A_X(E) = K_X(E)$, $K_X A_X(E) = K_X K_X(E) = K_X(E)$. Therefore, $K_X A_X(E) = A_X(E)$. ■

Proposition 7 When $X \neq P$ and $E \nsubseteq \Omega_X$, the following properties of knowledge and awareness/unawareness are obtained:

① KU Introspection: $K_X U_X(E) = \emptyset$.

② AU Introspection: $U_X(E) = U_X U_X(E)$.

③ Weak Necessitation: $A_X(E) \subseteq K_X(\Omega_X)$.

④ Strong Plausibility: $U_X(E) = \bigcap_{n=1}^{\infty}(\neg K_X)^n\,(E)$.

⑤ Weak Negative Introspection: $\neg K_X(E) \cap A_X \neg K_X(E) = K_X \neg K_X(E)$.

⑥ Reverse Symmetry: $A_X(\neg E) \supseteq A_X(E)$.

⑦ A-Conjunction: $A_X(\bigcap_{\lambda\in\Lambda} E_\lambda) = \bigcap_{\lambda\in\Lambda} A_X(E_\lambda)$.

⑧ AK-Self Reflection: $A_X K_X(E) = A_X(E)$.

⑨ AA-Self Reflection: $A_X A_X(E) = A_X(E)$.

⑩ A-Introspection: $K_X A_X(E) = A_X(E)$.

Proof. Suppose that $X \neq P$ and that for any $E \subseteq \Omega$, $E \nsubseteq \Omega_X$. By Condition 3 in Lemma 4, $A_X(E) = \emptyset$. Therefore, $U_X(E) = \Omega$.

1) $K_X U_X(E) = K_X(\Omega) = \emptyset$.

2) Because $U_X U_X(E) = U_X(\Omega)$. $\Omega \nsubseteq \Omega_X$ is obvious, $U_X(\Omega) = \Omega$. Therefore, as $U_X U_X(E) = \Omega$, $U_X(E) = U_X U_X(E)$.

3) By Remark 5, $A_X(E) = \emptyset \subseteq \Omega_X = K_X(\Omega_X)$.

4) By AU Introspection, $U_X U_X U_X(E) = U_X(\Omega) = \Omega$. By repetition, $U_X(E) = \bigcap_{n=1}^{\infty}(\neg K_X)^n\,(E)$.

5) By $E \nsubseteq \Omega_X$, because $K_X(E) = \emptyset$, $\neg K_X(E) = \Omega$. $K_X \neg K_X(E) = K_X(\Omega) = \emptyset$. Therefore, because $A_X \neg K_X(E) = A_X(\Omega) = K_X(\Omega) \cup K_X \neg K_X(\Omega) = \emptyset$, $\neg K_X(E) \cap A_X \neg K_X(E) = K_X \neg K_X(E)$.

6) Because $A_X(E) = \emptyset$, $A_X(\neg E) \supseteq A_X(E)$.

7) For any $E \nsubseteq \Omega_X$, as $A_X(E) = \emptyset$, $A_X(\bigcap_{\lambda \in \Lambda} E_\lambda) = \emptyset$. Moreover, for any $\lambda \in \Lambda$, because $A_X(E_\lambda) = \emptyset$, $\bigcap_{\lambda \in \Lambda} A_X(E_\lambda) = \emptyset$. Hence, $A_X(\bigcap_{\lambda \in \Lambda} E_\lambda) = \bigcap_{\lambda \in \Lambda} A_X(E_\lambda)$.

8) Because $K_X(E) = \emptyset$, $A_X K_X(E) = A_X(\emptyset) = K_X(\emptyset) \cup K_X \neg K_X(\emptyset) = \emptyset \cup K_X(\Omega) = \emptyset$. Therefore, $A_X K_X(E) = A_X(E)$.

9) $A_X A_X(E) = A_X(\emptyset) = \emptyset$. Therefore, $A_X A_X(E) = A_X(E)$.

10) $K_X A_X(E) = K_X(\emptyset) = \emptyset$. ■

Remark 6 Suppose $X \neq P$. For any $\lambda \in \Lambda$, let $E_\lambda \subseteq \Omega_X$, and for any $\delta \in \Delta$, let $E_\delta \nsubseteq \Omega_X$. Then, $A_X(\bigcap_{\lambda \in \Lambda} E_\lambda \bigcap_{\delta \in \Delta} E_\delta) \supseteq \bigcap_{\lambda \in \Lambda} A_X(E_\lambda) \bigcap_{\delta \in \Delta} A_X(E_\delta)$.

KU Introspection, AU Introspection, Weak Necessitation and Strong Plausibility have been proposed by Dekel et al. (1998); Symmetry, A-Conjunction, AK-Self Reflection and AA-Self Reflection by Modica and Rustichini (1999); Weak Negative Introspection, Symmetry, A-Conjunction, AK-Self Reflection, and AA-Self Reflection by Halpern (2001); and A-Introspection was proposed by Heifetz et al. (2006). However, when $X \neq P$ and $E \subseteq \Omega_X$, AU Introspection, Weak Necessitation and Strong Plausibility may not satisfy equality, and when $X \neq P$ and $E \nsubseteq \Omega_X$, Weak

Necessitation may not satisfy equality. Moreover, the A-Conjunction is satisfied only when every event satisfies $E \subseteq \Omega_X$ or $E \nsubseteq \Omega_X$. By Remark 6, if the condition does not hold, $\bigcap_{\lambda \in \Lambda} A_X(E_\lambda) \bigcap_{\delta \in \Delta} A_X(E_\delta)$ may be empty.

Interestingly, Symmetry crashes with Non-triviality. Previous studies discussing properties of awareness/unawareness prove Symmetry, e.g., Heifetz et al. (2006; 2013a) and Li (2009), or assume it, e.g., Modica and Rustichini (1994; 1999), Halpern (2001), Heifetz et al. (2008), and Sadzik (2021). In contrast, in our model, when the awareness/unawareness operator is non-trivial, Symmetry does not hold. Although Fukuda (2020) suggests that Symmetry may not hold in infinite higher-order unawareness, he shows that Symmetry holds in second order unknown, i.e., in first order unawareness. In contrast with his result, we show that Symmetry does not hold in first order unawareness, i.e., in second order unknown. Let us call the properties a *Reverse Symmetry*. Moreover, the inclusion relations in Reverse Symmetry are different, whether $E \subseteq \Omega_X$ or $E \nsubseteq \Omega_X$.

As shown in the proofs, when the agent cannot perceive a part of basic propositions, by definition of the knowledge operator, if the agent knows E, she may not perceive the negation, as it is not in her subjective state space. Therefore, the knowledge operator with the negation is empty. When E is not in her subjective state space, $K_X(E) = \emptyset$ and the negations, i.e., $\neg E$ and $\neg K_X(E)$ are empty or not. Therefore, she can perceive them or not.

It seems that the features of Reverse Symmetry are important and that there are at least two implications. One hand suggests that we should not discuss alike the event that the agent can perceive and the negation that she cannot perceive. The other implication is related with modal logics. Modica and Rustichini (1994) shows S4 with Symmetry equals to S5. In contrast, our model Symmetry with Non-triviality does not hold. Reverse Symmetry may suggest that in discussions about unawareness, we should exclude S5, and that we should discuss S4 with Reverse Symmetry in modal logics.

Finally, in this subsection, we show Awareness leads to Knowledge, and the inverse inclusion. Galanis (2013) proposes the property: For any $X, Y \subseteq P$ with $Y \subseteq X$ and $E \subseteq \Omega$, $K_Y(E_Y) \subseteq \left(K_X(E_Y)\right)_Y \cap A_Y(E_Y)$. He suggests that if PPK is not assumed, then the inverse inclusion may not hold. However, this paper shows PPK induced from Subjective Nondelusion and Stationarity of the possibility correspondence. Therefore, the inverse inclusion of Awareness leads to Knowledge hold. It means that our model does not fit Galanis's (2013) framework.

Proposition 8 For any $X, Y \subseteq P$ with $Y \subseteq X$ and $E \subseteq \Omega$, $K_Y(E_Y) = \left(K_X(E_Y)\right)_Y \cap A_Y(E_Y)$.

Proof. Given $X, Y \subseteq P$ with $Y \subseteq X$ and $E \subseteq \Omega$. It is obvious that $K_Y(E_Y) \subseteq A_Y(E_Y)$. By K4, because $K_X(E_Y) \subseteq E_Y$, $K_X(E_Y) \subseteq \Omega_Y$. Given $\omega \in \Omega \setminus E_Y$. Then, for any $\omega' \in \Omega$, $\omega \notin \Pi(\omega', X)$ and $\omega \notin \Pi(\omega', Y)$. It means $\omega \notin K_X(E_Y)$ and $\omega \notin K_Y(E_Y)$. That is, $\omega \in \neg K_X(E_Y)$ and $\omega \in \neg K_Y(E_Y)$. because ω is arbitrary, $\neg K_X(E_Y) = \neg K_Y(E_Y)$.

Hence, $K_X(E_Y) = K_Y(E_Y)$. because $E_Y \subseteq \Omega_Y$, $\left(K_X(E_Y)\right)_Y = K_X(E_Y)$. Therefore, $K_Y(E_Y) = K_Y(E_Y) \cap A_Y(E_Y) = K_X(E_Y) \cap A_Y(E_Y) = \left(K_X(E_Y)\right)_Y \cap A_Y(E_Y)$. ■

4-4 Relationships with Standard Aumann Structure

In our models if $X = P$, the main theorem in Dekel et al. (1998), that unawareness is trivial, is satisfied as follows.

Theorem 1 In any constructive Aumann structure, the following are equivalent.

1. $X = P$.
2. For any $E \subseteq \Omega$, $U_X(E) = \emptyset$.
3. For any $E, F \subseteq \Omega$, $E \subseteq F$, $U_X(E) \subseteq \neg K_X(F)$.

Proof. (1 $\Longrightarrow$ 2) It is obvious by Condition 1 in Lemma 4.

(2 $\Longrightarrow$ 3) Given $E \subseteq \Omega$, $U_X(E) = \emptyset$. Then, for any $F \subseteq \Omega$, $\emptyset = U_X(E) \subseteq \neg K_X(F)$.

(3 $\Longrightarrow$ 1) Suppose that for every $E, F \subseteq \Omega$, if $E \subseteq F$, then $U_X(E) \subseteq \neg K_X(F)$. Here, assume that $X \neq P$ and given $E = \emptyset$ and $\emptyset \neq F \subseteq \Omega_X$. Then, because $\neg K_X(\emptyset) = \Omega$ and $K_X(\Omega) = \emptyset$, $U_X(\emptyset) = \neg K_X(E) \cap \neg K_X \neg K_X(E) = \Omega \cap \neg K_X(\Omega) = \Omega \cap \left(\Omega \setminus K_X(\Omega)\right) = \Omega$. Because $\neg K_X(F) \subsetneq \Omega$ is obvious, $\neg K_X(F) \subsetneq U_X(\emptyset)$. This is a contradiction. Hence, $X = P$. ■

Dekel et al. (1998) show that if the unawareness operator satisfies Plausibility, AU Introspection and KU Introspection, and the knowledge operator satisfies Necessitation, then unawareness is trivial. Moreover, they show that under the above assumptions of the unawareness operator, if the knowledge operator satisfies Monotonicity, the agent is unaware of everything. In our model, where $X = P$, we show that the knowledge operator and unawareness operator satisfy the above properties. Hence, their main theorem must be satisfied when $X = P$, and vise versa.

Chen et al. (2012) show that if the knowledge operator satisfies Necessitation, and the unawareness operator satisfies Plausibility, then Negative Introspection is equivalent to AU Introspection and KU Introspection, and that if the assumptions adding Monotonicity and Truth are satisfied, Negative Introspection is equivalent to AU Introspection. In our model, where $X = P$, Negative Introspection and AU Introspection are equivalent. Moreover, Negative Introspection is equivalent to Symmetry, as shown by Modica and Rustichini (1994). Therefore, we can generalize the main theorem in Chen et al. (2012) and Modica and Rustichini (1994) as follows.

Theorem 2 In any constructive Aumann structure, the following are equivalent.

1. $X = P$.
2. Negative Introspection if and only if AU Introspection if and only if Symmetry.

Proof. ($1 \Longrightarrow 2$) Suppose that $X = P$. Then, by Proposition 3, Negative Introspection holds. Moreover, by Proposition 5, AU Introspection and Symmetry hold.

($2 \Longrightarrow 1$) Suppose that Negative Introspection, AU Introspection and Symmetry are equivalent. Here, assume that $X \neq P$. Then, by Proposition 3, Negative Introspection does not hold, and by Proposition 6 and Proposition 7, Symmetry does not hold. However, by Proposition 6 and Proposition 7, AU Introspection holds. This contradicts that the three properties are equivalent. Therefore, $X = P$. ■

Finally, we consider a relationship with Fukuda (2020). He suggests that non-trivial unawareness can be discussed in (non-partitional) standard state space models, and that Necessitation crashes AU Introspection. Hence, as pointed out by him, if AU introspection does not hold where Necessitation is satisfied, non-trivial unawareness can be discussed. Subsequently, he proposes Reverse AU Introspection ($U_X(E) \supseteq U_X U_X(E)$) instead of AU Introspection. He suggests two points: one is that AU Introspection is not necessary discussing non-trivial unawareness; the other is that if AU Introspection does not hold, (non-partitional) standard state space models represent awareness of unawareness. In contrast, our Reverse AU Introspection with Non-triviality may not hold in our model when the equation of the inclusion relation does not hold. Our model must induce AU Introspection, even if the agent cannot perceive some part of basic propositions. The different results imply that different features exist between Fukuda (2020) and this study.

V Concluding Remarks

This paper presents a constructive Aumann structure where a state space is a complete lattice. In contrast with Heifets et al. (2006) and Li (2009), the family of disjoint state spaces is not necessary in our model. However, unlike in the case of standard state space models, our models are multi-attribute models, similarly to those of Heifets et al. and Li. Note that our same states between different state spaces have different attributes. This means that a property of a state in each subjective state space turn on relationships with the others state in the state space.

Our model is a single agent model, and we do not discuss higher-order perceptions. However, our results differ from those of previous studies, as shown in Proposition 6, Proposition 7, and Remark 6. Possibly, other properties do not hold in multi-agent models or higher-order perceptions.

In particular, Symmetry with non-trivial unawareness is not satisfied. Previous studies prove or assume the property, but we show the impossibility. In the result (Reverse Symmetry), the implication is that we must not discuss alike the event that the agent can perceive and the negation that she cannot perceive, and that S5 in modal logics must be excluded discussing unawareness.

In contrast, constructive state spaces, HMS-state spaces, and Li-state spaces are equivalent, and we show generalizations of results in Deket et al. (1998) as Theorem 1 in this paper, and in Chen et al. (2012) (and Modica and Rustichini 1994) as Theorem

2 in this paper. This implies that our model is an intermediate between Heifetz et al. (2006) and Li's (2009) models and non-partitional standard state space models.

Previous studies discuss choice theories with unawareness, e.g., Karni and Vierø (2013; 2017) and Piermont (2017), interactive situations with unawareness, e.g., Auster (2013), Heifetz et al. (2013a) and Galanis (2013; 2018), and games with unawareness, e.g., Heifetz et al. (2013b), Halpern and Rêgo (2014), Perea (2018) and Feinberg (2020). In future studies, we must be able to introduce Aumann structures with complete lattices to their studies. For example, our model must be applied to Bayesian games with unawareness. Previous studies discussing games with unawareness are Sadzik (2021) and Meier and Schipper (2014). Sadzik discusses probabilistic beliefs with unawareness based on Heifetz et al. (2006) and defines Bayesian equilibrium in normal-form games with unawareness. Meier and Schipper discuss probabilistic beliefs based on Heifetz et al. (2013a), define Bayesian equilibrium and prove the existence.[7] We will discuss probabilistic beliefs based on our Aumann structure with a complete lattice and introduce it to Bayesian games with unawareness.

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[7] There exists a difference between Sadzik (2021) and Meier and Schipper (2014). The former assumes common prior, while the latter does not assume it.

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