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Abstract

This paper evaluates the impact of stochastic disturbances on a deterministic limit cycle in a stochastic post-Keynesian model. It presents an approximation formula for solution paths near the limit cycle and derives an approximated stationary distribution of the limit cycle.

Keywords: Limit cycles; Keynesian theory; Stationary distribution; Stochastic differential equations

JEL classification: C62; E12; E32

1 Introduction

It is well known that there are two approaches in the theory of business cycles: the "endogenous" and "exogenous" approaches.[1] The endogenous approach attributes the mechanism of business cycles to internal (endogenous) factors in economic systems (e.g., characteristics of consumption or investment behavior). In this approach, the phenomena of business cycles are usually explained by the existence of periodic motions in nonlinear economic models.[2] The exogenous approach, on the other hand, takes business cycles as phenomena induced by external (exogenous) factors outside economic systems (e.g., shocks in demand or supply). As such, this approach focuses on the propagation mechanism of random shocks in economic systems.[3] Although both of the approaches provide insightful expositions for business cycles, there have not been many attempts to integrate them until recently.

In recent years, several theoretical works have studied the response of deterministic business cycles to random shocks. Bashkirtseva et al. [6], Bashkirtseva et al. [7] and Bashkirtseva et al. [4] examined stochastic versions of Kaldor's [17] model;[4] Li et al. [24], Li et al. [22], Li et al. [23], Nguyen Huu and Costa-Lima [35], Lin et al. [25] and Jungeilges and Ryazanova [15, 16] explored stochastic versions of Goodwin's [9, 10] models. In these studies, the influence of stochastic disturbances is mostly evaluated by the properties of approximated stationary distributions.

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[1]For details on them, see also Murakami and Zimka [33].

[2]Since the classical contributions of Kaldor [17], Hicks [13] and Goodwin [9], nonlinearity has been considered the most essential factor for verifying the existence of periodic motions in economic dynamics.

[3]Slutzky [38] is one of the classic works in this approach.

[4]Kosobud and O'Neill [21] and Grasman and Wentzel [11] are classical works studying the implications of stochastic disturbances in Kaldor's [17] model.

The stochastic sensitivity function approach of Bashkirtseva and Ryashko [5] is, among others, a popular method for this purpose. It is a useful and convenient way to assess stochastic implications because it can eschew complexity associated with the Fokker-Planck equations making appropriate approximations for them.[5] Without relying on the Fokker-Planck equations, however, it is possible to evaluate the effect of stochastic disturbances. Indeed, making proper approximations for stochastic models themselves, we can gain much more information on the characteristics of stochastic models.

The purpose of this paper is to explore the influence of random shocks on deterministic business cycles in a stochastic version of Murakami's [29, 32] post-Keynesian model of growth cycles. For this purpose, we employ Bonnin's [8] method to evaluate the response of the unique limit cycle to stochastic disturbances. By so doing, we aim to pave a road to the synthesis of the endogenous and exogenous approaches in the theory of business cycles.

This paper is organized as follows. Sect. 2 sets up a stochastic post-Keynesian model based on Murakami [29, 32]. The model consists of stochastic differential equations. Sect. 3 analyzes our post-Keynesian model. Based on Murakami [32], we first confirm that our model, without stochastic disturbances, possesses a unique limit cycle. Following Bonnin [8], we then present an explicit approximation formula for solution paths near the unique limit cycle and derive an approximated stationary distribution of the limit cycle in our model with small stochastic disturbances. The (approximation) formula for solution paths is a distinguished feature of our analysis because such a formula has not been derived in the related literature. Sect. 4 performs numerical simulations. Sect. 5 concludes this paper.

2 The model

This section presents a stochastic post-Keynesian model. The model is based on Murakami [29, 32], but it consists of stochastic differential equations.

2.1 Aggregate saving

Following the post-Keynesian theory of income distribution (cf. Kaldor [18]; Pasinetti [36]), we postulate that capitalists have a higher propensity to save than workers. Specifically, aggregate saving is assumed to be written as follows:

$$S = s_c\Pi + s_w(Y - \Pi), \tag{1}$$

where s_c and s_w are, respectively, a positive constant and a nonnegative constant with $s_w \leq s_c < 1$. In (1), Y, S and Π stand for aggregate income, aggregate saving and aggregate profits (capitalists' income), respectively; s_c and

[5] For the Fokker-Planck equations in stochastic differential equations, see Arnold [3, sect. 9.4] for instance.

s_w represent the (average) propensities to save of capitalists and of workers, respectively.[6]

Reflecting Kaldor's [19] stylized fact (cf. Jones [14]), aggregate share of capital (or the ratio of aggregate profits to aggregate income) Π/Y is assumed to be constant as follows:

$$\Pi = \pi Y, \tag{2}$$

where π is a positive constant less than unity which represents aggregate share of capital.

It follows from (1) and (2) that aggregate saving can be written as follow:

$$S = s\Pi, \tag{3}$$

where s is a positive constant defined as follows:

$$s = \frac{s_c \pi + s_w(1-\pi)}{\pi} = s_c + s_w \frac{1-\pi}{\pi}.$$

2.2 Aggregate investment

Based on Keynes' [20, chap. 11] theory of investment, we postulate that the rate of gross capital formation (or the ratio of investment to capital stock) is positively influenced by the expected rate of profit on capital, which can be identified with the marginal efficiency of capital, in the following fashion:

$$\frac{I}{K} = f(r^e). \tag{4}$$

In (4), I, K and r^e stand for aggregate (gross) investment, aggregate capital stock and the expected rate of profit on capital, respectively; f is the (gross) capital formation function.[7]

The following is a reasonable assumption on the capital formation function. (cf. Murakami [29, 32]).

Assumption 1. *The nonnegative-valued function $f : \mathbb{R}_+ \to \mathbb{R}_+$ is twice continuously differentiable with*

$$f'(r^e) > 0$$

for every $r^e \in \mathbb{R}_+$.[8]

[6] It was empirically confirmed by Naastepad and Storm [34] that the (average) propensity to save of capitalists was higher than that of workers in eight developed countries in the period of 1960-2000.

[7] For this type of capital formation (or investment) function, it is possible to provide a microeconomic foundation from a viewpoint of dynamic optimization (cf. Murakami [28]).

[8] The twice continuous differentiability of f is assumed for the application of Ito's formula to our model, but only the continuous differentiability of f suffices for the proof of Theorem 1 (cf. Murakami [32]).

2.3 The rate of profit

According to the Keynesian principle of effective demand, aggregate saving is adjusted to the sum of aggregate investment and aggregate autonomous demand (or the exogenous component of effective demand) (cf. Hicks [13]; Serrano [37]; Allain [1]) in the following way:[9]

$$S = I + A. \tag{5}$$

In (5), A stands for aggregate autonomous demand, which includes government expenditure and net exports.

It follows from (3) and (4) that the rate of profit can be given as follows:

$$r = \frac{1}{s}[f(r^e) + a]. \tag{6}$$

In (6), r and a stand for the rate of profit (or the ratio of profits to capital stock) Π/K and the demand-capital ratio (or the ratio of autonomous demand to capital stock) A/K, respectively.

2.4 Revisions of the expected rate of profit

We postulate that the expected rate of profit is adaptively revised in response to the (realized) rate of profit but that this revision is subjected to stochastic disturbances. Specifically, the revision process is assumed to be described by

$$dr^e = \alpha(r - r^e)dt + \sigma_e dz_e. \tag{7}$$

Due to (6), this reduces to

$$dr^e = \alpha\Big\{\frac{1}{s}\Big[f(r^e) + \frac{1}{k}\Big] - r^e\Big\}dt + \sigma_e dz_e, \tag{8}$$

where α and σ_e are a positive constant and a nonnegative constant, respectively; dz_e is a Wiener process (or a Brownian motion) such that, for $t_1 > t_0$,

$$E[z_e(t_1) - z_e(t_0)] = 0,$$
$$E[(z_e(t_1) - z_e(t_0))^2] = t_1 - t_0.$$

Equation (8) is a stochastic differential equation in Ito's sense.[10]

In (8), α represents the (mean) speed of revisions of the expected rate of profit, while σ_e measures the size

[9]Equation (5) is equivalent to

$$Y = C + I + A,$$

where C stands for aggregate consumption.

[10]For stochastic differential equations, see Arnold [3] for example.

of volatility in the expectations formation. In the right-hand side, the first term expresses the usual (deterministic) adaptive expectations formation, while the second term describes stochastic disturbances in the expectations formation.

One reasonable interpretation can be given to (8) as follows.[11] Suppose that there exist N firms in the economy and that they are identical, and denote each of them by $i,\ i = 1, ..., N$. Firm i is assumed to revise its expected rate of profit r_i^e during the short time interval $(t, t+h)$ in the following way:

$$r_i^e(t+h) - r_i^e(t) = \alpha[r(t) - r_i^e(t)]h + \sigma_e e(t) + \nu\eta_i(t), \tag{9}$$

where ν is a nonnegative constant; $e(t)$ and $\eta_i(t),\ i = 1, ..., N$, are random variables such that for every t and every i and j with $i \neq j$

$$E[e(t)] = E[\eta_i(t)] = E[e(t)\eta_i(t)] = E[\eta_i(t)\eta_j(t)] = 0,$$
$$E[e^2(t)] = E[\eta_i^2(t)] = h.$$

Firm i's adaptive expectations formation is expressed by the first term of the right-hand side of (9), while the second and third terms describe the macro (economy-wide) shock occurring to firm i and the shock idiosyncratic to firm i, respectively. The expected rate of profit can be defined as follows:

$$r^e = \frac{1}{N}\sum_{i=1}^{N} r_i^e.$$

Then, the conditional mean and variance of the change in the expected rate of profit at t can be given by

$$E_t[r^e(t+h) - r^e(t)] = \alpha[r(t) - r^e(t)]h,$$
$$\mathrm{Var}_t[r^e(t+h) - r^e(t)] = \Big(\sigma_e^2 + \frac{\nu^2}{N}\Big)h.$$

Equation (8) is thus obtained as the limit of (9) as $N \to \infty$ and $h \to 0$.

2.5 Aggregate capital formation

Aggregate capital formation is assumed to be carried out through aggregate investment without stochastic disturbances in the following fashion:

$$d(\ln K) = \Big(\frac{I}{K} - \delta\Big)dt,$$

[11] Similar interpretations can be found in Merton [26] and Yoshikawa [39].

which, due to (4), reduces to

$$d(\ln K) = [f(r^e) - \delta]dt, \tag{10}$$

where δ is a positive constant which represents the constant rate of capital depreciation.[12]

2.6 The rate of change in autonomous demand

We postulate that the natural logarithm of autonomous demand changes in the following fashion:

$$d(\ln A) = \mu dt + \sigma_d dz_d, \tag{11}$$

where μ and σ_d are a real constant and a nonnegative constant, respectively; dz_d is a Wiener process such that, for $t_1 > t_0$,

$$E[z_d(t_1) - z_d(t_0)] = 0,$$
$$E[(z_d(t_1) - z_d(t_0))^2] = t_1 - t_0.$$

Equation (11) is also a stochastic differential equation in Ito's sense.

In (11), μ represents the (mean) rate of change in autonomous demand, while σ_d measures the size of "(aggregate) demand shock."[13] The first and second terms of the right-hand side characterize the deterministic (steady) and stochastic (random) natures of the rate of change in autonomous demand.

Due to (10), we can obtain the following stochastic differential equation for the demand-capital ratio:

$$d(\ln a) = [\delta + \mu - f(r^e)]dt + \sigma_d dz_d. \tag{12}$$

[12] Equation (10) is equivalent to the following ordinary differential equation:

$$\frac{\dot{K}}{K} = f(r^e) - \delta.$$

We may introduce stochastic disturbances (due to unpredicted capital depreciation etc.) in capital formation in the following fashion:

$$d(\ln K) = [f(r^e) - \delta]dt + \sigma_s dz_s,$$

where σ_s is a nonnegative constant and dz_s is a standard Wiener process. In this case, equation (12) becomes

$$d(\ln a) = [\delta + \mu - f(r^e)]dt + \sigma_d dz_d - \sigma_s dz_s.$$

The qualitative nature of our analysis will, however, be unaltered even in this case.

[13] If $\sigma_d = 0$, equation (11) reduces to

$$\frac{\dot{A}}{A} = \mu.$$

2.7 Full model: Model (K)

The post-Keynesian model to be analyzed can be summarized as follows:

$$dr^e = \alpha\Big\{\frac{1}{s}[f(r^e) + a] - r^e\Big\}dt + \sigma_e dz_e, \tag{8}$$

$$d(\ln a) = [\delta + \mu - f(r^e)]dt + \sigma_d dz_d. \tag{12}$$

In what follows, this model is denoted by "Model (K)" (to symbolize "Keynesian").[14]

3 Analysis

This section analyzes Model (K) to derive an approximation formula for solution paths near the unique limit cycle observed in the deterministic case with $\sigma_e = \sigma_d = 0$ and an approximated stationary distribution of the limit cycle. By so doing, we evaluate the impact of random shock on deterministic business cycles.

3.1 Existence and uniqueness of equilibrium

We define (deterministic) equilibrium of Model (K) as $(r^e, a) = (r^*, a^*) \in \mathbb{R}^2_{++}$ that satisfies $\dot{r}^e = \dot{a} = 0$. It can also be defined as a solution of the following simultaneous equations:

$$0 = f(r^e) + a - sr^e,$$

$$0 = \delta + \mu - f(r^e).$$

Then, the unique equilibrium of Model (K), if it exists, can be given as follows:

$$(r^*, a^*) = (f^{-1}(\delta + \mu), sf^{-1}(\delta + \mu) - (\delta + \mu)). \tag{13}$$

This unique equilibrium can be shown to exist by the following assumption together with Assumption 1 (cf. Murakami [29, 32]).

Assumption 2. *The following conditions are satisfied:*

$$f(0) < \delta + \mu < \lim_{r^e \to \infty} f(r^e),$$

$$f\Big(\frac{\delta + \mu}{s}\Big) < \delta + \mu.$$

[14] Model (K) can be regarded as a stochastic version of Murakami's [29, 32].

3.2 Reformulation of Model (K)

For our analysis, we reformulate Model (K) introducing the following variables:

$$x = r^e - r^*, \tag{14}$$

$$y = \ln a - \ln a^*. \tag{15}$$

Model (K) can then be transformed as follows:

$$dx = [\phi(y) - F(x)]dt + \sigma_e dz_e, \tag{16}$$

$$dy = -g(x)dt + \sigma_d dz_d, \tag{17}$$

where

$$g(x) = f(f^{-1}(\delta + \mu) + x) - (\delta + \mu), \tag{18}$$

$$F(x) = \alpha\Big[x - \frac{1}{s}g(x)\Big], \tag{19}$$

$$\phi(y) = \alpha\Big[f^{-1}(\delta + \mu) - \frac{\delta + \mu}{s}\Big][\exp(y) - 1]. \tag{20}$$

In what follows, Model (K) is redefined as the system of equations (16) and (17) with (18)-(20), equivalent to the original Model (K).[15]

It is easily seen from (14) and (15) that (the redefined) Model (K) possesses unique (deterministic) equilibrium at the origin $(0, 0)$.

3.3 Existence, uniqueness and stability of a limit cycle

To establish the existence, uniqueness and stability of a limit cycle in the deterministic case ($\sigma_e = \sigma_d = 0$) of Model (K), we make the following assumptions (cf. Murakami [32]).

Assumption 3. *The following condition is satisfied:*

$$f'(f^{-1}(\delta + \mu)) > s.$$

Assumption 4. *The following condition is satisfied:*

$$\lim_{r^e \to \infty}[sr^e - f(r^e)] = \infty.$$

[15] The redefined Model (K) can be taken as a stochastic version of a generalized Liénard system. For applications of (deterministic) generalized Liénard systems to economics, see Murakami [29, 30, 31, 32].

The following equation (with respect to r^e) possesses exactly two nonnegative roots $\underline{r}^e$ and $\overline{r}^e$ with $\underline{r}^e < \overline{r}^e$:

$$f'(r^e) = s.$$

Furthermore, the following condition is satisfied:

$$f(\overline{r}^e) < s\overline{r}^e.$$

Assumption 3 says that the unique equilibrium is locally asymptotically totally unstable in the deterministic case.[16] In this sense, this assumption is concerned with a local nature (at the equilibrium value r^*) of the capital formation function f. Assumption 4 is, on the other hand, related to the global nature of this function. Indeed, the function f can be illustrated as in figure 1 under Assumptions 1-4. Thus, Model (K), with $\sigma_e = \sigma_d = 0$, has quite a lot in common with Kaldor [17] (cf. Murakami [27]) in that the investment (or capital formation) function has a sigmoid shape.

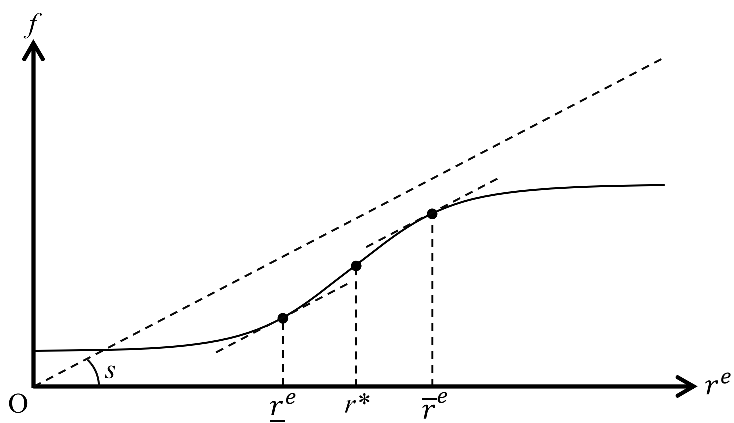

Figure 1: $f(r^e)$

We can also explore the implications of Assumptions 3 and 4 in (the reformulated) Model (K) (cf. Murakami [32]). Assumption 3 is equivalent to

$$F'(0) < 0.$$

Assumption 4, together with Assumption 3, implies that

$$\lim_{x \to \infty} F(x) = \infty,$$

[16]In this paper, the unique equilibrium is said to be "locally asymptotically totally unstable" if it is an unstable node or an unstable focus.

and that $F'(x) = 0$ has exactly two real roots $\underline{x} = \underline{r}^e - r^* < 0$ and $\overline{x} = \overline{r}^e - r^* > 0$ with

$$F(\overline{x}) < 0.$$

The following theorem ensures the existence, uniqueness and stability of a limit cycle in the deterministic case.

Theorem 1. *Let Assumptions 1-4 hold. Assume that α is sufficiently large and that $\sigma_e = \sigma_d = 0$. Then, for every initial condition $(x(0), y(0)) \in D \setminus (0,0)$, there exists a unique solution path of Model (K), $(x(t), y(t))$, for all $t \geq 0$, and it converges to a unique and stable limit cycle on D as $t \to \infty$, where*[17]

$$D = \{(x, y) \in \mathbb{R}^2 : x > -r^*\}.$$

Proof. See Murakami [32]. □

The unique and stable limit cycle established in Theorem 1 can be taken as the unique growth cycle in the economy without stochastic disturbances. Thus, this theorem implies that the economy, in the absence of random shocks, almost surely undergoes persistent cyclical fluctuations along the convergence to the unique growth cycle. This consequence may be interpreted to mean that persistent business cycles are inevitable in capitalist economies, characterized by frequent revisions of prospected profits, and taken as a strong theoretical support for Keynes' [20] view on business cycles (cf. Murakami [32]).

3.4 Small stochastic disturbances

To examine in detail the influence of stochastic disturbances on the unique limit cycle, we consider the case in which the sizes of them, σ_e and σ_d, are small enough. Specifically, we assume that they can be written as follows:

$$\sigma_e = \epsilon \tilde{\sigma}_e,$$

$$\sigma_d = \epsilon \tilde{\sigma}_d,$$

where $\tilde{\sigma}_e$ and $\tilde{\sigma}_d$ are positive constants; ϵ is a small positive constant. In this case, Model (K) reduces to the following:

$$dx = [\phi(y) - F(x)]dt + \epsilon \tilde{\sigma}_e dz_e, \tag{21}$$

$$dy = -g(x)dt + \epsilon \tilde{\sigma}_d dz_d. \tag{22}$$

In what follows, the system of equations (21) and (22) with (18)-(20) is denoted by Model (K*).

[17] It is seen from (14) and (15) that $(x, y) \in D$ is equivalent to $(r^e, a) \in \mathbb{R}^2_{++}$.

3.5 Behavior near the unique limit cycle

We introduce local coordinates in a neighborhood of the unique limit cycle for Model (K*) to explore the behavior of solution paths in the neighborhood.[18] To this end, denote the unique limit cycle by Γ and let $(x^*(t), y^*(t))$ and $T > 0$ be the solution path and (smallest) period of Γ (as in Proof of Theorem 1), respectively. We make the following transformation in a neighborhood of Γ:

$$x = x^*(\theta) + g(x^*(\theta))R, \tag{23}$$

$$y = y^*(\theta) + [\phi(y^*(\theta)) - F(x^*(\theta))]R. \tag{24}$$

Noting that $(x - x^*, y - y^*)$ is orthogonal to the vector field along Γ or $(\phi(y^*) - F(x^*), -g(x^*))$, it can be seen that θ is the phase (time) of the point on Γ nearest to (x, y) and that the absolute value of R corresponds to the "distance" between (x, y) and Γ.[19] Thus, every point (x, y) near the limit cycle Γ can be represented by the "phase" θ and "amplitude" R.

It is seen from Bonnin [8, pp. 638-639, Theorem 1] that the following stochastic differential equations of θ and R can be derived from (16), (17), (23) and (24):

$$d\theta = [1 + a_\theta(\theta, R) + \epsilon^2 \tilde{a}_\theta(\theta, R)]dt + \epsilon[\tilde{\sigma}_e \tilde{b}_e(\theta, R)dz_e + \tilde{\sigma}_d \tilde{b}_d(\theta, R)dz_d], \tag{25}$$

$$dR = [L(\theta)R + a_R(\theta, R) + \epsilon^2 \tilde{a}_R(\theta, R)]dt + \epsilon[\tilde{\sigma}_e b_e(\theta, R)dz_e + \tilde{\sigma}_d b_d(\theta, R)dz_d], \tag{26}$$

where L, a_θ, a_R, $\tilde{a}_\theta$ and $\tilde{a}_R$ are continuously differentiable and b_e, b_d, $\tilde{b}_e$ and $\tilde{b}_d$ are twice continuously differentiable (due to Assumption 1) with

$$L(\theta) = -F'(x^*(\theta)) - \frac{d}{d\theta}\ln([\phi(y^*(\theta)) - F(x^*(\theta))]^2 + g^2(x^*(\theta))). \tag{27}$$

$$a_\theta(\theta, 0) = a_R(\theta, 0) = \frac{\partial a_R}{\partial R}(\theta, 0) = 0, \tag{28}$$

$$b_e(\theta, 0) = \frac{g(x^*(\theta))}{[\phi(y^*(\theta)) - F(x^*(\theta))]^2 + g^2(x^*(\theta))}, \tag{29}$$

$$b_d(\theta, 0) = \frac{\phi(y^*(\theta)) - F(x^*(\theta))}{[\phi(y^*(\theta)) - F(x^*(\theta))]^2 + g^2(x^*(\theta))}. \tag{30}$$

The behavior of solution paths of Model (K*) near the unique limit cycle Γ can be described by (25) and (26).

[18] In Andronov et al. [2, chap. V, sect. 7] and Hale [12, chap. VI], similar methods can be found for analyzing the behavior of solution paths near periodic orbits in ordinary differential equations. Our method, following Bonnin [8], may be taken as a stochastic differential equation version of them.

[19] Note that the actual distance between (x, y) and Γ, denoted by ρ, is given by

$$\rho = \frac{|R|}{\sqrt{\phi(y^*(\theta)) - F(x^*(\theta))]^2 + g^2(x^*(\theta))}}.$$

3.6 Approximation of solution paths

We consider the deviation of R around the unique limit cycle Γ. Since $R = 0$ on Γ due to (23) and (24), we may write

$$R = 0 + \epsilon\tilde{R} = \epsilon\tilde{R}. \tag{31}$$

Then, equations (25) and (26) can be reduced to the following stochastic differential equations of θ and $\tilde{R}$:

$$\begin{aligned} d\theta &= [1 + a_\theta(\theta,\epsilon\tilde{R}) + \epsilon^2\tilde{a}_\theta(\theta,\epsilon\tilde{R})]dt + \epsilon[\tilde{\sigma}_e\tilde{b}_e(\theta,\epsilon\tilde{R})dz_e + \tilde{\sigma}_d\tilde{b}_d(\theta,\epsilon\tilde{R})dz_d], \\ d\tilde{R} &= \Big[L(\theta)\tilde{R} + \frac{1}{\epsilon}a_R(\theta,\epsilon\tilde{R}) + \epsilon\tilde{a}_R(\theta,\epsilon\tilde{R})\Big]dt + \tilde{\sigma}_e b_e(\theta,\epsilon\tilde{R})dz_e + \tilde{\sigma}_d b_d(\theta,\epsilon\tilde{R})dz_d. \end{aligned}$$

It is seen from (28) that letting $\epsilon \to 0$, they can be approximated as follows:[20]

$$d\theta = dt, \tag{32}$$

$$d\tilde{R} = L(\theta)\tilde{R}dt + \tilde{\sigma}_e b_e(\theta,0)dz_e + \tilde{\sigma}_d b_d(\theta,0)dz_d. \tag{33}$$

Solving (32) and (33), we can establish the following fact on solution paths near the unique limit cycle.

Theorem 2. *Let Assumptions 1-4 hold. Assume that α is sufficiently large and that ϵ is sufficiently small. Assuming that $\theta(0) = 0$ (without loss of generality), solution paths of Model (K*) in a neighborhood of the unique limit cycle can be approximated as follows:*

$$\theta(t) = t, \tag{34}$$

$$R(t) = R(0)P(t)\exp(\lambda t) + \epsilon\tilde{\sigma}_e\int_0^t \frac{P(t)}{P(s)}\exp(\lambda(t-s))b_e(t,0)dz_e + \epsilon\tilde{\sigma}_d\int_0^t \frac{P(t)}{P(s)}\exp(\lambda(t-s))b_d(t,0)dz_d, \tag{35}$$

where[21]

$$\lambda = -\frac{1}{T}\int_0^T F'(x^*(\theta))d\theta, \tag{36}$$

$$\begin{aligned} P(t) &= \exp\Big(\int_0^t [L(\theta) - \lambda]d\theta\Big) \\ &= \frac{[\phi(y^*(0)) - F(x^*(0))]^2 + g^2(x^*(0))}{[\phi(y^*(t)) - F(x^*(t))]^2 + g^2(x^*(t))}\exp\Big(\frac{t}{T}\int_0^T F'(x^*(\theta))d\theta - \int_0^t F'(x^*(\theta))d\theta\Big). \end{aligned} \tag{37}$$

[20] Note, in particular, that due to (28),

$$\frac{1}{\epsilon}a_R(\theta,\epsilon\tilde{R}) = \frac{1}{\epsilon}\Big[a_R(\theta,0) + \frac{\partial a_R}{\partial R}(\theta,0) + o(\epsilon)\Big] = \frac{1}{\epsilon}o(\epsilon) \to 0,$$

as $\epsilon \to 0$.

[21] It is seen from (27), (34), (36) and (37) that $P(t) > 0$ and $P(t+T) = P(t)$ for all $t \geq 0$.

Proof. The solution of (32) with $\theta(0) = 0$ is obviously (34). After substitution of (34), we can solve (33) to obtain

$$\tilde{R}(t) = \tilde{R}(0)P(t)\exp(\lambda t) + \tilde{\sigma}_e \int_0^t \frac{P(t)}{P(s)} \exp(\lambda(t-s))b_e(\theta(t),0)dz_e + \tilde{\sigma}_d \int_0^t \frac{P(t)}{P(s)} \exp(\lambda(t-s))b_d(\theta(t),0)dz_d.$$

This implies (35) due to (31). □

Theorem 2 presents an explicit (approximation) formula for solution paths near the unique limit cycle. Since such a formula has not been derived in the related literature, this theorem may be taken as one of the notable contributions of our study.[22]

3.7 Stationary distribution for the unique limit cycle

We derive the (approximated) stationary distribution of solution paths starting on the unique limit cycle. It follows from (32) and (33) that

$$dE[\tilde{R}] = L(\theta)E[\tilde{R}]d\theta, \tag{38}$$

With $\theta(0) = 0$, this be solved as follows:

$$E[\tilde{R}] = \tilde{R}(0)\exp\Big(\int_0^\theta L(\theta')d\theta'\Big) = \tilde{R}(0)P(\theta)\exp(\lambda\theta). \tag{39}$$

This is the (approximated) mean of $\tilde{R}$.

It is seen from (32), (33) and (38) that

$$d(\tilde{R} - E[\tilde{R}]) = L(\theta)(\tilde{R} - E[\tilde{R}])d\theta + \tilde{\sigma}_e b_e(\theta,0)dz_e + \tilde{\sigma}_d b_d(\theta,0)dz_d.$$

It follows from Ito's formula that

$$d(\tilde{R} - E[\tilde{R}])^2 = \{2L(\theta)(\tilde{R} - E[\tilde{R}])^2 + B(\theta)\}d\theta + \tilde{\sigma}_e b_e(\theta,0)dz_e + \tilde{\sigma}_d b_d(\theta,0)dz_d,$$

which implies that

$$d\mathrm{Var}[\tilde{R}] = \{2L(\theta)\mathrm{Var}[\tilde{R}] + B(\theta)\}d\theta, \tag{40}$$

[22]In Bashkirtseva et al. [6], Bashkirtseva et al. [7], Bashkirtseva et al. [4] and Jungeilges and Ryazanova [15, 16], the stochastic sensitivity approach of Bashkirtseva and Ryashko [5] is employed to discuss the (approximated) stationary distribution of deterministic stable limit cycles (for the stationary distribution in Model (K*), see Theorem 3), but explicit (approximated) solutions are not derived.

where

$$B(\theta) = \tilde{\sigma}_e^2 b_e^2(\theta, 0) + \tilde{\sigma}_d^2 b_d^2(\theta, 0) = \frac{\tilde{\sigma}_e^2 g^2(x^*(\theta)) + \tilde{\sigma}_d^2[\phi(y^*(\theta)) - F(x^*(\theta))]^2}{\{[\phi(y^*(\theta)) - F(x^*(\theta))]^2 + g^2(x^*(\theta))\}^2}. \tag{41}$$

Thus, we have

$$\sigma(\theta) \equiv \mathrm{Var}[\tilde{R}] = \sqrt{\int_0^\theta B(\theta') \exp\Big(2\int_{\theta'}^\theta L(\theta'')d\theta''\Big)d\theta'} = \sqrt{\int_0^\theta B(\theta') \int_{\theta'}^\theta \frac{P^2(\theta)}{P^2(\theta'')} \exp(2\lambda(\theta - \theta''))d\theta'' d\theta'}. \tag{42}$$

This is the (approximated) variance of $\tilde{R}$.

The following theorem characterizes the (approximated) stationary probability distribution of solution paths of Model (K*) starting on the unique limit cycle Γ.

Theorem 3. *Under the hypotheses of Theorem 2, the stationary probability density function for solution paths of Model (K*) with the initial conditions on the unique limit cycle (i.e., with $(x(0), y(0)) \in \Gamma$), denoted by $p^*(\theta, R)$, can be approximated as follows:*

$$p^*(\theta, R) = \frac{1}{\sqrt{2\pi}\epsilon\sigma(\theta)} \exp\Big(-\frac{R^2}{2\epsilon^2\sigma^2(\theta)}\Big). \tag{43}$$

Proof. It can be found from (39)-(42) that if $(x(0), y(0)) \in \Gamma$, i.e., if $R(0) = 0$ or $\tilde{R}(0) = 0$, the stationary distribution of $(\theta, \tilde{R})$ can be characterized by the following probability density function:[23]

$$p(\theta, \tilde{R}) = \frac{1}{\sqrt{2\pi}\sigma(\theta)} \exp\Big(-\frac{\tilde{R}^2}{2\sigma^2(\theta)}\Big),$$

because $E[\tilde{R}] = 0$. Then, the probability density function (43) follows from (31). □

Theorem 3 provides the distribution of the unique growth cycle (limit cycle) when the economy is exposed to small random shocks (stochastic disturbances). In this sense, the stationary probability density function p^* evaluates the impact of external shocks and the robustness of (deterministic) business cycles.[24]

[23] This function can be obtained as a stationary solution of the Fokker-Planck equation for (32) and (33):

$$\frac{\partial p}{\partial t}(\theta, \tilde{R}, t) = -\frac{\partial p}{\partial \theta}(\theta, \tilde{R}, t) - \frac{\partial}{\partial \tilde{R}}[L(\theta)p(\theta, \tilde{R}, t)] + \frac{1}{2}\frac{\partial^2}{\partial \tilde{R}^2}[B(\theta)p(\theta, \tilde{R}, t)].$$

For details, see Bonnin [8].

[24] In the stochastic sensitivity function approach of Bashkirtseva and Ryashko [5], the stationary probability density function of stable limit cycles, corresponding to our $p^*(\theta, R)$, is discussed under the restriction of $p^*(\theta + T, R) = p^*(\theta, R)$ because the variance $\mathrm{Var}[\tilde{R}(\theta)]$, as a solution of (40), is evaluated under the constraint of $\mathrm{Var}[\tilde{R}(T)] = \mathrm{Var}[\tilde{R}(0)]$. The difference between their approach and ours is due to whether the phases θ and $\theta + T$ are identified or not; in our Theorems 2 and 3, they are distinguished.

4 Numerical analysis

This section performs numerical simulations to confirm the analytical results in Sect. 3. To this end, we specify the parameter values and the capital formation function based on the empirical data in Japan.[25]

4.1 Setup

Based on the average value of the ratio of aggregate saving S to aggregate profits Π, we set the propensity to save s to[26]

$$s = 0.7. \tag{44}$$

To formulate the capital formation function f consistent with Assumptions 1-4, we assume that it is written in the following logistic form:[27]

$$\frac{I}{K} = f(r^e) = \frac{\beta_l + \beta_u \exp(\beta_0 + \beta r^e)}{1 + \exp(\beta_0 + \beta_r r^e)},$$

where β_0 and β_r are, respectively, a real constant and a positive constant; β_l and β_u are a real constant and a nonnegative constant with $\beta_l < \beta_u$. Note that the upper and lower limits of f are β_u and β_l, respectively. It can thus be seen that

$$\ln\Big(\frac{I/K - \beta_l}{\beta_u - I/K}\Big) = \beta_0 + \beta_r r^e.$$

For estimation, we may write

$$\ln\Big(\frac{(I/K)_t - \beta_l}{\beta_u - (I/K)_t}\Big) = \beta_0 + \beta_r r^e_t. \tag{45}$$

In (45) and what follows, t represents the year of the data.

Since the expected rate of profit r^e is unobservable, we conduct estimations for (45) using the rate of profit $r = \Pi/K$, which is observable. For this purpose, we write a discrete version of (7), without stocahastic disturbances, in the following form:

$$r^e_t - r^e_{t-1} = \alpha(r_{t-1} - r^e_{t-1}),$$

[25] For details on the data, see Appendix.

[26] It is seen from (3) that s can be obtained from S and Π alone. The empirical average of S/Π is 0.718.

[27] For logistic capital formation functions, it is possible to provide a theoretical foundation from a viewpoint of econophysics (cf. Murakami [30, 31]).

It then follows from (45) that

$$\ln\Big(\frac{(I/K)_t-\beta_l}{\beta_u-(I/K)_t}\Big)-(1-\alpha)\ln\Big(\frac{(I/K)_{t-1}-\beta_l}{\beta_u-(I/K)_{t-1}}\Big)=\alpha(\beta_0+\beta_r r_{t-1}), \tag{46}$$

which implies that

$$\Big(\frac{I}{K}\Big)_t=\frac{\beta_l[\beta_u-(I/K)_{t-1}]^{1-\alpha}+\beta_u\exp(\alpha(\beta_0+\beta_r r_{t-1}))[(I/K)_{t-1}-\beta_l]^{1-\alpha}}{[\beta_u-(I/K)_{t-1}]^{1-\alpha}+\exp(\alpha(\beta_0+\beta_r r_{t-1}))[(I/K)_{t-1}-\beta_l]^{1-\alpha}}. \tag{47}$$

We estimate the parameters in (47) by way of the method of nonlinear least squares.[28] The estimates are given as follows:[29]

$$(\hat{\alpha},\hat{\beta}_0,\hat{\beta}_r,\hat{\beta}_l,\hat{\beta}_u)=(0.59683,-2.03917,4.90721,-0.08947,0.61235).$$

Then, we set the parameters as follows:

$$(\alpha,\beta_0,\beta_r,\beta_l,\beta_u)=(0.6,-2.0,5.0,-0.08,0.6). \tag{48}$$

The capital formation function can thus be formalized as follows:

$$f(r^e)=\frac{-0.08+0.6\exp(-2.0+5.0r^e)}{1+\exp(-2.0+5.0r^e)}. \tag{49}$$

Based on the average of the ratio of aggregate depreciation of capital to aggregate stock of capital,[30] we set the rate of capital depreciation δ to

$$\delta=0.1. \tag{50}$$

Finally, we set the mean rate of change in autonomous demand μ to

$$\mu=0.03. \tag{51}$$

It can be confirmed that Assumptions 1-4 are all satisfied with (44) and (49)-(51).

In what follows, Model (K*) with (44) and (48)-(51) is denoted by "Model (NK*)."

[28] To determine the parameter values, it is possible to estimate (46) by the method of maximum likelihood. But the computational load is much higher in this method.

[29] For the nonlinear least squares estimation, we set the initial values to

$$(\alpha,\beta_0,\beta_r,\beta_l,\beta_u)=(0.6,-2.8,8.2,0,0.4).$$

[30] The empirical average of this ratio is 0.127.

4.2 Simulations

We first confirm that Model (NK*) possesses a (deterministic) unique limit cycle if $\epsilon = 0$. The red and blue curves in figure 2 illustrate the solution paths of Model (NK*) with $\epsilon = 0$, respectively, for $(x(0), y(0)) = (0.1, 0)$ and for $(x(0), y(0)) = (-0.2, 0)$. It can be seen that there exists a unique limit cycle in Model (NK*) with $\epsilon = 0$.

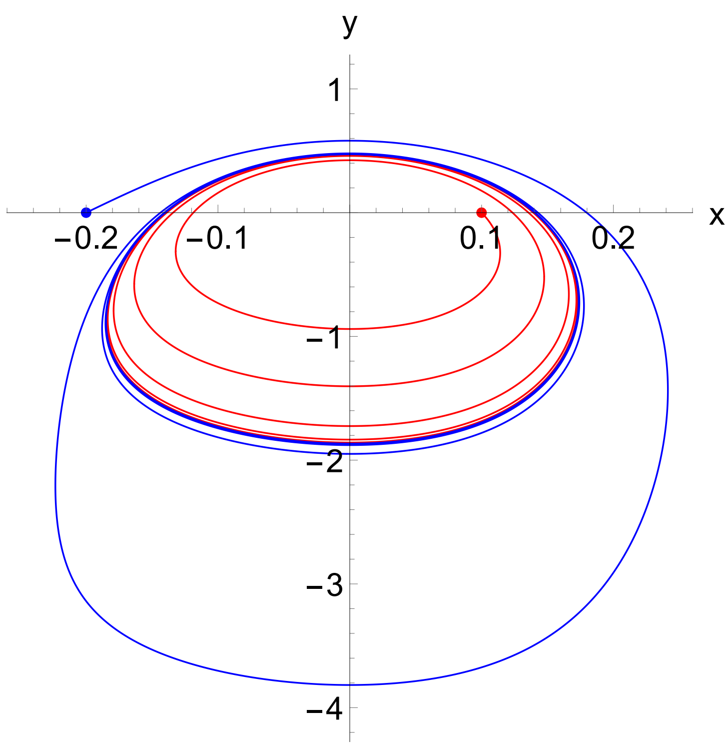

Figure 2: Model (NK*) with $\epsilon = 0$

We then examine Model (NK*) in the case of $\epsilon > 0$. In figure 3, the blue curve describes a sample solution path of Model (NK*) with $\epsilon\tilde{\sigma}_e = \epsilon\tilde{\sigma}_d = 0.01$ for $(x(0), y(0)) = (0, -1.863703)$, which lies on the unique limit cycle illustrated by the dashed black curve. In figures 4 and 5, respectively, the blue curves describe the time series of x and y along the same sample solution path, and the area enclosed by the two red curves represent the approximated 95% confidence intervals of x and y calculated by (14), (15) and (43) in Theorem 3, while the dashed black curves correspond to the unique limit cycle paths. It is seen that the prediction based on our analysis shows a reliable performance.

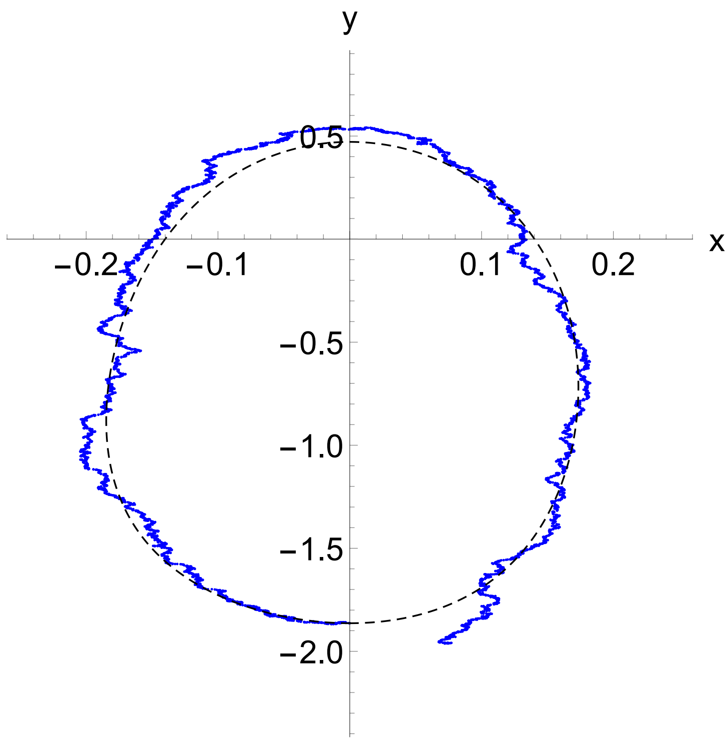

Figure 3: Model (NK*) with $\epsilon\tilde{\sigma}_e = \epsilon\tilde{\sigma}_d = 0.01$

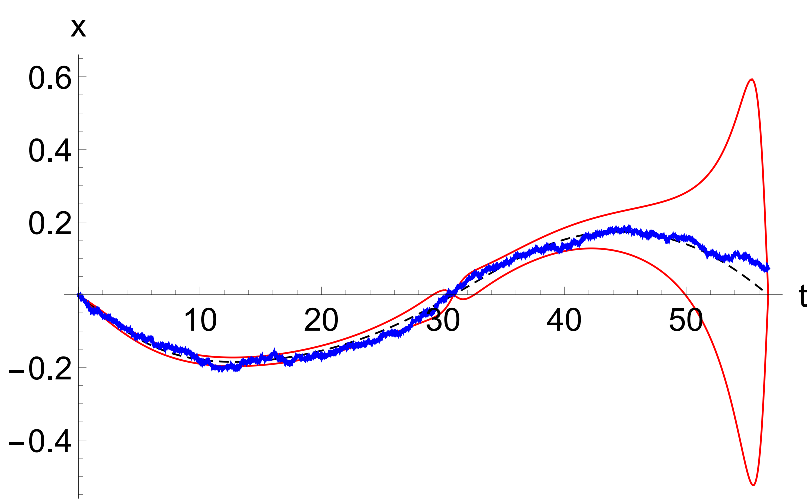

Figure 4: Sample path and 95% confidence interval of x

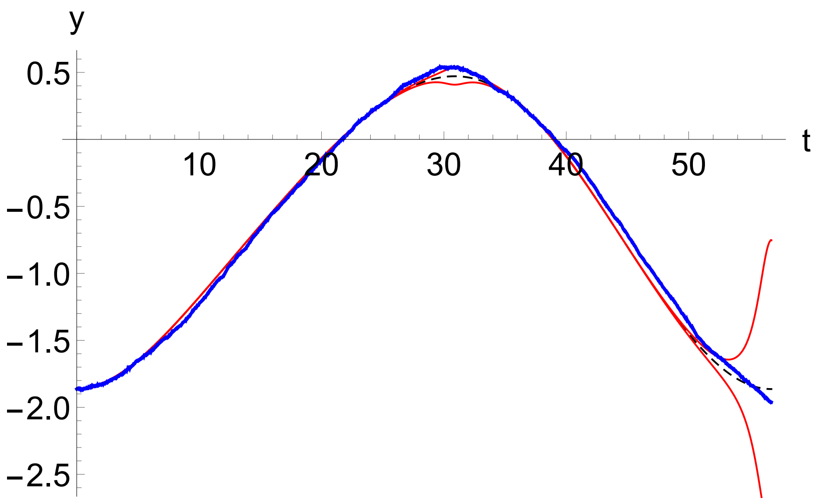

Figure 5: Sample path and 95% confidence interval of y

5 Conclusion

This paper has evaluated the impact of stochastic disturbances on deterministic limit cycles in a stochastic post-Keynesian model of growth cycles. We have employed a new method for the evaluation, different from those adopted in the related literature, and presented an explicit approximation formula for the stochastic model. By so doing, we have attempted to integrate the endogenous and exogenous approaches in the theory of business cycles. We hope that our analysis is helpful for better understanding of business cycles.

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Appendix: Data

For the estimation of the propensity to save s, we use the data taken from Annual Report on National Accounts, Cabinet Office, Japan, for the period of FY1980-FY2020. Aggregate gross saving S and aggregate gross profits Π are defined as follows:

$S =$ "Saving, gross",

$\Pi =$ "Operating surplus and mixed income" + "Consumption of fixed capital".

For the estimation of the capital formation function f, we use the data taken from Financial Statements Statics of Corporations by Industry, Ministry of Finance, Japan, for the period of FY1962-FY2020. Aggregate stock of capital K, aggregate gross investment I and aggregate profits Π are defined as follows:

K = the average of "Other tangible fixed assets" + "Construction in progress" at the beginning and end of the period,

I = the change in "Other tangible fixed assets" + "Construction in progress" during the period + "Depreciation expenses",

Π = "Operating profits" + "Depreciation expenses".

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