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Is “Unawareness Leads to Ignorance” Trivial?

Yoshihiko Tada
Chuo University
Economic Reserch Institute,
Associate Researcher

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INSTITUTE OF ECONOMIC RESEARCH
Chuo University
Tokyo, Japan

Note and Comment

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Yoshihiko Tada[1]

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1. Introduction

Dekel, Lipman, and Rustichini (1998, p. 166) show two properties of unawareness: the first is that "the agent is never unaware of anything," (let us call it Triviality), and the second is that "if the agent is unaware of anything, he knows nothing," (let us call it Unawareness Leads to Ignorance). They assert that "clearly, then, either property leaves us with only a trivial form of unawareness at best." It is certainly agreed that Triviality is a property of trivial-unawareness. However, is it Unawareness Leads to Ignorance a property of trivial-unawareness? This article addresses this question.

2. Preliminary

Let us first define the standard information structure $\langle \Omega, P \rangle$ and the (standard) knowledge operator. We define Ω as the state space and $P: \Omega \to 2^{\Omega} \setminus \{\emptyset\}$ is the

[1] Graduate School of Economics, Chuo University, Japan

yoshihiko.tada.4@gmail.com

agent's (standard) information function. We do not assume whether P is partitional. Here, the knowledge operator $K: 2^{\Omega} \to 2^{\Omega}$ is defined as the following: Given any event $E \subseteq \Omega$,

$$\begin{cases} \omega \in K_i(E) \text{ if } P_i(\omega) \subseteq E; \text{ and} \\ \omega \notin K_i(E) \text{ otherwise.} \end{cases}$$

Then, by the definition, the following properties hold:

Necessitation $\quad K(\Omega) = \Omega$, and

Monotonicity $\quad E \subseteq F \implies K(E) \subseteq K(F)$.

Next, we assume three properties of the unawareness operator $U: 2^{\Omega} \to 2^{\Omega}$ as proposed by Dekel, Lipman, and Rustichini (1998) as follows:

Plausibility $\quad U(E) \subseteq \neg K(E) \cap \neg K \neg K(E)$,

KU Introspection $\quad KU(E) = \emptyset$, and

AU Introspection $\quad U(E) \subseteq UU(E)$.

Then, Dekel, Lipman, and Rustichini (1998) show the following theorem:

Theorem 1 (Dekel, Lipman, and Rustichini 1998): In a standard information structure $\langle \Omega, P \rangle$, suppose that the unawareness operator U satisfies Plausibility, KU Introspection, and AU Introspection. Then,

- (Triviality) if the knowledge operator K satisfies Necessitation, then for any event $E \subseteq \Omega$, $U(E) = \emptyset$; and

- (Unawareness Leads to Ignorance)[2] if K satisfies Monotonicity, then for all events $E, F \subseteq \Omega$, $U(E) \subseteq \neg K(F)$.

Proof: By AU Introspection, $U(E) \subseteq UU(E)$, by Plausibility, $UU(E) \subseteq \neg KU(E) \cap \neg K \neg KU(E)$, and by KU Introspection, $\neg KU(E) \cap \neg K \neg KU(E) = \neg K(\Omega)$.

First, suppose that Necessitation holds. Then, because $\neg K(\Omega) = \emptyset$, $U(E) \subseteq \emptyset$. Second, suppose that Monotonicity holds. Then, for any $F \subseteq \Omega$, $K(F) \subseteq K(\Omega)$,

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[2] Galanis (2013) defines "Awareness Leads to Knowledge" as a property of unawareness in unawareness structures proposed by Heifetz, Meier, and Schipper (2006). The property is similar to the inverse of Unawareness Leads to Ignorance. However, he first focuses only on Awareness Leads to Knowledge in unawareness structures but not in (non)standard information structures. Unawareness structures assume a family of disjoint subjective state spaces. Some spaces might be related with and more expressive than another space. It means that the description of one space is greater than that of another. Awareness Leads to Ignorance suggests that an agent's knowledge in a more expressive state space is more descriptive than her knowledge in a less expressive state space. By contrast, the inverse of Unawareness Leads to Ignorance does not mean that in (non)standard information structures, because the state space in the information structure does not assume another disjoint state space. Moreover, he does not consider the property of Awareness Leads to Knowledge in standard information structures: "...the standard model assumes an agent who is aware of everything and knows all relevant theorems." (Galanis 2013, P52)

that is, $\neg K(\Omega) \subseteq \neg K(F)$. Therefore, $U(E) \subseteq \neg K(E)$ for any $E, F \subseteq \Omega$. ■

That is, when Plausibility, KU Introspection, and AU Introspection hold, Necessitation implies Triviality, and Monotonicity implies Unawareness Leads to Ignorance. The former means that the agent cannot be aware of anything, and the latter means that the agent cannot get any knowledge where the agent is unaware of some event. Hence, they assert that both properties are trivial, and say that "a nontrivial model of unawareness requires us to abandon both necessitation and monotonicity" (Dekel, Lipman, and Rustichini 1998, p. 166). However, the two properties, Necessitation and Monotonicity, must be held by a definition of the knowledge operator. Therefore, we cannot discuss nontrivial unawareness in a standard information structure.

3. Unawareness Leads to Ignorance without Triviality

This section addresses the question "Is Unawareness Leads to Ignorance trivial?" In some cases, the answer is no. Let us consider an unemployed person, Alice, who cannot access information about the unemployment insurance program. Then, she may not know that she can receive a jobless insurance, or how to apply for the jobless insurance. In other words, she cannot know everything about the unemployed insurance program.

We mathematically model this situation. Let $\Omega = \{\omega_1, \omega_2\}$, and let us interpret ω_1 as "Alice can receive jobless insurance," and ω_2 as "Alice cannot receive jobless insurance." Each event is interpreted as follows.

$E_1 = \{\omega_1\}$: Alice can receive jobless insurance

$E_2 = \{\omega_2\}$: Alice cannot receive jobless insurance

$\Omega = \{\omega_1, \omega_2\}$: Alice's country has an unemployed insurance program.

Here, let $K: 2^{\{\omega_1,\omega_2\}} \to 2^{\{\omega_1,\omega_2\}}$ be Alice's knowledge operator, and for any event $E \subseteq \Omega$, $K(E)$ is interpreted as "Alice knows the event E." Let $U: 2^{\{\omega_1,\omega_2\}} \to 2^{\{\omega_1,\omega_2\}}$ be Alice's unawareness operator, and for any event $E \subseteq \Omega$, $U(E)$ is interpreted as "Alice is unaware of the event E." Suppose that for any event $E \subseteq \Omega$, $K(E) = \emptyset$, that is, she does not know everything. Then, K does not satisfy Necessitation, but it satisfies Monotonicity. Moreover, U obviously satisfies Plausibility, KU Introspection, and AU Introspection because $U(E) = \Omega$ for any E. Then, it represents that if she is unaware of something about the unemployed insurance program, she does not know everything about it.

This example represents Unawareness Leads to Ignorance. In this, the property is interpreted as Alice is unaware of the true event related to the decision making, therefore she cannot be aware that she faces a decision. In other words, Alice cannot be aware that she faces the decision to apply for jobless insurance. A similar situation often occurs in the real world. Hence, in this case, Unawareness Leads to Ignorance is not a property of trivial unawareness.

However, this case has a crucial problem. In the standard information structure, both Necessitation and Monotonicity must hold. Hence, we cannot discuss Monotonicity without Necessitation. In other words, we cannot consider Unawareness Leads to Ignorance without Triviality. Can we not explore whether nontrivial Unawareness Leads to Ignorance? To answer this question, we consider a nonstandard information structure in the next section.

4. Information structure without Necessitation

To consider Unawareness Leads to Ignorance without Triviality, let us first define a nonstandard information function. The standard information function P does not imply that the information set is empty. Every information set is a nonempty set. In contrast, we allow some information sets to be empty. Let $\hat{P}: \Omega \to 2^{\Omega}$ be the agent's nonstandard information function. We do not assume whether it is partitional or not. Some state $\omega \in \Omega$ may satisfy $\hat{P}(\omega) = \emptyset$. It is interpreted as "the agent cannot access information (set) about ω." Let us reconsider Alice's example: suppose that the true state is ω_1, and $\hat{P}(\omega_1) = \emptyset$. Then, the information set is interpreted as "Alice cannot access information about her applying for jobless insurance." Let $\langle \Omega, \hat{P} \rangle$ be a nonstandard information structure.

Next, let us define the nonstandard knowledge operator $\hat{K}: 2^{\Omega} \to 2^{\Omega}$ in $\langle \Omega, \hat{P} \rangle$. Given any event E, $K(E)$ is defined as follows:

$$\begin{cases} \omega \in \hat{K}(E) \text{ if } \hat{P}(\omega) \subseteq E \text{ and } \hat{P}(\omega) \neq \emptyset; \text{ and} \\ \omega \notin \hat{K}(E) \text{ otherwise.} \end{cases}$$

At ω, the agent knows E if not only the information set given ω is a subset of E, but also the information set is not empty. That is, to know E, she must access information about ω. It is obvious that $\hat{P}$ and $\hat{K}$ are generalizations of P and K.

Here, we show the following properties.

Theorem 2: In a nonstandard information structure $\langle \Omega, \hat{P} \rangle$ possessing a nonstandard knowledge operator $\hat{K}$, $\hat{K}$ satisfies the following properties:

- (Necessitation) for any $\omega \in \Omega$, $\hat{P}(\omega) \neq \emptyset$ if and only if $\hat{K}(\Omega) = \Omega$.
- (Monotonicity) $E \subseteq F \implies \hat{K}(E) \subseteq \hat{K}(F)$

Proof. Suppose for any $\omega \in \Omega$, $\hat{P}(\omega) \neq \emptyset$. Then, $\hat{K}$ is equivalent to the standard knowledge operator K. Hence, Necessitation holds. Next, suppose that there exists $\omega \in \Omega$ satisfying $\hat{P}(\omega) = \emptyset$, then $\omega \notin \hat{K}(E)$ for any $E \subseteq \Omega$. That is, $\omega \notin \hat{K}(\Omega)$. Hence, $\hat{K}(\Omega) \subsetneq \Omega$.

Next, suppose that $E \subseteq F$, given $\omega \in \hat{K}(E)$. Then, by definition of $\hat{K}$, $\hat{P}(\omega) \subseteq E$ and $\hat{P}(\omega) \neq \emptyset$. Because $E \subseteq F$, $\hat{P}(\omega) \subseteq E \subseteq F$. Hence, $\omega \in \hat{K}(F)$, that is, $\hat{K}(E) \subseteq \hat{K}(F)$. ■

Interestingly, the nonstandard knowledge operator $\hat{K}$ may not satisfy Necessitation. If there exists some information set that is empty, then Necessitation does not hold, and vice versa. By contrast, Monotonicity holds. Hence, it is obvious that given the unawareness operator based on $\hat{K}$ satisfying Plausibility, KU Introspection, and AU Introspection, Unawareness Leads to Ignorance holds, whereas Triviality holds if and only if there is empty no information set.

Corollary 1: In a nonstandard information structure $\langle \Omega, \hat{P} \rangle$ possessing the nonstandard knowledge operator $\hat{K}$, let the nonstandard unawareness operator $\hat{U}: 2^{\Omega} \to 2^{\Omega}$ satisfy Plausibility ($\hat{U}(E) \subseteq \neg\hat{K}(E) \cap \neg\hat{K}\neg\hat{K}(E)$), KU Introspection

($\widehat{K}\widehat{U}(E) = \emptyset$), and AU Introspection ($\widehat{U}(E) \subseteq \widehat{U}\widehat{U}(E)$). Then, $\widehat{U}$ satisfies the following:

- (Triviality) If $\hat{P}(\omega) \neq \emptyset$, then $\widehat{U}(E) = \emptyset$.
- (Unawareness Leads to Ignorance) For any $E, F \subseteq \Omega$, $\widehat{U}(E) \subseteq \neg\widehat{K}(F)$.

Let us reconsider Unawareness Leads to Ignorance. $\widehat{K}$ and $\widehat{U}$ are based on the nonstandard information function, $\hat{P}$. It allows some information set to be empty, and given such an information set, the agent cannot know the relevant event. Here, given $\omega \in \Omega$, suppose $\hat{P}(\omega) = \emptyset$. Then, for any E, $\omega \notin \widehat{K}(E)$, that is, $\omega \in \neg\widehat{K}(E)$. This means that at ω, if the agent cannot receive relevant information (set), then she cannot know every event. Let $F = \neg\widehat{K}(E)$, then $\omega \in \neg\widehat{K}(F) = \neg\widehat{K}\neg\widehat{K}(E)$ is obvious. This means that, at ω, the agent cannot know the ignorance. Therefore, $\omega \in \widehat{U}(E)$ for any E. From the above, $\omega \in \widehat{U}(E)$ means that at ω, because she cannot receive any relevant information (set), she cannot perceive or understand that she faces some (nonstandard) information structure. Hence, she cannot perceive every event in the nonstandard information structure. Then, Unawareness Leads to Ignorance means that the agent who cannot perceive her (nonstandard) information structure cannot know everything.

5. Conclusion

This note characterizes Unawareness Leads to Ignorance by allowing an empty information set. Dekel, Lipman, and Rustichini (1998) suggest that Unawareness Leads to Ignorance is trivial in a standard information structure. That is true. However, in a nonstandard information structure that allows the information set to be empty, Unawareness Leads to Ignorance means that because the agent cannot perceive her information structure, the agent cannot know everything in the information structure. Hence, the answer to the question, "Is Unawareness Leads to Ignorance trivial?" is no in the case where some information set is empty.

Reference

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