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Unawareness and Reverse Symmetry: Aumann Structure with Complete Lattice

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Abstract

This paper models Aumann structures with complete lattices and discusses unawareness. In multi-attribute state space models, although previous studies discussing unawareness assume a family of spaces with a complete lattice, there is no model in which a standard state space is a complete lattice. Without formulating the family of spaces, this paper models a state space with a complete lattice using a set-theoretic and a constructive approach. However, in the presented model, although almost all the properties of the unawareness operator as those of previous studies hold, Symmetry with non-trivial unawareness does not. Consequently, this paper proposes a property, the Reverse Symmetry, to show that non-trivial unawareness holds if and only if this Reverse Symmetry holds, and discusses the implications.

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1 Introduction

Unawareness was proposed by Fagin and Halpern (1988) as a higher-order unknown. However, in partitional standard state space models (or standard Aumann structure models), e.g., Aumann (1976), if the knowledge operator satisfies Necessitation ($K(\Omega) = \Omega$), and the unawareness operator satisfies Plausibility ($U(E) = \neg K(E) \cap \neg K \neg K(E)$), KU Introspection ($KU(E) = \emptyset$)and AU Introspection ($U(E) = UU(E)$), non-trivial unawareness ($U(E) \neq \emptyset$) cannot be modeled (Dekel et al. 1998).

To avoid this issue, there are two approaches to discussing unawareness: one is the unawareness structures proposed by Heifetz et al. (2006) and Li (2009). In their models, unawareness indicates a lack of conception. The other approach indicates non-partitional standard state space models (e.g., Modica and Rustichini 1994, 1999; Geanakoplos 1989). State spaces in unawareness structures are complete lattices, while those in the standard state space models are not. A basic difference between standard state spaces and state spaces with unawareness is as follows: The former spaces are flat—which means that each state is independent—whereas the latter spaces are not.

This means that some states may have an order relation as a lattice.[1,2]

In Heifetz et al. (2006) and Li (2009), the family of disjoint state spaces is a complete lattice. However, there is no model in which a standard state space itself is a complete lattice. This paper models the state spaces, called *constructive state spaces*, via a set-theoretic and constructive approach. Our state space is equivalent to the space presented by Heifetz et al. (2006) and Li (2009).

In our approach, a standard operator is not convenient, because the operator cannot define a state related with ϕ in the state space of Heifetz et al. (2006) or Li (2009). To avoid this issue, we must formulate the *overloaded operator*. Our operator has multiple arities. Although our state space is similar to that of Heifetz et al. (2008), our discussions and assumptions about unawareness are different from theirs.

In our state space, any (subjective) state space is the subset of the constructive state space. The meaning of a state that belongs to a particular state space is not the same as the meaning of a state that belongs to a different space, because some of the attributes that may be included in the former, may not be included in the latter. The meaning of a state depends on the state to which the state space belongs. Thus, the meaning of a state is decided in relation to the other states in a given state space.

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1 Recently, Fukuda (2020) presented generalized state space models that nest both unawareness structures and non-partitional standard state space models.

2 Schipper (2014) provides a historical survey of unawareness. Schipper (2015) provides a mathematical survey of it in modal logic.

This paper models constructive Aumann structures with constructive state spaces and defines the possibility correspondence, knowledge operator, and awareness/unawareness operator in the state space. Almost the same properties of the knowledge and awareness/unawareness operators were employed in previous studies.

However, interestingly, Symmetry ($U(E) = U(\neg E)$) crashes non-trivial unawareness, although previous studies have proved the property (e.g., Heifetz et al 2006, 2013a; Li 2009; Fukuda 2020) or assumed it (e.g., Modica and Rustichini 1994, 1999; Halpern 2001; Heifetz et al. 2008; Sadzik 2021). We call the property in which Symmetry does not hold *Reverse Symmetry*. Reverse Symmetry has two implications. One is that we must not discuss the event that the agent can perceive and the negation that it cannot perceive using the same approach. The other is that S5 in modal logics may not be necessary in discussing unawareness. Modica and Rustichini (1994) assumed Symmetry and showed that S4 with Symmetry is equal to S5. In contrast, because we show Reverse Symmetry in our model, we must consider S4 with Reverse Symmetry.

Moreover, Awareness Leads to Knowledge does not hold. Galanis (2013) proposed a property of Awareness Leads to Knowledge and proved the property in unawareness structures. In contrast, this study shows the inverse of the property. The assumptions made in this study are the same as those made by Galanis (2013). However, the opposite result is obtained. These contrasting results emphasize the difference between unawareness structures and our constructive Aumann structures.

A constructive Aumann structure induces generalization of the main theorems in Dekel et al. (1998) and Chen et al. (2012). Because several properties of the knowledge and the awareness/unawareness operators are the same in previous studies, our models are intermediate between the standard state space and unawareness structure.

The remainder of this paper is organized as follows: The modeling of the state spaces in Heifetz et al. (2006) and Li (2009) is discussed in Section 2. The modeling of a constructive state space and its comparison with the state spaces in Heifetz et al. (2006) and Li (2009) are discussed in Section 3. The modeling of a constructive Aumann structure is discussed in Section 4, and the possibility correspondence, knowledge operator, and awareness/unawareness operator are defined and discussed as well. The concluding remarks are presented in Section 5.

2 State Space in Previous Studies about Unawareness

This section formulates the state spaces defined by Heifetz et al. (2006) and Li (2009).[3]

2.1 State Space in Heifetz et al. (2006)

First, we formulate the state spaces proposed by Heifetz et al. (2006), which we denote HMS-state spaces. Let $\mathcal{S} = \{S_\lambda\}_{\lambda \in \Lambda}$ be a complete lattice of disjoint state spaces, and

[3] This paper does not discuss unawareness structures. Related studies discussing their related models are Heifetz et al. (2006, 2008, 2013a), Li (2009), Heinsalu (2012), Galanis (2013, 2018), Schipper (2014, 2015), Fukuda (2020), and Sadzik (2021).

let $\preccurlyeq$ be a partial order on $\mathcal{S}$. For any $S, S' \in \mathcal{S}$, $S \succcurlyeq S'$ is interpreted as "S is more expressive than S'." Then, there exists a surjective projection $R^S_{S'}: S \longrightarrow S'$, that is, for any $\omega \in S$, $R^S_{S'}(\omega) \in S'$. An HMS-state space is denoted by $\Sigma = \bigcup_{\lambda \in \Lambda} S_\lambda$. A standard state space is not Σ, but each state spate on $\mathcal{S}$. Hence, an HMS-state space is the union of disjoint standard state spaces.

Example 1 Suppose that $\mathcal{S} = \left\{S_{\{x,y\}}, S_{\{x\}}, S_{\{y\}}, S_{\{\phi\}}\right\}$ is a complete lattice of disjoint spaces, and let $S_{\{x,y\}} = \{xy, x\neg y, \neg xy, \neg x\neg y\}$, $S_{\{x\}} = \{x, \neg x\}$, $S_{\{y\}} = \{y, \neg y\}$ and $S_{\{\phi\}} = \{\phi\}$. For example, x indicates that x is true, while $\neg x$ means that x is false. Given two different spaces $S_{\{x,y\}}, S_{\{x\}}$, $R^{S_{\{x,y\}}}_{S_{\{x\}}}(xy) = R^{S_{\{x,y\}}}_{S_{\{x\}}}(x\neg y) = x$, and $R^{S_{\{x,y\}}}_{S_{\{x\}}}(\neg xy) = R^{S_{\{x,y\}}}_{S_{\{x\}}}(\neg x\neg y) = \neg x$. Then, an agent who can perceive only $S_{\{x\}}$ cannot perceive y. The agent perceives x for xy or $x\neg y$, while it perceives $\neg x$ for $\neg xy$ or $\neg x\neg y$. In the example, the HMS-state space is $\Sigma = S_{\{x,y\}} \cup S_{\{x\}} \cup S_{\{y\}} \cup S_{\{\phi\}}$, while the standard state space is each element of $\mathcal{S}$, that is, $S_{\{x,y\}}, S_{\{x\}}, S_{\{y\}}$, and $S_{\{\phi\}}$. The example is depicted in Figure 1. ■

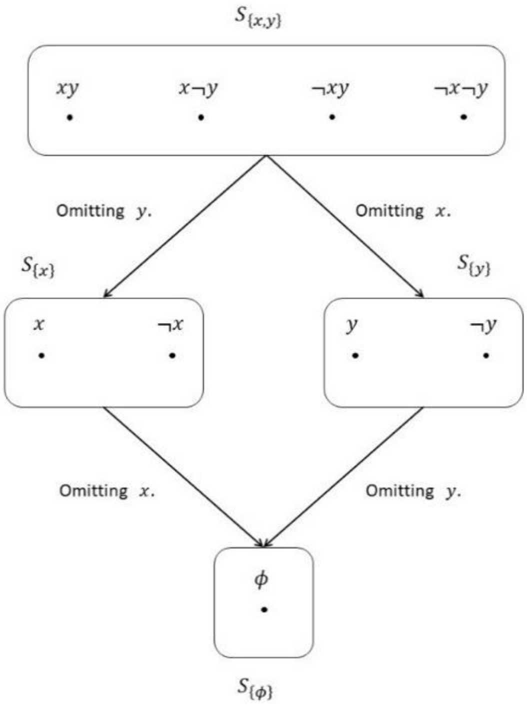

Fig. 1: HMS-state space.

2.2 State Space in Li (2009)

Here, we formulate the state spaces proposed by Li (2009) and denote them Li-state spaces. Let Q^* be the set of questions. $A_q = \{a_q, \neg a_q\}$ is the set of answers about $q \in Q^*$. Here, a Cartesian product $\prod_{q\in Q^*} A_q$ is the objective state space, and $\prod_{q\in Q} A_q$ is a subjective state space, where $Q \subseteq Q^*$. If $Q = \emptyset$, let $\prod_{q\in Q} A_q = \{\phi\}$. Evidently, two different spaces $A, A' \in \left\{\prod_{q\in Q} A_q \middle| Q \subseteq Q^*\right\}$ are disjoint. For any $Q, Q' \in 2^{Q^*} \setminus \{\emptyset\}$, such that $Q' \subseteq Q \subseteq Q^*$, there is a surjective projection $\pi^Q_{Q'}\colon \prod_{q\in Q} A_q \longrightarrow \prod_{q\in Q'} A_q$. Hence, for any $\omega \in \prod_{q\in Q} A_q$, $\pi^Q_{Q'}(\omega) \in \prod_{q\in Q'} A_q$. Denote by $\mathcal{A} = \bigcup_{Q\subseteq Q^*} \prod_{q\in Q} A_q$ a Li-state space. A standard state space is not $\mathcal{A}$, but each element on $\mathcal{A}$. Thus, a Li-state space is the union of all disjoint spaces on $\mathcal{A}$.

Example 2 Suppose that $Q^* = \{q(x), q(y)\}$ is the set of questions. Here, $q(x)$ is a question about an attribute x, and $q(y)$ is a question about an attribute y. Then, the sets of answers for each question are $A_{q(x)} = \{a_{q(x)}, \neg a_{q(x)}\}$ and $A_{q(y)} = \{a_{q(y)}, \neg a_{q(y)}\}$. Given x, $a_{q(x)}$ is interpreted as "the answer for $q(x)$ is *yes*," while $\neg a_{q(x)}$ is interpreted as "the answer for $q(x)$ is *no*." The objective state space is $A_{q(x)} \times A_{q(y)}$, while subjective state spaces are $A_{q(x)}$, $A_{q(y)}$, and $A_{q(\phi)}$. Given $\{q(x)\} \subseteq Q^*$, there is a surjective projection $\pi^{Q^*}_{\{q(x)\}}: A_{q(x)} \times A_{q(y)} \longrightarrow A_{q(x)}$. Then, $\pi^{Q^*}_{\{q(x)\}}(a_{q(x)}, a_{q(y)}) = \pi^{Q^*}_{\{q(x)\}}(a_{q(x)}, \neg a_{q(y)}) = a_{q(x)}$ and $\pi^{Q^*}_{\{q(x)\}}(\neg a_{q(x)}, a_{q(y)}) = \pi^{Q^*}_{\{q(x)\}}(\neg a_{q(x)}, \neg a_{q(y)}) = \neg a_{q(x)}$. An agent who can perceive only an attribute x perceives a state $a_{q(x)}$ for $(a_{q(x)}, a_{q(y)})$ and $(a_{q(x)}, \neg a_{q(y)})$, and the agent perceives a state $\neg a_{q(x)}$ for $(\neg a_{q(x)}, a_{q(y)})$ or $(\neg a_{q(x)}, \neg a_{q(y)})$. Then, a Li-state space is $\mathcal{A} = A_{q(x)} \times A_{q(y)} \cup A_{q(x)} \cup A_{q(y)} \cup A_{q(\phi)}$, while standard state spaces are $A_{q(x)} \times A_{q(y)}, A_{q(x)}, A_{q(y)}$, and $A_{q(\phi)}$. The example is shown in Figure 2. ■

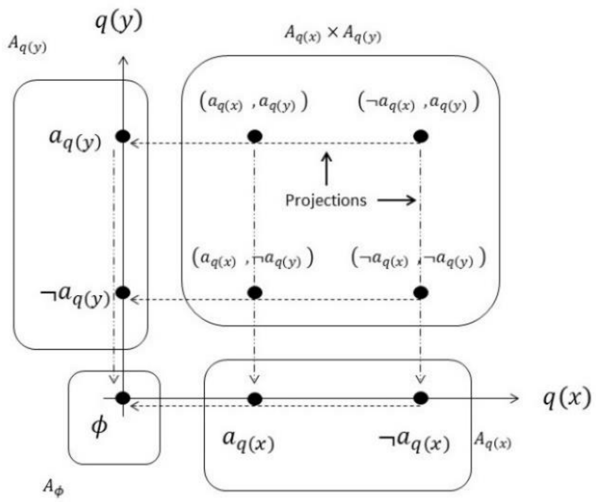

Fig. 2: Li-state space.

3 Constructive State Space

Standard state space models do not assume that they are semi-lattices, even if the spaces have multi-attribute properties, e.g., dice.[4] In contrast, state spaces in unawareness structures are complete lattices. However, because each element of the family of spaces in their models is a standard state space, each state space is not a semi-lattice. This section shows that a standard state space is a semi-lattice (or a complete lattice). Because our formulating approach is similar to those of Heifetz et al. (2008) and Li (2009), as a constructive approach, we call the space a constructive state space.

3.1 Overloaded Function

First, we define *functions overloading*. Given two sets X and Y and for any $k = 0, 1, \cdots, n$, X_k is defined as follows:

$$X_k = \begin{cases} \emptyset & \text{if } k = 0; \\ \times_k X & \text{otherwise.} \end{cases}$$

[4] We can represent multi-attribute for dices. For example, for "1," we represent "1, but not 2, not 3, not 4, not 5, and not 6." Then, the state space is not a complete lattice, but a semi-lattice.

Definition of overloaded functions: A function f is overloaded by $n+1$-tuple arities $(0, 1, \cdots, n)$ if f is satisfied as follows:

$$f\colon \bigcup_{k=0}^{n} X_k \to Y.$$

This paper assumes that overloading is applicable to operators.

3.2 Overloaded Operator and Constructive State Space

The state spaces are modeled using a constructive approach, as discussed in this section. Let P be the set of basic propositions or conceptions. Given an *overloaded operator with 3-tuple arities* $(0, 1, 2)$, $\vee$, and the following conditions are satisfied by the operator:

C1 For any $p \in P$, $p \vee\ = \ \vee p = p \vee p = p$.

C2 $\vee\ = \phi$.

C3 For any $p, p' \in P$, $p \vee p' = p' \vee p$.

C4 For any $p, p', p'' \in P$, $p \vee (p' \vee p'') = (p \vee p') \vee p''$.

C1 means that p can be led by itself when the arity of $\vee$ is not only 2 but also 1. C2 is a technical condition. When the arity is 0, $\vee$ leads ϕ. The ϕ is interpreted as "every proposition is not true." C3 means that $\vee$ satisfies a commutative law, and C4 means that $\vee$ satisfies an absorption law.

Here, for any subset of basic propositions $X \subseteq P$, where X may be an empty set, $\bigvee_{p \in X} p$ is a state. Let $\Omega = \left\{\bigvee_{p \in X} p \,\middle|\, X \subseteq P\right\}$ be the objective state space. Further, for any $X \subseteq P$, let $\Omega_X = \left\{\bigvee_{p \in Y} p \,\middle|\, Y \subseteq X\right\}$ be a subjective state space. For any $X, Y \in 2^P \setminus \{\emptyset\}$ such that $Y \subseteq X \subsetneq P$, $\bigvee_{p \in Y} p \in \Omega$ and $\bigvee_{p \in Y} p \in \Omega_X$ hold evidently. However, attributes between $\bigvee_{p \in Y} p$ in Ω and $\bigvee_{p \in Y} p$ in Ω_X are different. In Ω, $\bigvee_{p \in Y} p$ includes that any attribute $p' \in P \setminus Y$ does not hold. In contrast, in Ω_X, $\bigvee_{p \in Y} p$ includes that any $p' \in P \setminus Y$ does not hold, but it does not include a means that any $p'' \in P \setminus X$ holds or not. $\bigvee_{p \in Y} p$ in Ω_X is related with any element in the Ω_X, but it is not related with every element in $\Omega \setminus \Omega_X$. Then, every $\bigvee_{p \in Y} p$ in Ω_X does not have any attribute $p'' \in P \setminus X$.

Let us define projections. For any basic proposition sets $X, Y \subseteq P$, there is a projection $r_Y^X \colon \Omega_X \longrightarrow \Omega_Y$. This may not be surjective. Hence, for any $\bigvee_{p \in Z:\, Z \subseteq X} p \in \Omega_X$, $r_Y^X\left(\bigvee_{p \in Z:\, Z \subseteq X} p\right) = \bigvee_{p \in Z \cap Y:\, Z \subseteq X} p \in \Omega_Y$. Then $r_Y^X \circ r_X^P = r_Y^P$. Below, let $\bigvee_{p \in Z:\, Z \subseteq P} p = \omega$ and for any $\omega \in \Omega$ and $X \subseteq P$, let $r_X^P(\omega) = \omega_X$. For any $X \subseteq P$, r_X^X is the identity, that is, for any $\omega \in \Omega_X$, $r_X^X(\omega) = \omega$.

The objective state space Ω is a complete lattice. Although it is a standard state space with a complete lattice, let us call Ω the *constructive state space*.

Remark 1 For any subsets X, Y such that $Y \subseteq X \subseteq P$, $\Omega_Y \subseteq \Omega_X$.

Remark 1 means that our (subjective) state spaces are subsets on the objective state space, unlike in Heifetz et al. (2006) and Li (2009). The feature differs from unawareness structures, and the feature is the same to (non-partitional) standard state space models. Moreover, different state spaces have an intersection, and all intersections must have ϕ.

Our formulation is similar to that of Heifetz et al. (2008). However, our formulations are set-theoretic approaches, whereas theirs are logic approaches. Further, there is a crucial difference between our discussion and theirs about unawareness. Heifetz et al. assumed that the awareness/unawareness operator satisfies Symmetry. In contrast, we show that the operator does not satisfy Symmetry for Non-triviality. Because our results are different from theirs, although both the state spaces have the same formulations, we would like to assert that our framework is different from theirs.

Example 3 Let $P = \{x, y\}$ be the set of basic propositions, and let $\Omega = \{x \vee y, x, y, \phi\}$, $\Omega_{\{x\}} = \{x, \phi\}$, $\Omega_{\{y\}} = \{y, \phi\}$, and $\Omega_{\{\phi\}} = \{\phi\}$ be state spaces. Each state space is a subset of Ω. Because the projection must not be surjective, given two sets $\{x\}$ and $\{y\}$, $r_{\{y\}}^{\{x\}}(x) = r_{\{y\}}^{\{x\}}(\phi) = \phi$. Evidently, the intersection has ϕ. Then, Ω is a constructive state space. The example is depicted in Figure 3.

Here, let us focus on ϕ. If $\phi \in \Omega$, a state ϕ indicates that it does not represent $x \vee y$, x or y. In contrast, if $\phi \in \Omega_X$, ϕ means only that it does not represent x, but it does not mean that it represents or not $x \vee y$ and y. ϕ does not imply a conception

y. That is, if any two state spaces are different, then the same state does not have same attribute between them. ■

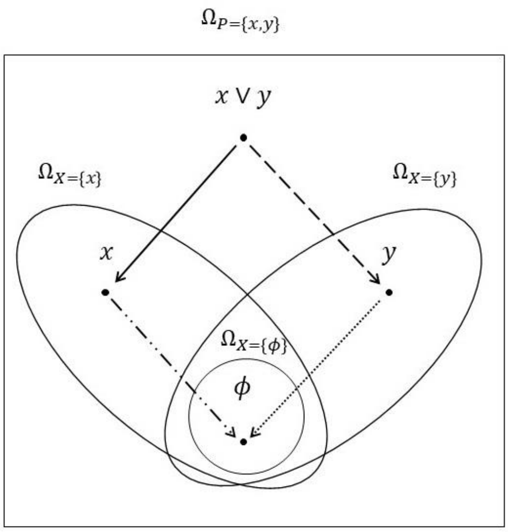

Fig. 3: Constructive state space.

3.3 Relationships with Other State Spaces

Our constructive state spaces are related to HMS-state spaces and Li-state spaces by the following lemmas.

Lemma 1 The following are equivalent:

1. A constructive state space can be constructed.
2. An HMS-state space can be constructed.

Proof. (1⟹2) Any constructive state space Ω has the set of basic propositions P and for any subset $X \subseteq P$, there is $\Omega_X = \left\{\bigvee_{p \in Y} p \,\middle|\, Y \subseteq X\right\}$. Here, let us define the family of disjoint sets $\mathcal{S}$ and bijective mapping $f : \{\Omega_X | X \subseteq P\} \longrightarrow \mathcal{S}$. Then, for any $X, Y \subseteq P$, if $X \neq Y$, then $f(\Omega_X) \cap f(\Omega_Y) = \emptyset$. Let $\preccurlyeq$ be a partial order on $\mathcal{S}$ and be defined as follows: if $X \subseteq Y$, then $f(\Omega_X) \preccurlyeq f(\Omega_Y)$. Then, suppose that there exists a surjective projection $r_{\Omega_X}^{\Omega_Y} : \Omega_Y \longrightarrow \Omega_X$. Then, $\mathcal{S} = \{f(\Omega_X) | X \subseteq P\}$ is a complete lattice, and $\Sigma = \bigcup_{X \subseteq P} f(\Omega_X)$ is an HMS-state space.

(2⟹1) Any HMS-state space Σ has a complete lattice with disjoint spaces $\mathcal{S} = \{S_\lambda\}_{\lambda \in \Lambda}$. Here, let us define $\mathcal{S}^{min} = \left\{S \in \mathcal{S} \middle| S \textit{ is minimal element on } \mathcal{S} \setminus \{\emptyset\}\right\}$, let P be some set and be given a bijective mapping $\hat{f} : \mathcal{S}^{min} \longrightarrow P$. Then, an overloaded operator with 3-tuple arities $(0, 1, 2)$, $\vee$, satisfies the following.

- For any $S \in \mathcal{S}^{min}$, $\hat{f}(S) \vee = \vee \hat{f}(S) = \hat{f}(S) \vee \hat{f}(S) = \hat{f}(S)$.
- $\vee = \emptyset$.
- For any $S, S' \in \mathcal{S}^{min}$, $\hat{f}(S) \vee \hat{f}(S') = \hat{f}(S') \vee \hat{f}(S)$.
- For any $S, S', S'' \in \mathcal{S}^{min}$, $\hat{f}(S) \vee \left(\hat{f}(S') \vee \hat{f}(S'')\right) = \left(\hat{f}(S) \vee \hat{f}(S')\right) \vee \hat{f}(S'')$.

Then, for any $\mathcal{X} \subseteq \mathcal{S}^{min}$, $\Omega_{\mathcal{X}} = \left\{\bigvee_{S \in \mathcal{X}} \hat{f}(S) \,\middle|\, \mathcal{X} \subseteq \mathcal{S}^{min}\right\}$. Let us define that for any $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{S}^{min}$, if we define a projection $r_{\mathcal{Y}}^{\mathcal{X}} : \Omega_{\mathcal{X}} \longrightarrow \Omega_{\mathcal{Y}}$, then for any $\bigvee_{S \in \mathcal{Z} : \mathcal{Z} \subseteq \mathcal{X}} S$, $r_{\mathcal{Y}}^{\mathcal{X}}(\bigvee_{S \in \mathcal{Z} : \mathcal{Z} \subseteq \mathcal{X}} S) = \bigvee_{S \in \mathcal{Z} \cap \mathcal{Y} : \mathcal{Z} \subseteq \mathcal{X}} S$. Then, $\left\{\hat{f}(S) \middle| S \in \mathcal{S}^{min}\right\}$ is the set of basic propositions, and $\Omega = \left\{\bigvee_{S \in \mathcal{X}} \hat{f}(S) \,\middle|\, \mathcal{X} \subseteq \mathcal{S}^{min}\right\}$ is a constructive state space. ■

Lemma 2 The following are equivalent:

1. A constructive state space can be constructed.

2. A Li-state space can be constructed.

Proof. (1⟹2) Any constructive state space Ω has the set of basic propositions P, and for any $X \subseteq P$, there is $\Omega_X = \{\bigvee_{p\in Y} p \,|Y \subseteq X\}$. Here, given some set Q^* and a bijection mapping $g: P \longrightarrow Q^*$. Then, for any $p, p' \in P$, $g(p) \neq g(p')$. Moreover, for any $p \in P$, let us define $A_{g(p)} = \{a_{g(p)}, \neg a_{g(p)}\}$ and for any $X \subseteq P$, given $\prod_{p\in X} A_{g(p)}$. Note that, if $X = \emptyset$, we denote it by $\prod_{p\in X} A_{g(p)} = \{\phi\}$. Here, for any $X, Y \subseteq P$, such that $X \neq Y$, $\prod_{p\in X} A_{g(p)} \neq \prod_{p\in Y} A_{g(p)}$ is evident. When $Y \subseteq X$, suppose that there is a surjective projection $r_{\mathrm{Y}}^{\mathrm{X}}: \prod_{p\in X} A_{g(p)} \longrightarrow \prod_{p\in Y} A_{g(p)}$. Then, $\{g(p)|p \in P\}$ is the set of questions, and $\mathcal{A} = \bigcup_{X\subseteq P} \prod_{p\in X} A_{g(p)}$ is a Li-state space.

(2⟹1) Any Li-state space $\mathcal{A}$ has the set of questions Q^*. Here, given some set P a bijection mapping $\hat{g}: Q^* \longrightarrow P$; then, for any $q, q' \in Q^*$, if $q \neq q'$, $\hat{g}(q) \neq \hat{g}(q')$. Let us define an overloaded operator $\vee$ as follows.

- For any $q \in Q^*$, $\hat{g}(q) \vee = \vee\, \hat{g}(q) = \hat{g}(q) \vee \hat{g}(q) = \hat{g}(q)$.
- $\vee = \phi$.
- For any $q, q' \in Q^*$, $\hat{g}(q) \vee \hat{g}(q') = \hat{g}(q') \vee \hat{g}(q)$.
- For any $q, q', q'' \in Q^*$, $\hat{g}(q) \vee (\hat{g}(q') \vee \hat{g}(q'')) = (\hat{g}(q) \vee \hat{g}(q')) \vee \hat{g}(q'')$.

Then, for any $Q \subseteq Q^*$, let $\Omega_Q = \{\bigvee_{q\in Q} \hat{g}(q) \,|Q \subseteq Q^*\}$. For any $Q, Q' \subseteq Q^*$, given a projection $r_{Q'}^{Q}: \Omega_Q \longrightarrow \Omega_{Q'}$ and for any $\bigvee_{q\in Q'':\, Q''\subseteq Q} \hat{g}(q) \in \Omega_Q$, let us define $r_{Q'}^{Q}(\bigvee_{q\in Q'':\, Q''\subseteq Q} \hat{g}(q)) = \bigvee_{q\in Q''\cap Q':\, Q''\subseteq Q} \hat{g}(q)$. Then, $\{\hat{g}(q)|q \in Q^*\}$ is the set of basic propositions, and $\Omega = \{\bigvee_{q\in Q} \hat{g}(q) \,|Q \subseteq Q^*\}$ is a constructive state space. ■

The lemmas indicate that any constructive state space can construct an HMS-state space and Li-state space, and vice versa.

Proposition 1 The following are equivalent.

1. A constructive state space can be constructed.
2. HMS-state space can be constructed.
3. Li-state space can be constructed.

Let us consider constructive state spaces related to HMS-state spaces. We compare Example 2 with 3. Their relations are the following:

$$S_{\{x,y\}} \Leftrightarrow \Omega$$

$$S_{\{x\}} \Leftrightarrow \Omega_{\{x\}}$$

$$S_{\{y\}} \Leftrightarrow \Omega_{\{y\}}$$

$$S_{\{\phi\}} \Leftrightarrow \Omega_{\{\phi\}}$$

When $S_{\{x,y\}}$ is compared with Ω, the states between the spaces are represented as follows:

$$xy \Leftrightarrow x \vee y$$

$$x\neg y \Leftrightarrow x$$

$$\neg xy \Leftrightarrow y$$

$$\neg x\neg y \Leftrightarrow \phi$$

In contrast, when $S_{\{x\}}$ is compared with $\Omega_{\{x\}}$, the states between the spaces are represented as follows:

$$x \Leftrightarrow x$$

$$\neg x \Leftrightarrow \phi$$

By these comparisons, x in Ω and x in $\Omega_{\{x\}}$ have different implications and ϕ in Ω and ϕ in $\Omega_{\{x\}}$ are different as well. $\Omega_{\{x\}}$ is the lack of y.

Moreover, the following are relationships between Ω and Σ:

$$xy \Leftrightarrow x \vee y$$

$$x \Leftrightarrow x$$

$$y \Leftrightarrow y$$

$$\phi \Leftrightarrow \phi$$

This means that each element of Ω is related with each element without negations of Σ. That is, not only is Ω related with $S_{\{x,y\}} \subseteq \Sigma$, but also Ω is related with $\{xy, x, y, \phi\} \subseteq \Sigma$. Hence, Ω has a dual structure for Σ.

4 Constructive Aumann Structure

The *Constructive Aumann structures* are modeled based on constructive state spaces, as discussed in this section. We focus on only a single agent, formulate possibility correspondences on constructive state spaces, and knowledge operators and awareness/unawareness operators on constructive Aumann structures, and we discuss their properties. Finally, we provide a generalization of the main theorems proposed by

Dekel et al. (1998) and Chen et al. (2012).

4.1 Possibility Correspondence

Possibility correspondences in standard Aumann structures are only defined in state spaces. In contrast, because our state spaces are semi-lattices, a domain of possibility correspondences is not only the state space, but also the power set of basic propositions. Let $\langle P, \Omega, \Pi \rangle$ be the constructive Aumann structure, where Ω is constructed by P. Then, $\Pi: \Omega \times 2^{\mathrm{P}} \longrightarrow 2^{\Omega} \setminus \{\emptyset\}$ is the possibility correspondence. Suppose that an agent can perceive every basic proposition in the subset of basic propositions $X \subseteq P$, but not in $Y \subseteq P \setminus X$. Then, for any $\omega \in \Omega$, $\Pi(\omega, X) \subseteq \Omega_X$. Let us assume that the possibility correspondence satisfies the following properties.

1. Subjective Nondelusion: For any $\omega \in \Omega$, and any $X \subseteq P$, $\omega_X \in \Pi(\omega, X)$.
2. Stationarity: For any $\omega, \omega' \in \Omega$ and any $X \subseteq P$, if $\omega' \in \Pi(\omega, X)$, then $\Pi(\omega', X) = \Pi(\omega, X)$.

Example 3 (Continued.) Let $\omega_1 = x \vee y$, $\omega_2 = x$, $\omega_3 = y$ and $\omega_4 = \phi$. Suppose that an agent can perceive the basic proposition set $X = \{x\}$. By Subjective Nondelusion, $\omega_2 \in \Pi(\omega_1, X)$, $\omega_2 \in \Pi(\omega_2, X)$, $\omega_4 \in \Pi(\omega_3, X)$, and $\omega_4 \in \Pi(\omega_4, X)$. By Stationarity, $\Pi(\omega_1, X) = \Pi(\omega_2, X)$ and $\Pi(\omega_3, X) = \Pi(\omega_4, X)$. Note that whether $\Pi(\omega_1, X) = \Pi(\omega_3, X)$ or not may depend on how Π is formulated. ■

Subjective Nondelusion and Stationarity are the analogues of the partitional information function in a standard Aumann structure. When X is a proper subset of P, Π is evidently not partitional on Ω. However, it may be partitional on Ω_X.

Definition 1 (Partial Partition) Given any $X \subseteq P$. $\Pi: \Omega \times 2^{\mathrm{P}} \longrightarrow 2^{\Omega} \setminus \{\emptyset\}$ is *partially partitional* on Ω_X if there exists $\mathcal{P} = \{P_\lambda\}_{\lambda \in \Lambda}$ such that

1. $\bigcup_{\lambda \in \Lambda} P_\lambda = \Omega_X$;
2. For any $\omega \in \Omega$, there exists P_λ such that $\omega_X \in P_\lambda$ and $\Pi(\omega, X) = P_\lambda$; and
3. For any $P_\lambda, P_{\lambda'} \in \mathcal{P}$, if $P_\lambda \neq P_{\lambda'}$, then $P_\lambda \cap P_{\lambda'} = \emptyset$.

A partial partition is the analog of the partition in a standard Aumann structure. We can induce the following proposition.

Proposition 2 Given any $X \subseteq P$. Π is partially partitional on Ω_X if and only if Π satisfies Subjective Nondelusion and Stationarity.

Proof. ($\Longrightarrow$) Suppose that the possibility correspondence Π is partially partitional on Ω_X. Then, by Condition 1 in Definition 1, $\bigcup_{\lambda \in \Lambda} P_\lambda = \Omega_X$ and by Condition 2 in Definition 1, for any $\omega \in \Omega$, because there exists P_λ with $\omega_X \in P_\lambda$ such that $\Pi(\omega, X) = P_\lambda$, $\omega_X \in \Pi(\omega, X)$. That is, Π satisfies Subjective Nondelusion. Moreover, by Condition 3 in Definition 1, for any $P_\lambda, P_{\lambda'} \in \mathcal{P}$, if $P_\lambda \neq P_{\lambda'}$, then $P_\lambda \cap P_{\lambda'} = \emptyset$. This satisfies that for any $\omega, \omega' \in \Omega$ if $\Pi(\omega, X) \neq \Pi(\omega', X)$, then

$\Pi(\omega, X) \cap \Pi(\omega', X) = \emptyset$, that is, $\omega' \notin \Pi(\omega, X)$. The contraposition is that if $\omega' \in \Pi(\omega, X)$, then $\Pi(\omega', X) = \Pi(\omega, X)$. Hence, Π satisfies Stationarity.

($\Longleftarrow$) Suppose that Π satisfies Subjective Nondelusion and Stationarity. Given P_λ with $\Pi(\omega, X) = P_\lambda$ for some $\omega \in \Omega$. By Subjective Nondelusion and the assumption of projection, for any $\omega \in \Omega_X$, $\omega \in \Pi(\omega, X)$. Therefore, $\bigcup_{\omega \in \Omega_X} \Pi(\omega, X) = \bigcup_{\lambda:\, P_\lambda = \Pi(\omega, X)} P_\lambda = \Omega_X$ is evident, that is, Condition 1 in Definition 1 holds. By Subjective Nondelusion and the definition of P_λ, Condition 2 in Definition 1 holds. By Stationarity, for any $\omega, \omega' \in \Omega$, if $\Pi(\omega, X) \neq \Pi(\omega', X)$, then $\omega' \notin \Pi(\omega, X)$. That is, $\Pi(\omega, X) \cap \Pi(\omega', X) = \emptyset$. By the definition of P_λ, if $P_\lambda \neq P'_\lambda$, then $P_\lambda \cap P'_\lambda = \emptyset$. That is, Condition 3 in Definition 1 holds. Therefore, Π is partially partitional on Ω_X. ■

Heifetz et al. (2006) proposed five assumptions. Three of them, Confinedness, Generalized Reflexivity, Projections Preserve Awareness (PPA), can be induced from the Subjective Nondelusion and Stationarity in our model, while Projections Preserve Ignorance (PPI), and Projections Preserve Knowledge (PPK) cannot. Therefore, we assume or do not do their properties. However, when we relax the PPK, that let us call *Partially Projections Preserve Knowledge* (Partially PPK), the relaxing property can be induced from Subjective Nondelusion and Stationarity. Given $E \subseteq \Omega$, for any $X \subseteq P$, let $E_X = \{\omega_X \in \Omega | \omega \in E\}$ and let $E^X = \left\{\omega' \in \Omega \middle| \forall \omega \in E \quad \omega' = \omega \vee_{p \in Z: Z \subseteq X} p\right\}$.

Then, the abovementioned properties are formulated and shown as follows.[5]

Remark 2 If a possibility correspondence Π satisfies Subjective Nondelusion and Stationarity, then it satisfies the following.

1. Confinedness: For any $\omega \in \Omega_X$ and any $X \subseteq P$, $\Pi(\omega, X) \subseteq \Omega_X$.
2. Generalized Reflexivity: For any $\omega \in \Omega$ and $X \subseteq P$, $\omega \in \left(\Pi(\omega, X)\right)^P$.
3. PPA: For any $\omega \in \Omega$ and $X \subseteq P$, if $\omega \in \Pi(\omega, X)$, then $\omega_X \in \Pi(\omega_X, X)$.
4. Partially PPK: For any $\omega \in \Omega$ and $X, Y \subseteq P$, if $\Pi(\omega, X) \subseteq \Omega_Y$, then $\left(\Pi(\omega, X)\right)_Y = \Pi(\omega_Y, X)$.

Proof. (Property 1) By Subjective Nondelusion, $\omega_X \in \Pi(\omega, X)$. By Stationarity, if $\omega' \in \Pi(\omega, X)$, then $\Pi(\omega', X) = \Pi(\omega, X)$. That is, $\omega'_X = \omega'$. Therefore, for any $\omega' \in \Pi(\omega, X)$, $\omega' \in \Omega_X$. Hence, $\Pi(\omega, X) \subseteq \Omega_X$.

(Property 2) Given any $\omega \in \Omega$ and $X \subseteq P$. Then, $\left(\Pi(\omega, X)\right)^P = \{\omega'' \in \Omega \mid \forall \omega' \in \Pi(\omega, X) \quad \omega'' = \omega' \vee_{p \in Z: Z \subseteq X} p\}$. By Subjective Nondelusion, $\omega_X \in \Pi(\omega, X)$ and there exists $Z \subseteq X$ with $\omega = \omega_X \vee_{p \in Z} p$. Hence, $\omega \in \left(\Pi(\omega, X)\right)^P$.

(Property 3) It is evident by Subjective Nondelusion.

(Property 4) Given $\omega \in \Omega$, $X, Y \subseteq P$ and $\Pi(\omega, X) \subseteq \Omega_Y$. For any $\omega' \in$

[5] In the first version of this paper, the author called the statement of Property 4 of Remark 2 not Partially PPK but PPK, which is wrong. Although in this paper we distinguish between Partially PPK and PPK, Partially PPK is PPK under some condition, even if there exists a difference. See Proposition 8.

$\Pi(\omega\,, X)$, because $\omega' \in \Omega_Y$, $r_Y^X(\omega') = \omega'$. That is, $\left(\Pi(\omega\,, X)\right)_Y = \Pi(\omega\,, X)$. Hence, by Subjective Nondelusion and Stationarity, $\Pi(\omega_Y\,, X) = \Pi(\omega_X\,, X) = \Pi(\omega\,, X)$. ■

In our model, PPK and PPI are formulated as follows:

- (PPK) For any $\omega \in \Omega$ and $X, Y \subseteq P$ $(Y \subseteq X)$, $\left(\Pi(\omega\,, X)\right)_Y = \Pi(\omega_Y\,, X)$;
- (PPI) For any $\omega \in \Omega$ and $X, Y \subseteq P$, $\left(\Pi(\omega\,, X)\right)^P \subseteq \left(\Pi(\omega_Y\,, X)\right)^P$.

They cannot be induced from Subjective Nondelusion and Stationarity. The following are countering examples.

Example 4 Given $P = \{x, y\}$. Then, $\Omega = \{x \vee y, x, y, \phi\}$. Suppose that an agent can perceive all the basic propositions, that is, $X = P$, that $Y = \{x, \phi\}$, and that $\Pi(x \vee y\,, X) = \Pi(y\,, X) = \{x \vee y, y\}$, $\Pi(x\,, X) = \{x\}$, and $\Pi(y\,, X) = \{y\}$. Then, because $\left(\Pi(\omega\,, X)_{\omega = x \vee y, y}\right)_Y = Y$, $\left(\Pi(\omega\,, X)_{\omega = x \vee y, y}\right)_Y \neq \Pi(x\,, X)$ and $\left(\Pi(\omega\,, X)_{\omega = x \vee y, y}\right)_Y \neq \Pi(\phi\,, X)$. That is, this case does not satisfy PPK. ■

Example 5 Given $P = \{x, y, z\}$. Then, $\Omega = \{x \vee y \vee z, x \vee y, y \vee z, z \vee x, x, y, z, \phi\}$. Suppose that an agent can perceive all the basic propositions, that is, $X = P$, and that $\Pi(x \vee y \vee z\,, P) = \Pi(x \vee y\,, P) = \{x \vee y \vee z, x \vee y\}$, $\Pi(y \vee z\,, P) = \Pi(z \vee x\,, P) = \Pi(x\,, P) = \{y \vee z, z \vee x, x\}$, $\Pi(y\,, P) = \Pi(z\,, P) = \{y, z\}$, and $\Pi(\phi\,, P) = \{\phi\}$. The partitions are shown in Figure 4. Then, the possibility correspondence satisfies Subjective Nondelusion and Stationarity. Let $Y = \{y\}$ and $\omega = x \vee y$. Then,

$\left(\Pi(\omega\,,P)\right)^P = \{x \vee y \vee z, x \vee y, y \vee z, z \vee x, x\}$, whereas $\left(\Pi(\omega_Y\,,P)\right)^P = \left(\Pi(\omega_Y\,,P)\right)^P = \{x \vee y \vee z, x \vee y, y \vee z, z \vee x\}$. Then, evidently, $\left(\Pi(\omega\,,X)\right)^P \nsubseteq \left(\Pi(\omega_Y\,,X)\right)^P$ because there exists some element $x \in \left(\Pi(\omega\,,P)\right)^P \setminus \left(\Pi(\omega_Y\,,P)\right)^P$. This case does not satisfy PPI. ■

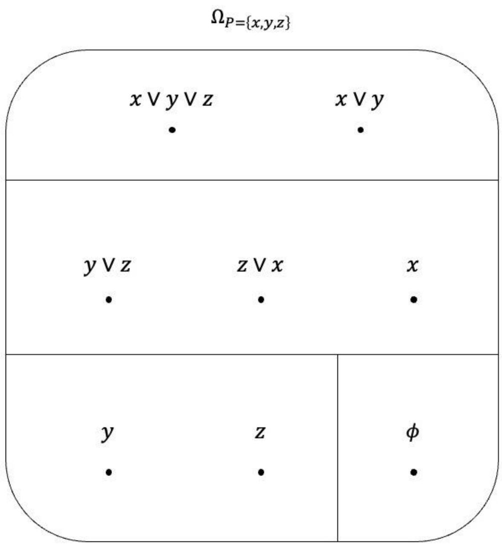

Fig. 4: Projections Preserve Ignorance is not satisfied.

The above properties have the following relations.

Remark 3 The possibility correspondence Π satisfies the following properties.

A) Generalized Reflexivity implies Subjective Nondelusion.

B) PPK implies PPA.

C) Partially PPK implies PPA

Proof. (A) Suppose that Π satisfies Generalized Reflexivity. Given any $\omega \in \Omega$ and

$X \subseteq P$ with $\omega \in \left(\Pi(\omega, X)\right)^P = \{\omega'' \in \Omega \,|\forall \omega' \in \Pi(\omega, X) \quad \omega'' = \omega' \vee_{p \in Z : Z \subseteq X} p\}$. Then, there must exist $\omega' \in \Pi(\omega, X)$ such that $\omega = \omega' \vee_{p \in Z : Z \subseteq X} p$. That is, $r_X^Y(\omega) = r_X^Y\left(\omega' \vee_{p \in Z : Z \subseteq P} p\right) = \omega'$. Hence, $\omega_X \in \Pi(\omega, X)$.

(B) Suppose that Π satisfies Confinedness and PPK and that $\omega \in \Pi(\omega, X)$. Then, $\omega_X \in \left(\Pi(\omega, X)\right)_X$. By PPK, because $\left(\Pi(\omega, X)\right)_X = \Pi(\omega_X, X)$, $\omega_X \in \Pi(\omega_X, X)$.

(C) Suppose that Π satisfies Partially PPK and that for any $\omega \in \Omega$ and $X \subseteq P$, $\omega \in \Pi(\omega, X)$. Then, because $\Pi(\omega, X) \subseteq \Omega_X$, $\omega = \omega_X \in \left(\Pi(\omega, X)\right)_X$. By Partially PPK, because $\left(\Pi(\omega, X)\right)_X = \Pi(\omega_X, X)$, $\omega_X \in \Pi(\omega_X, X)$. ■

Although Properties (A) and (B) of Remark 3 are shown by Heifetz et al. (2006), there is a difference between their statement and our statement about (B). In contrast with our statement, Heifetz et al. state that Confinedness and PPK imply PPA. In unawareness structures, Confinedness is necessary, while it is not necessary in constructive Aumann structures.

Some previous studies have referred to PPK and PPI for interactive situations, e.g., Heifetz et al. (2006, 2008) and Galanis (2013, 2018). In contrast, because our model is a single-agent model, we did not have to make the assumption of the aforementioned property.[6]

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6 Constructive Aumann structures are non-partitional standard possibility correspondence models with multiple attributes. In non-partitional standard models, it does not seem to assume PPK and PPI. This paper conjectures that PPK and PPI are not necessary in constructive Aumann structures with multiple agents too.

4.2 Knowledge Operator

Let us define a knowledge operator. Let an event E be the subset of Ω. When an agent can perceive the subset of basic propositions $X \subseteq P$, the knowledge operator $K_X: 2^\Omega \longrightarrow 2^\Omega$ is defined as follows: $K_X(E) = \{\omega \in \Omega | \Pi(\omega, X) \subseteq E\}$ if $E \subseteq \Omega_X$; and $K_X(E) = \emptyset$ otherwise. $K_X(E)$ is interpreted as "An agent who can perceive X knows the event E." If $K_X(E) = \emptyset$, it is false that the agent knows E.

Example 3 (Continued.) Suppose that $X = \{x\}$ and that $\Pi(\omega_1, X) = \{\omega_2\}$, $\Pi(\omega_2, X) = \{\omega_2\}$, $\Pi(\omega_3, X) = \{\omega_4\}$, and $\Pi(\omega_4, X) = \{\omega_4\}$. Let $E_1 = \{\omega_2\}$. Then, $E_1 \subseteq \Omega_X$ and $\Pi(\omega_2, X) \subseteq E_1$. Therefore, $K_X(E_1) = \{\omega_2\}$, hence, the agent knows E_1. Let $E_2 = \{\omega_1, \omega_2\}$. Then, because $E_2 \nsubseteq \Omega_X$, $K_X(E_2) = \emptyset$. This means that it is false that the agent knows E_2. ■

$E \subseteq \Omega_X$ in the definition of the knowledge operator is important. When $X \neq P$, $\Pi(\omega, X) \subseteq \Omega$ is evident. Therefore, if $K_X(E) = \{\omega \in \Omega | \Pi(\omega, X) \subseteq E\}$ for every E, then $K_X(\Omega)$ is not empty, and it allows that the agent knows Ω.[7] Notably, $K_X(E)$ may be empty for some $E \subseteq \Omega_X$.

Remark 4 Given $E \subseteq \Omega_X$, the following are equivalent.

[7] In this case, Monotonicity is satisfied even if $X \neq P$.

1. For any $\omega \in \Omega$, $\Pi(\omega, X) \nsubseteq E$.
2. $K_X(E) = \emptyset$.

Example 3 (Continued.) Let $E_3 = \{\omega_4\}$. Then, because $\Pi(\omega_2, X) \nsubseteq E_3$, $K_X(E_3) = \emptyset$. That is, at ω_2, it is false that the agent knows E_3. ■

It is evident that $K_X(E)$ is an event on Ω_X.

Proposition 2 (Heifetz et al. 2006) For any $E \subseteq \Omega$, $K_X(E) \subseteq \Omega_X$.

Proof. Given any $E \subseteq \Omega$ and $\omega \in K_X(E)$. By the definition of knowledge operator and Subjective Nondelusion, $\omega \in \Pi(\omega, X) \subseteq E$. Then, by Confinedness, as $\Pi(\omega, X) \subseteq \Omega_X$, $\omega \in \Omega_X$. Therefore, $K_X(E) \subseteq \Omega_X$. ■

Let $\neg K_X(E) = \Omega \setminus K_X(E)$ be the negation of $K_X(E)$. It is interpreted as "An agent who can perceive only X does not know the event E." Here, we can show the generalization of properties of knowledge operators in Heifets et al. (2006).

Proposition 3 A knowledge operator K_X has the following properties.

K1 (Necessitation) $X = P$ if and only if $K_X(\Omega) = \Omega$.

K2 (Monotonicity) $X = P$ if and only if $E \subseteq F \Longrightarrow K_X(E) \subseteq K_X(F)$.

K3 (Conjunction) $\forall\lambda \in \Lambda \quad E_\lambda \subseteq \Omega_X$ or $\forall\lambda \in \Lambda \quad E_\lambda \nsubseteq \Omega_X \implies K_X(\bigcap_{\lambda\in\Lambda} E_\lambda) = \bigcap_{\lambda\in\Lambda} K_X(E_\lambda)$.

K4 (Truth) $K_X(E) \subseteq E$.

K5 (Positive Introspection) $K_X(E) = K_X K_X(E)$.

K6 (Negative Introspection) $X = P$ if and only if $\neg K_X(E) \subseteq K_X \neg K_X(E)$.

Proof. (K1) ($\Longrightarrow$) When $X = P$, by Subjective Nondelusion, for any $\omega \in \Omega$, $\omega \in \Pi(\omega, P) \subseteq \Omega$. That is, $\Omega \subseteq K_P(\Omega)$. Moreover, by Proposition 2, because $K_P(E) \subseteq \Omega$, $K_P(\Omega) = \Omega$.

($\Longleftarrow$) Suppose that $K_X(\Omega) = \Omega$. Assume that $X \neq P$. Then, $\Omega_X \subsetneq \Omega$. However, by the definition of the knowledge operator, $K_X(\Omega) = \emptyset$. This is a contradiction. Therefore, $X = P$.

(K2) ($\Longrightarrow$) When $X = P$, $K_P(E) = \{\omega \in \Omega | \Pi(\omega, P) \subseteq E\} \subseteq \{\omega \in \Omega | \Pi(\omega, P) \subseteq F\} = K_P(F)$.

($\Longleftarrow$) Suppose that $E \subseteq F \Longrightarrow K_X(E) \subseteq K_X(F)$. Assume that $X \neq P$. Then, $\Omega_X \subsetneq \Omega$ and $K_X(\Omega) = \emptyset$. For any $\emptyset \neq E \subseteq \Omega_X$, because $K_X(E) \supsetneq K_X(\Omega)$, this is a contradiction. Therefore, $X = P$.

(K3) Given any $\lambda \in \Lambda$, suppose that $E_\lambda \subseteq \Omega_X$. Given any $\omega \in K_X(\bigcap_{\lambda\in\Lambda} E_\lambda)$. Then, $\Pi(\omega, P) \subseteq \bigcap_{\lambda\in\Lambda} E_\lambda$. This means that for any $\lambda \in \Lambda$, $\Pi(\omega, P) \subseteq E_\lambda$. That is, for any $\lambda \in \Lambda$, because $\omega \in K_X(E_\lambda)$, $\omega \in \bigcap_{\lambda\in\Lambda} K_X(E_\lambda)$. For any $\lambda \in \Lambda$, suppose that $E_\lambda \nsubseteq \Omega_X$. Then, $K_X(E_\lambda) = \emptyset$. That is, $K_X(\bigcap_{\lambda\in\Lambda} E_\lambda) = \bigcap_{\lambda\in\Lambda} K_X(E_\lambda) = \emptyset$.

(K4) Given any $\omega' \in K_X(E)$, $\omega' \in \Pi(\omega, P) \subseteq E$. Therefore, $K_X(E) \subseteq E$.

(K5) By K4, $K_X K_X(E) \subseteq K_X$. Given any $\omega \in K_X(E)$, $\Pi(\omega, P) \subseteq E$. Here, for any $\omega' \in \Pi(\omega, P)$, $\Pi(\omega', P) \subseteq E$. Thus, $\omega' \in K_X(E)$. Hence, because $\Pi(\omega, P) \subseteq K_X(E)$, $\omega' \in K_X K_X(E)$. Thus, $K_X(E) \subseteq K_X K_X(E)$. Hence $K_X(E) = K_X K_X(E)$.

(K6) ($\Longrightarrow$) Assume that $X = P$. Given any $\omega \in \neg K_P(E)$, $\omega \notin K_P(E)$. Thus, $\Pi(\omega, P) \nsubseteq E$. Given $\omega' \in \Pi(\omega, P)$, by Stationarity, because $\Pi(\omega, P) = \Pi(\omega', P)$, $\omega' \in \neg K_P(E)$. That is, $\Pi(\omega, P) \subseteq \neg K_P(E)$. Therefore, $\neg K_X(E) \subseteq K_X \neg K_X(E)$.

($\Longleftarrow$) Suppose that $\neg K_X(E) \subseteq K_X \neg K_X(E)$ and $X \neq P$. Then, $K_X(E) \subseteq \Omega_X \subsetneq \Omega$. Because $\neg K_X(E) \nsubseteq \Omega_X$, this must be $K_X \neg K_X(E) = \emptyset$. This is a contradiction. Therefore, $X = P$. ■

In our model, the knowledge operator satisfies Necessitation, Monotonicity, and Negative Introspection if and only if the agent can perceive all basic propositions in P.

Remark 5 $K_X(\Omega_X) = \Omega_X$.

Although the remark is evident, the agent that can perceive X believes that it faces the Aumann structure with the Ω_X. Thus, if we define some correspondence on only Ω_X, we can define the standard knowledge operator on Ω_X.

Finally, we show the following proposition proposed by Heifetz et al. (2006).

Proposition 4 (HMS 2006) $\neg K_X(E) \cap \neg K_X \neg K_X(E) \subseteq \neg K_X \neg K_X \neg K_X(E)$.

Proof. See each property 2 in Propositions 5–7 below. ■

4.3 Awareness/Unawareness Operator

In this section, we define the unawareness operator. Suppose that an agent can perceive X. Then, the unawareness operator is defined as $U_X(E) = \neg K_X(E) \cap \neg K_X \neg K_X(E)$, and the awareness operator is defined as $A_X(E) = \neg U_X(E) = K_X(E) \cup K_X \neg K_X(E)$.

Example 3 (Continued.) For $E_1 = \{\omega_2\}$, because $\neg K_X(E_1) = \{\omega_1, \omega_3, \omega_4\}$ and $\neg K_X \neg K_X(E_1) = \emptyset$, $U_X(E_1) = \emptyset$. Therefore, $A_X(E_1) = \{\omega_2, \omega_4\}$. In contrast, for $E_2 = \{\omega_1, \omega_2\}$, because $\neg K_X(E_2) = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $\neg K_X \neg K_X(E_2) = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, $U_X(E_2) = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and $A_X(E_1) = \emptyset$. ■

Before we discuss the properties of the knowledge and awareness/unawareness operators, we must consider three cases: the agent can perceive all basic propositions; the agent cannot perceive some non-empty subset of the basic proposition set, and an event is the subset of the state space which it can perceive; or the agent cannot perceive some non-empty subset of the basic proposition set, and an event is not the subset of the state space which it can perceive.

Let us examine the following lemmas, prior to showing their properties.

Lemma 3 (Heifetz et al. 2006) $E, F \subseteq \Omega_X \quad \Longrightarrow \quad K_X(E \cup K_X(F)) = K_X(E) \cup K_X(F)$.

Proof. First, given any $\omega \in K_X(E \cup K_X(F))$. Then, $\Pi(\omega, X) \subseteq E \cup K_X(F)$. This means that $\Pi(\omega, X) \subseteq E$ or $\Pi(\omega, X) \subseteq K_X(F)$. Hence, by K5, because $K_X(F) = K_X K_X(F)$, $K_X(E) \cup K_X K_X(F) = K_X(E) \cup K_X(F)$, and $\omega \in K_X(E) \cup K_X(F)$. That is, $K_X(E \cup K_X(F)) \subseteq K_X(E) \cup K_X(F)$. Next, given any $\omega \in K_X(E) \cup K_X(F)$. By K5, $K_X(E) \cup K_X(F) = K_X(E) \cup K_X K_X(F)$. Then, $\Pi(\omega, X) \subseteq E$ or $\Pi(\omega, X) \subseteq K_X(F)$. This means that $\Pi(\omega, X) \subseteq E \cup K_X(F)$. Therefore, because $\omega \in K_X(E \cup K_X(F))$, $K_X(E) \cup K_X(F) \subseteq K_X(E \cup K_X(F))$. Thus, $K_X(E \cup K_X(F)) = K_X(E) \cup K_X(F)$. ■

Lemma 4 An awareness operator has the following properties.

1. (Triviality) If $X = P$, then $A_X(E) = \Omega$.
2. (Non-triviality) If $X \neq P$ and $E \subseteq \Omega_X$, then $A_X(E) = K_X(E)$.
3. (Non-triviality) If $X \neq P$ and $E \nsubseteq \Omega_X$, then $A_X(E) = \emptyset$.

Proof. (1) Suppose that $X = P$. Then, $A_P(E) = K_P(E) \cup K_P \neg K_P(E)$. By K5, $K_P(E) \cup K_P \neg K_P(E) = K_P K_P(E) \cup K_P \neg K_P(E)$. By Lemma 3, $K_P K_P(E) \cup K_P \neg K_P(E) = K_P(K_P(E) \cup \neg K_P(E)) = K_P(\Omega) = \Omega$. Therefore, $A_P(E) = \Omega$.

(2) Suppose that $X \neq P$ and $E \subseteq \Omega_X$. Then, by Proposition 2, because $K_X(E) \subseteq \Omega_X$, $\neg K_X(E) \nsubseteq \Omega_X$. Therefore, $K_X \neg K_X(E) = K_X(\emptyset) = \emptyset$. Thus, $A_X(E) = K_X(E) \cup$

$K_X \neg K_X(E) = K_X(E)$.

(3) Suppose that $X \neq P$ and $E \nsubseteq \Omega_X$. Then, by the definition of the knowledge operator, $K_X(E) = \emptyset$. Then, $\neg K_X(E) = \Omega$ and $K_X(\Omega) = \emptyset$. Therefore, $A_X(E) = K_X(E) \cup K_X \neg K_X(E) = \emptyset$. ■

The properties of the knowledge and awareness/unawareness operators are as follows.

Proposition 5 When $X = P$, the following properties of knowledge and awareness/unawareness are obtained:

① KU Introspection: $K_X U_X(E) = \emptyset$.

② AU Introspection: $U_X(E) = U_X U_X(E)$.

③ Weak Necessitation: $A_X(E) = K_X(\Omega_X)$.

④ Strong Plausibility: $U_X(E) = \bigcap_{n=1}^{\infty} (\neg K_X)^n (E)$.

⑤ Weak Negative Introspection: $\neg K_X(E) \cap A_X \neg K_X(E) = K_X \neg K_X(E)$.

⑥ Symmetry: $A_X(\neg E) = A_X(E)$.

⑦ A-Conjunction: $A_X(\bigcap_{\lambda \in \Lambda} E_\lambda) = \bigcap_{\lambda \in \Lambda} A_X(E_\lambda)$.

⑧ AK-Self Reflection: $A_X K_X(E) = A_X(E)$.

⑨ AA-Self Reflection: $A_X A_X(E) = A_X(E)$.

⑩ A-Introspection: $K_X A_X(E) = A_X(E)$.

Proof. Assume that $X = P$. By condition 1 in Lemma 4, $A_X(E) = \Omega$, i.e., $U_X(E) = \emptyset$.

1) $K_X U_X(E) = K_X\left(\neg K_X(E) \cap \neg K_X \neg K_X(E)\right) = K_X \neg K_X(E) \cap K_X \neg K_X \neg K_X(E) \subseteq K_X \neg K_X(E) \cap \neg K_X \neg K_X(E) = \emptyset$.

2) By condition 1 in Lemma 4, because $A_X(E) = \Omega$ and $U_X(E) = \neg A_X(E) = \emptyset$, $A_X U_X(E) = A_X(\emptyset) = K_X(\emptyset) \cup K_X \neg K_X(\emptyset) = \emptyset \cup K_X(\Omega) = \Omega$. Therefore, $A_X(E) = A_X U_X(E)$ and $U_X(E) = U_X U_X(E)$.

3) By $X = P$, $K_X(\Omega) = \Omega$. By Lemma 4, because $A_X(E) = \Omega$, $A_X(E) = K_X(\Omega)$.

4) By Lemma 4, $U_X(E) = \neg A_X(E) = \emptyset$. By Lemma 4 and AU Introspection, $U_X(E) = U_X U_X(E) = \emptyset$ and $U_X U_X U_X(E) = U_X(\emptyset)$. Then, as $A_X(\emptyset) = \Omega$, $U_X U_X U_X(E) = U_X(\emptyset) = \emptyset$. By repeating it, $U_X(E) = \bigcap_{n=1}^{\infty} (\neg K_X)^n\,(E)$.

5) $A_X \neg K_X(E) = K_X \neg K_X(E) \cup K_X \neg K_X \neg K_X(E) = K_X K_X \neg K_X(E) \cup K_X \neg K_X \neg K_X(E) = K_X\left(K_X \neg K_X(E) \cup \neg K_X \neg K_X(E)\right) = K_X(\Omega) = \Omega$. Therefore, $\neg K_X(E) \cap A_X \neg K_X(E) = \neg K_X(E) \cap \Omega = \neg K_X(E)$. By $X = P$, the knowledge operator satisfies K6, i.e., $\neg K_X(E) \subseteq K_X \neg K_X(E)$. Moreover, by K4, $K_X \neg K_X(E) \subseteq \neg K_X(E)$. Therefore, as $\neg K_X(E) = K_X \neg K_X(E)$, $\neg K_X(E) \cap A_X \neg K_X(E) = K_X \neg K_X(E)$.

6) Given $E \subseteq \Omega$, because $A_X(E) = \Omega$, $A_X(\neg E) = \Omega$. Hence, $A_X(\neg E) = A_X(E)$.

7) For any $E \subseteq \Omega$ and any $\lambda \in \Lambda$, as $A_X(E) = \Omega$, $A_X(\bigcap_{\lambda \in \Lambda} E_\lambda) = \Omega$ and $A_X(E_\lambda) = \Omega$, $\bigcap_{\lambda \in \Lambda} A_X(E_\lambda) = \Omega$. Therefore, $A_X(\bigcap_{\lambda \in \Lambda} E_\lambda) = \bigcap_{\lambda \in \Lambda} A_X(E_\lambda)$.

8) Given $E \subseteq \Omega$, because $A_X(E) = \Omega$, $A_X K_X(E) = \Omega$. Thus, $A_X K_X(E) = A_X(E)$.

9) Given $E \subseteq \Omega$, because $A_X(E) = \Omega$, $A_X A_X(E) = \Omega$. Thus, $A_X A_X(E) =$

$A_X(E)$.

10) For any $E \subseteq \Omega$, because $A_X(E) = \Omega$, $K_X A_X(E) = K_X(\Omega)$. By K1, $K_X(\Omega) = \Omega$. Therefore, $K_X A_X(E) = A_X(E)$. ■

Proposition 6 When $X \neq P$ and $E \subseteq \Omega_X$, the following properties of knowledge and awareness/unawareness are obtained:

① KU Introspection: $K_X U_X(E) = \emptyset$.

② AU Introspection: $U_X(E) \subseteq U_X U_X(E)$.

③ Weak Necessitation: $A_X(E) \subseteq K_X(\Omega_X)$.

④ Strong Plausibility: $U_X(E) \subseteq \bigcap_{n=1}^{\infty} (\neg K_X)^n (E)$.

⑤ Weak Negative Introspection: $\neg K_X(E) \cap A_X \neg K_X(E) = K_X \neg K_X(E)$.

⑥ Reverse Symmetry: $A_X(\neg E) \subseteq A_X(E)$.

⑦ A-Conjunction: $A_X(\bigcap_{\lambda \in \Lambda} E_\lambda) = \bigcap_{\lambda \in \Lambda} A_X(E_\lambda)$.

⑧ AK-Self Reflection: $A_X K_X(E) = A_X(E)$.

⑨ AA-Self Reflection: $A_X A_X(E) = A_X(E)$.

⑩ A-Introspection: $K_X A_X(E) = A_X(E)$.

Proof. Suppose that $X \neq P$ and that for any $E \subseteq \Omega$, $E \subseteq \Omega_X$. Then, by Condition 2 in Lemma 4, $A_X(E) = K_X(E)$. Therefore, $U_X(E) = \neg K_X(E)$.

1) By $A_X(E) = K_X(E)$, $U_X(E) = \neg K_X(E)$. By Proposition 2, because $K_X(E) \subseteq \Omega_X$, $\neg K_X(E) \nsubseteq \Omega_X$. Therefore, $K_X\left(\neg K_X(E)\right) = \emptyset$.

2) $U_X(E) = \neg K_X(E) \subseteq \Omega$. $U_X U_X(E) = \neg K_X U_X(E)$. By KU Introspection, because $K_X U_X(E) = \emptyset$, $\neg K_X U_X(E) = \Omega$. Therefore, $U_X(E) \subseteq U_X U_X(E)$.

3) As $A_X(E) = K_X(E)$ by Condition 2 in Lemma 4 and $K_X(\Omega_X) = \Omega_X$ by Remark 5, $A_X(E) \subseteq K_X(\Omega_X)$.

4) By AU Introspection, $U_X U_X(E) = \neg K_X U_X(E) = \Omega$. $U_X U_X U_X(E) = U_X \neg K_X U_X(E) = U_X(\Omega) = \neg K_X(\Omega)$. By the definition of the knowledge operator, because $K_X(\Omega) = \emptyset$, $U_X(\Omega) = \Omega$. By repetition, $\bigcap_{n=1}^{\infty}(\neg K_X)^n(E) = \Omega$. Therefore, $U_X(E) \subseteq \bigcap_{n=1}^{\infty}(\neg K_X)^n(E)$.

5) By Condition 2 in Lemma 4, $A_X \neg K_X(E) = K_X \neg K_X(E)$. By K4, because $K_X(E) \subseteq E \subseteq \Omega_X$, $\neg K_X(E) \nsubseteq \Omega_X$. Therefore, $K_X \neg K_X(E) = \emptyset$. Thus, $\neg K_X(E) \cap A_X \neg K_X(E) = \neg K_X(E) \cap \emptyset = \emptyset = K_X \neg K_X(E)$.

6) $A_X(\neg E) = K_X(\neg E) \cup K_X \neg K_X(\neg E)$. By $E \subseteq \Omega_X$, $\neg E \nsubseteq \Omega_X$. Therefore, $K_X(\neg E) = \emptyset$. By $\neg K_X(\neg E) = \Omega$, $K_X \neg K_X(\neg E) = \emptyset$. Therefore, $A_X(\neg E) = \emptyset$. By Condition 2 in Lemma 4, because $A_X(E) = K_X(E)$, $A_X(\neg E) \subseteq A_X(E)$.

7) For any $E \subseteq \Omega_X$, because $A_X(E) = K_X(E)$, $A_X(\bigcap_{\lambda \in \Lambda} E_\lambda) = K_X(\bigcap_{\lambda \in \Lambda} E_\lambda)$. Moreover, for any $\lambda \in \Lambda$, because $A_X(E_\lambda) = K_X(E_\lambda)$, $\bigcap_{\lambda \in \Lambda} A_X(E_\lambda) = \bigcap_{\lambda \in \Lambda} K_X(E_\lambda)$. Therefore, by K3, $A_X(\bigcap_{\lambda \in \Lambda} E_\lambda) = \bigcap_{\lambda \in \Lambda} A_X(E_\lambda)$.

8) For any $E \subseteq \Omega_X$, by K4, $K_X(E) \subseteq E$. Therefore, by $A_X K_X(E) = K_X K_X(E)$, $K_X K_X(E) = K_X(E)$. Thus, $A_X K_X(E) = A_X(E)$.

9) For any $E \subseteq \Omega_X$, by K4, as $K_X(E) \subseteq E$. $A_X(E) = K_X(E)$, $A_X A_X(E) = A_X K_X(E) = K_X K_X(E) = K_X(E)$. Therefore, $A_X A_X(E) = A_X(E)$.

10) For any $E \subseteq \Omega_X$, because $A_X(E) = K_X(E)$, $K_X A_X(E) = K_X K_X(E) = K_X(E)$. Therefore, $K_X A_X(E) = A_X(E)$. ■

Proposition 7 When $X \neq P$ and $E \nsubseteq \Omega_X$, the following properties of knowledge and awareness/unawareness are obtained:

① KU Introspection: $K_X U_X(E) = \emptyset$.

② AU Introspection: $U_X(E) = U_X U_X(E)$.

③ Weak Necessitation: $A_X(E) \subseteq K_X(\Omega_X)$.

④ Strong Plausibility: $U_X(E) = \bigcap_{n=1}^{\infty} (\neg K_X)^n (E)$.

⑤ Weak Negative Introspection: $\neg K_X(E) \cap A_X \neg K_X(E) = K_X \neg K_X(E)$.

⑥ Reverse Symmetry: $A_X(\neg E) \supseteq A_X(E)$.

⑦ A-Conjunction: $A_X(\bigcap_{\lambda \in \Lambda} E_\lambda) = \bigcap_{\lambda \in \Lambda} A_X(E_\lambda)$.

⑧ AK-Self Reflection: $A_X K_X(E) = A_X(E)$.

⑨ AA-Self Reflection: $A_X A_X(E) = A_X(E)$.

⑩ A-Introspection: $K_X A_X(E) = A_X(E)$.

Proof. Suppose that $X \neq P$ and that for any $E \subseteq \Omega$, $E \nsubseteq \Omega_X$. By Condition 3 in Lemma 4, $A_X(E) = \emptyset$. Therefore, $U_X(E) = \Omega$.

1) $K_X U_X(E) = K_X(\Omega) = \emptyset$.

2) Because $U_X U_X(E) = U_X(\Omega)$. $\Omega \nsubseteq \Omega_X$ is evident, $U_X(\Omega) = \Omega$. Therefore, as $U_X U_X(E) = \Omega$, $U_X(E) = U_X U_X(E)$.

3) By Remark 5, $A_X(E) = \emptyset \subseteq \Omega_X = K_X(\Omega_X)$.

4) By AU Introspection, $U_X U_X U_X(E) = U_X(\Omega) = \Omega$. By repetition, $U_X(E) = \bigcap_{n=1}^{\infty} (\neg K_X)^n (E)$.

5) By $E \nsubseteq \Omega_X$, because $K_X(E) = \emptyset$, $\neg K_X(E) = \Omega$. $K_X \neg K_X(E) = K_X(\Omega) = \emptyset$. Therefore, because $A_X \neg K_X(E) = A_X(\Omega) = K_X(\Omega) \cup K_X \neg K_X(\Omega) = \emptyset$, $\neg K_X(E) \cap A_X \neg K_X(E) = K_X \neg K_X(E)$.

6) Because $A_X(E) = \emptyset$, $A_X(\neg E) \supseteq A_X(E)$.

7) For any $E \nsubseteq \Omega_X$, as $A_X(E) = \emptyset$, $A_X(\bigcap_{\lambda \in \Lambda} E_\lambda) = \emptyset$. Moreover, for any $\lambda \in \Lambda$, because $A_X(E_\lambda) = \emptyset$, $\bigcap_{\lambda \in \Lambda} A_X(E_\lambda) = \emptyset$. Hence, $A_X(\bigcap_{\lambda \in \Lambda} E_\lambda) = \bigcap_{\lambda \in \Lambda} A_X(E_\lambda)$.

8) Because $K_X(E) = \emptyset$, $A_X K_X(E) = A_X(\emptyset) = K_X(\emptyset) \cup K_X \neg K_X(\emptyset) = \emptyset \cup K_X(\Omega) = \emptyset$. Therefore, $A_X K_X(E) = A_X(E)$.

9) $A_X A_X(E) = A_X(\emptyset) = \emptyset$. Therefore, $A_X A_X(E) = A_X(E)$.

10) $K_X A_X(E) = K_X(\emptyset) = \emptyset$. ■

Remark 6 Suppose $X \neq P$. For any $\lambda \in \Lambda$, let $E_\lambda \subseteq \Omega_X$, and for any $\delta \in \Delta$, let $E_\delta \nsubseteq \Omega_X$. Then, $A_X(\bigcap_{\lambda \in \Lambda} E_\lambda \bigcap_{\delta \in \Delta} E_\delta) \supseteq \bigcap_{\lambda \in \Lambda} A_X(E_\lambda) \bigcap_{\delta \in \Delta} A_X(E_\delta)$.

KU Introspection, AU Introspection, Weak Necessitation, and Strong Plausibility were proposed by Dekel et al. (1998); Symmetry, A-Conjunction, AK-Self Reflection, and AA-Self Reflection by Modica and Rustichini (1999); Weak Negative Introspection, Symmetry, A-Conjunction, AK-Self Reflection, and AA-Self Reflection by Halpern (2001); and A-Introspection by Heifetz et al. (2006). However, when $X \neq P$ and $E \subseteq \Omega_X$, AU Introspection, Weak Necessitation, and Strong Plausibility may not satisfy the equality. Furthermore, the equality may not be satisfied by Weak Necessitation when $X \neq P$ and $E \nsubseteq \Omega_X$. Moreover, the A-Conjunction is satisfied only when every event satisfies $E \subseteq \Omega_X$ or $E \nsubseteq \Omega_X$. Based on Remark 6, if the condition does not hold, $\bigcap_{\lambda \in \Lambda} A_X(E_\lambda) \bigcap_{\delta \in \Delta} A_X(E_\delta)$ may be empty.

Interestingly, Symmetry crashes with Non-triviality. Previous studies discussing the properties of awareness/unawareness have proved Symmetry, e.g., Heifetz et al. (2006, 2013a) and Li (2009), or assumed it, e.g., Modica and Rustichini (1994, 1999), Halpern (2001), Heifetz et al. (2008), and Sadzik (2021). In contrast, in our model, when the awareness/unawareness operator is non-trivial, Symmetry does not hold. Although Fukuda (2020) suggested that Symmetry may not hold in infinite higher-order unawareness, he showed that it does hold in second order unknown, i.e., in first order unawareness. In contrast with his result, we showed that Symmetry does not hold in first order unawareness, i.e., in second order unknown. Therefore, we refer to these properties as *Reverse Symmetry*. Moreover, the inclusion relations in Reverse Symmetry are different, whether $E \subseteq \Omega_X$ or $E \nsubseteq \Omega_X$.

As shown in the proofs, when the agent cannot perceive a part of the basic propositions, by definition of the knowledge operator, if the agent knows E, it may not perceive the negation, as it is not in the agent's subjective state space. Therefore, the knowledge operator with the negation is empty. When E is not in the agent's subjective state space, $K_X(E) = \emptyset$ and the negations, i.e., $\neg E$ and $\neg K_X(E)$ are empty or not. Therefore, the agent can perceive them or not.

It appears that the features of Reverse Symmetry are important and that there are at least two implications. One suggests that the event in which the agent can perceive and the negation that it cannot perceive need not be discussed. The other implication is related to modal logic. Modica and Rustichini (1994) showed that S4 with Symmetry is equal to S5. In contrast, in our model, Symmetry with Non-triviality does not hold. Further, Reverse Symmetry may suggest that in discussions about unawareness, we should exclude S5, and that we should discuss S4 with Reverse Symmetry in modal logic.

Finally, in this subsection, we show that Awareness Leads to Knowledge and present the inverse inclusion. Galanis (2013) proposed the following property: For any $X, Y \subseteq P$ with $Y \subseteq X$ and $E \subseteq \Omega$, if $E \subseteq \Omega_Y$, then $K_Y(E) \subseteq \left(K_X(E)\right)_Y \cap A_Y(E)$. He suggested that if PPK is not assumed, then the inverse inclusion may not hold in unawareness structures. However, in constructive Aumann structures, Awareness Leads to Knowledge does not hold, whereas the inverse holds even if the PPK is not assumed.[8]

Proposition 8 For any $X, Y \subseteq P$ with $Y \subseteq X$ and $E \subseteq \Omega$, if $E \subseteq \Omega_Y$, then

[8] In the first draft of this manuscript, a statement and a proof of Proposition 8 were wrong. The first draft states that the equation hold, which is incorrect. The statement and the proof in this version suggest that either the equation or Awareness leads to Knowledge does not hold.

$K_Y(E) \supseteq (K_X(E))_Y \cap A_Y(E)$.

Proof. Suppose that for any $X, Y \subseteq P$, $Y \subseteq X$ and given $E \subseteq \Omega_Y$. Firstly, suppose that $Y = P$, i.e., $X = P$. Then, $A_Y(E) = \Omega$ by Condition 1 of Lemma 4, and $(K_X(E))_Y = (K_Y(E))_Y = K_Y(E)$ because $K_Y(E) = K_X(E) = K_P(E)$. Therefore, $K_Y(E) = (K_X(E))_Y \cap A_Y(E)$. Next, suppose that $Y \neq P$. Then, by Condition 2 of Lemma 4, $K_Y(E) = A_Y(E)$. Because $A_Y(E) \supseteq (K_X(E))_Y \cap A_Y(E)$, $K_Y(E) \supseteq (K_X(E))_Y \cap A_Y(E)$.

When $Y \neq P$, $K_Y(E) \subseteq (K_X(E))_Y \cap A_Y(E)$ may not hold. Suppose that $\Pi(\omega, Y) \subseteq E$. Then, $K_Y(E) \neq \emptyset$. Assume that $\Pi(\omega, X) \nsubseteq E$ for any $\omega \in \Omega$. Then, $K_X(E) = \emptyset$ by Remark 4. Then, $\emptyset = K_X(E) = K_X(E) \cap A_Y(E) = (K_X(E))_Y \cap A_Y(E) \subseteq K_Y(E)$. ■

This study does not assume PPK, but Awareness Leads to Knowledge does not hold. Proposition 8 induces the inverse of Awareness Leads to Knowledge. The result is opposite to the result reported by Galanis (2013). Note that the equation does not hold if and only if the agent is unaware of all basic propositions and it does not capture a knowledge operator on the extension of a state space that it is aware of. The inverse means that the unaware agent cannot be aware of a (partial) partition on a state space that it is unaware of. The different results mean that a feature of our constructive Aumann structure is different to a feature of an unawareness structure. It means that our model does not fit Galanis's discussion.

Proposition 8 implies Partially PPK. Proposition 8 induces that for any $\omega \in K_X(E)$, $\Pi(\omega, X) \subseteq E \subseteq \Omega_Y$. It also induces that for any $\omega \in K_Y(E)$, $\Pi(\omega, Y) \subseteq E \subseteq \Omega_Y \subseteq \Omega_X$. Their statements are conditions of Partially PPK. It means that a statement of PPK is satisfied under the condition of Proposition 8. However, the equation may not hold in the proposition. The equation and Awareness Leads to Knowledge need an additionally strict assumption opposite to that of Galanis (2013), who relaxes assumptions.

4.4 Relationships with Standard Aumann Structure

In our models if $X = P$, the main theorem in Dekel et al. (1998), that unawareness is trivial, is satisfied as follows.

Theorem 1 In any constructive Aumann structure, the following are equivalent.

1. $X = P$.
2. For any $E \subseteq \Omega$, $U_X(E) = \emptyset$.
3. For any $E, F \subseteq \Omega$, $E \subseteq F$, $U_X(E) \subseteq \neg K_X(F)$.

Proof. ($1 \Longrightarrow 2$) It is evident by Condition 1 in Lemma 4.

($2 \Longrightarrow 3$) Given $E \subseteq \Omega$, $U_X(E) = \emptyset$. Then, for any $F \subseteq \Omega$, $\emptyset = U_X(E) \subseteq \neg K_X(F)$.

($3 \Longrightarrow 1$) Suppose that for every $E, F \subseteq \Omega$, if $E \subseteq F$, then $U_X(E) \subseteq \neg K_X(F)$. Here, assume that $X \neq P$ and given $E = \emptyset$ and $\emptyset \neq F \subseteq \Omega_X$. Then, because $\neg K_X(\emptyset) =$

Ω and $K_X(\Omega) = \emptyset$, $U_X(\emptyset) = \neg K_X(E) \cap \neg K_X \neg K_X(E) = \Omega \cap \neg K_X(\Omega) = \Omega \cap \big(\Omega \setminus K_X(\Omega)\big) = \Omega$. Because $\neg K_X(F) \subsetneq \Omega$ is evident, $\neg K_X(F) \subsetneq U_X(\emptyset)$. This is a contradiction. Hence, $X = P$. ■

Dekel et al. (1998) showed that if the unawareness operator satisfies Plausibility, AU Introspection, and KU Introspection, and the knowledge operator satisfies Necessitation, then unawareness is trivial. Moreover, they showed that under the above assumptions of the unawareness operator, if the knowledge operator satisfies Monotonicity, the agent is unaware of everything. In our model, where $X = P$, we show that the knowledge operator and unawareness operator satisfy the above properties. Hence, their main theorem must be satisfied when $X = P$, and vice versa.

Chen et al. (2012) showed that if the knowledge operator satisfies Necessitation, and the unawareness operator satisfies Plausibility, then Negative Introspection is equivalent to AU Introspection and KU Introspection, and that if the assumptions adding Monotonicity and Truth are satisfied, Negative Introspection is equivalent to AU Introspection. In our model, where $X = P$, Negative Introspection and AU Introspection are equivalent. Moreover, Negative Introspection is equivalent to Symmetry, as shown by Modica and Rustichini (1994). Therefore, we can generalize the main theorem in Chen et al. (2012) and Modica and Rustichini (1994) as follows.

Theorem 2 In any constructive Aumann structure, the following are equivalent.

1. $X = P$.

2. Negative Introspection if and only if AU Introspection if and only if Symmetry.

Proof. ($1 \Longrightarrow 2$) Suppose that $X = P$. Then, by Proposition 3, Negative Introspection holds. Moreover, by Proposition 5, AU Introspection and Symmetry hold.

($2 \Longrightarrow 1$) Suppose that Negative Introspection, AU Introspection and Symmetry are equivalent. Here, assume that $X \neq P$. Then, by Proposition 3, Negative Introspection does not hold, and by Proposition 6 and Proposition 7, Symmetry does not hold. However, by Proposition 6 and Proposition 7, AU Introspection holds. This contradicts that the three properties are equivalent. Therefore, $X = P$. ■

Finally, we consider a relationship with Fukuda (2020). He suggested that non-trivial unawareness can be discussed in (non-partitional) standard state space models, and that Necessitation crashes AU Introspection. Hence, as pointed out by him, if AU Introspection does not hold where Necessitation is satisfied, non-trivial unawareness can be discussed. Subsequently, he proposed Reverse AU Introspection ($U_X(E) \supseteq U_X U_X(E)$) instead of AU Introspection. He suggested two points: one is that AU Introspection is not necessary in discussing non-trivial unawareness; the other is that if AU Introspection does not hold, (non-partitional) standard state space models can represent awareness of unawareness. In contrast, our Reverse AU Introspection with Non-triviality may not hold in our model when the equation of the inclusion relation does not hold. Our model must induce AU Introspection, even if the agent cannot perceive some part of the basic propositions. The different results imply that different features exist between Fukuda (2020) and this study.

5. Conclusion

This paper presented a constructive Aumann structure in which a state space is a complete lattice. In contrast with Heifets et al. (2006) and Li (2009), the family of disjoint state spaces is not necessary in our model. However, unlike in the case of standard state space models, our models are multi-attribute models, similar to those of Heifets et al. and Li. Note that the same states between different state spaces in our models have different attributes. This means that a property of a state in each subjective state space turns on the relationships with the other states in the state space.

Our model is a single-agent model, and we did not discuss higher-order perceptions. However, our results differ from those of previous studies, as shown in Proposition 6, Proposition 7, and Remark 6. Possibly, other properties do not hold in multi-agent models or higher-order perceptions.

In particular, Symmetry with non-trivial unawareness is not possible. Previous studies have proved or assumed this property but, in this study, we show the impossibility of achieving Symmetry with non-trivial unawareness. Based on the result obtained (Reverse Symmetry), the implication is that the event in which the agent can perceive and the negation that she cannot need not be discussed, and that S5 in modal logics must be excluded when discussing unawareness.

In contrast, constructive state spaces, HMS-state spaces, and Li-state spaces are equivalent, and we showed generalizations of the results obtained by Dekel et al. (1998) as Theorem 1 and those obtained by Chen et al. (2012) (and Modica and Rustichini 1994) as Theorem 2. This implies that our model is an intermediate between those presented by Heifetz et al. (2006) and Li (2009) and the non-partitional standard state space models.

Previous studies have discussed choice theories with unawareness, e.g., Karni and Vierø (2013, 2017) and Piermont (2017); interactive situations with unawareness, e.g., Auster (2013), Heifetz et al. (2013a), and Galanis (2013, 2018); and games with unawareness, e.g., Heifetz et al. (2013b), Halpern and Rêgo (2014), Perea (2018), and Feinberg (2020). Future studies should introduce Aumann structures with complete lattices. For example, our model must be applied to Bayesian games with unawareness. Sadzik (2021) and Meier and Schipper (2014) discussed Bayesian games with unawareness. Sadzik discussed probabilistic beliefs with unawareness based on the findings of Heifetz et al. (2006) and defined the Bayesian equilibrium in normal-form games with unawareness. Meier and Schipper discussed probabilistic beliefs also based on the findings of Heifetz et al. (2013a), defined the Bayesian equilibrium, and proved the existence of a Bayesian equilibrium.[9] In the future, we will discuss probabilistic

[9] There exists a difference between Sadzik (2021) and Meier and Schipper (2014). The former assumes common prior, while the latter does not assume it.

beliefs based on our Aumann structure with a complete lattice and introduce it to Bayesian games with unawareness.

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